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INTERMITTENT TRANSITIONS IN THE QUASIPERIODICALLY FORCED MAPS

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We study the mechanisms of dynamical transitions accompanied by intermittent behavior in the quasiperiodically forced Henon map. In terms of rational approximations, we show that the intermittent transition from smooth attracting invariant curve to a strange nonchaotic attractor occurs via phase-dependent saddle-node bifurcation between the invariant curve and a new kind of invariant «ring-shaped» unstable (saddle) set. We also investigate the mechanisms of interior and basin-boundary crises occurring to the strange nonchaotic and chaotic attractors in the model system. It is shown that a collision of the strange nonchaotic attractor or chaotic attractor with the ring-shaped unstable set may cause an interior or basin-boundary crisis, depending upon the present structure of the basin of the attractor.

Introduction

In recent years, dynamical transitions in the quasiperiodically forced systems have become the topic of instant interest of the researches. Much attention has been paid to investigation of different routes of transition from regular quasiperiodic motion to strange nonchaotic attractor (SNA) [1,2] and observation of crises of the SNAs and chaotic attractors (CAs). However, until the recent moment, the mechanisms of many dynamical transitions still remained unclear.

An intermittent route from smooth torus to SNA was first reported in the work [3], where the quasiperiodically forced logistic map was considered as a representative model. The mechanism of this transition was explained in the recent work [4]. In the last paper, authors used the methods of rational approximations (RAs) as the tool of investigation. In terms of RAs, they observed a new kind of invariant «ring-shaped» unstable sets, which are different from smooth unstable tori of the system. It was shown, that the intermittent transition from smooth torus to SNA in the quasiperiodically forced noninvertible 1D maps occurs via phase-dependent saddle-node bifurcation (SNB) between the invariant curve and the ring-shaped unstable set. Authors also discussed a possible role of the ring-shaped unstable sets in the mechanisms of the band-merging, interior and basin-boundary crises of strange attractors, which were observed in ref. [5].

In present paper we consider the quasiperiodically forced Henon map [5], which can be regarded as the model of Poincare map of a hypothetical nonlinear oscillator driven by external biharmonic signal with irrational rate of frequencies. We investigate the underlying mechanisms of the intermittent transition to SNA and interior and basin-boundary crises, which occur to regular and strange attractors in this system.

1. Ring-shaped unstable sets and intermittent route to SNA

We consider the model system in the form as in ref. [5]:

$$x_{n+1} = 1 - ax_n^2 + y_n + \varepsilon\cos 2\pi\theta_n,$$
$$y_{n+1} = bx_n, \quad (1)$$
$$\theta_{n+1} = \theta_n + \omega, (\mathrm{mod}1),$$

where $\theta \in S^1$. The frequency ω is traditionally set to be the reciprocal of the *golden mean* value: $\omega=(5^{1/2}-1)/2$.

The phase diagram of the system (1) is shown in the fig. 1, *a*. In order to characterize the dynamical regimes in any point of the diagram, we compute the nontrivial Lyapunov exponents $\lambda_{1,2}$ and the *phase sensitivity exponent* δ; the last one measures sensitivity with respect to the phase of the quasiperiodic forcing and characterizes strangeness of the attractor [6]. A smooth attracting invariant curve is characterized by the values $\lambda_{1,2}<0$ and $\delta=0$. The region where it exists is denoted by *T*. The region of double attracting curve is denoted by *2T*. On the other hand, the chaotic attractor has one positive Lyapunov exponent $\lambda_1>0$ ($\lambda_2<0$); its region is shown in black. On the border between regions of regular and chaotic dynamics, SNA exists, which has negative Lyapunov exponents ($\lambda_{1,2}<0$) and positive phase sensitivity exponent ($\delta>0$). The corresponding regions are denoted in gray and dark gray. In the thin gray region SNA appears due to intermittent mechanism, while in the dark gray regions other scenarios of transition to SNA (gradual fractalization or tori collision) take place. Note, that the chaotic region is separated into two parts by «tongue» of quasiperiodic regimes. Such structure is typical for the phase diagrams of the quasiperiodically forced period-doubling systems [4]. The intermittent transition to SNA (denoted by route **a** in the fig. 1, *a*) occurs along the border of the upper chaotic region.

At a=0.95 and ε=0.4761 the system (1) has a smooth attracting invariant curve, shown in fig. 1, *b*. Besides this attracting curve, there is a saddle invariant curve, which originally appeared together with the attracting one due to quasiperiodic saddle-node bifurcation. The structure of the phase space is determined by the *2D* invariant manifolds associated with the saddle curve. In the fig. 1, *c* we see the section of the phase space by the plane θ_0=0.2. The stable manifold W^S of the saddle invariant curve determines the boundary for the basin (shown in gray) of the attracting curve.

Let us proceed to the mechanism of the intermittent transition to SNA. As the parameter ε passes critical value $\varepsilon^*\approx 0.476148155$, the smooth attracting curve suddenly disappears, and SNA arises in some wide area of the phase space (see fig. 1, *d* for ε=0.476149). Now, the dynamics consists of laminar phases of motion in vicinity of the destroyed invariant curve and bursts away from it. Note, that the profile of the newly-born intermittent SNA is determined by the unstable manifold W^U of the saddle invariant curve. In order to illustrate this, in fig. 1, *e* we consider a section of the phase space by the plane θ_0=0.2, and draw a projection of the segment of SNA from the interval $\theta\in[\theta_0-0.01,\theta_0+0.01]$ upon this section. One can see, that the points of attractor (denoted by black dots in the fig. 1, *e*) are disposed along the unstable manifold W^U.

In order to explain the underlying mechanisms of the transition described above, we use the method of rational approximations (RAs). For the case of *golden mean* value of ω, the RAs can be obtained as the ratios of the Fibonacci numbers: $\omega_k=F_{k-1}/F_k$, where the sequence of $\{F_k\}$ is determined as $F_{k+1}=F_k+F_{k-1}$ with $F_0=0$ and $F_1=1$. Instead of the quasiperiodically forced system (1), we consider an infinite sequence of periodically forced maps with rational frequencies ω_k; the properties of the original system can be obtained in the quasiperiodic limit at $k\to\infty$. For the RA of level k, each periodically

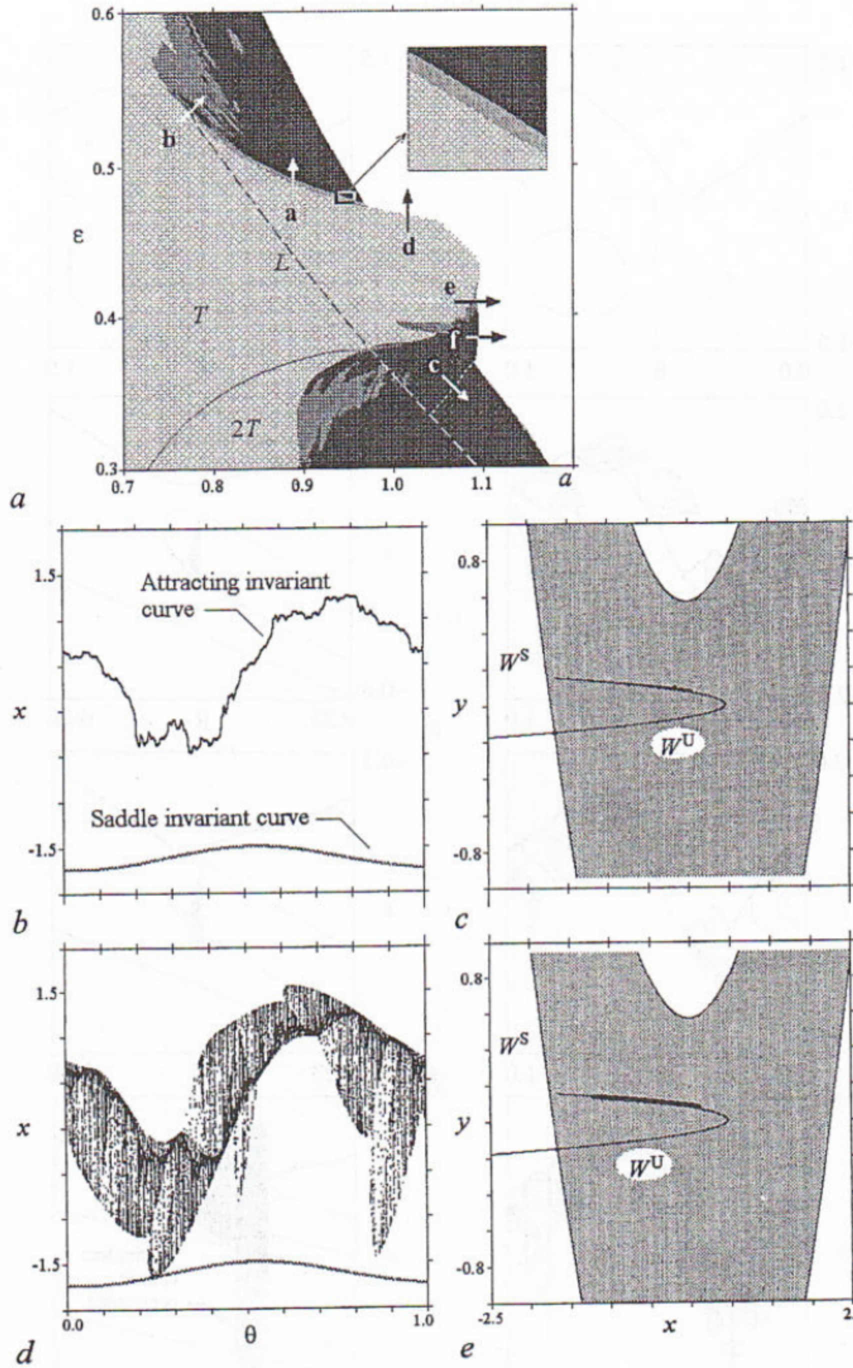

Fig. 1. a - The phase diagram of the system (1) (here and thereafter we set the parameter b=0.1). Regular, chaotic, SNA, and divergence regimes are shown in light gray, black, gray (or dark gray), and white, respectively. To show the existence of intermittent SNA (gray), a small segment near (a,ε)=[0.95,ε*] is magnified. Solid line denotes the doubling bifurcation of invariant curve. Due to interaction with the ring-shaped unstable sets born when passing the dashed line L, different dynamical transition such as intermittency (route **a**), interior (routes **b** and **c**), and basin-boundary (routes **d, e** and **f**) crises can occur. b - The smooth attractor (black) and saddle invariant curve (gray) at a=0.95 and ε=0.4761; c - the section of the phase space by plane θ_0=0.2 at the same parameter values; the basin of attractor is shown in gray; the black dot embedded into unstable manifold W^U is a section of smooth attractor. d - SNA at a=0.95 and ε=0.476149; e - the plane θ_0=0.2 at the same parameter values. Black dots distributed along W^U denote projection of SNA from the interval $\theta \in [\theta_0-0.01, \theta_0+0.01]$ on the plane

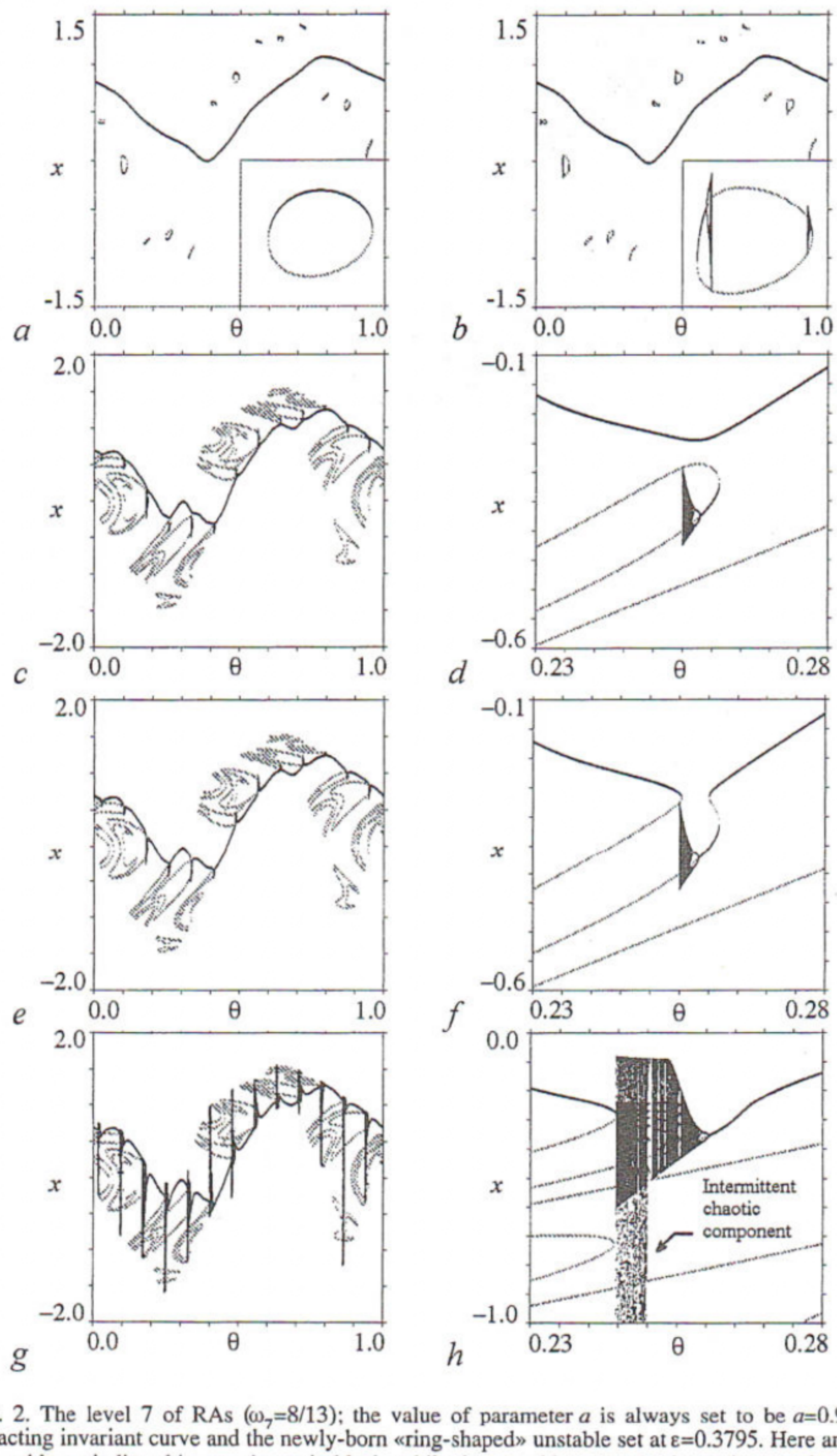

Fig. 2. The level 7 of RAs (ω_7=8/13); the value of parameter a is always set to be a=0.95. a - The attracting invariant curve and the newly-born «ring-shaped» unstable set at ε=0.3795. Here and thereafter the stable periodic orbits are shown in black, while the unstable ones are shown in gray. b - The typical structure of the ring-shaped unstable set after appearance of a chaotic component at ε=0.383. c, d - The smooth attracting invariant curve and the ring-shaped unstable set on the threshold of phase-dependent SNB at ε=0.4585; e, f - the destruction of the smooth attractor via phase-dependent SNB (ε=0.4588); g, h - the appearance of intermittent chaotic component of approximation of the SNA (ε=0.4610)

forced map has periodic or chaotic orbits, which depend upon the initial phase θ_0. As θ_0 varies within the interval $[0, 1/F_k)$, the union of such orbits forms full approximations of the attractors and saddles of the system (1). For instance, an approximation of a smooth attracting invariant curve at k-th level represents a uniform set of stable periodic orbits of period F_k. For the case of SNA, the rational approximation may contain orbits of different periods and even chaotic ones. An existence of bifurcations in the structure of rational approximation is a characteristic of strangeness of the attractor.

In terms of RAs, we observe invariant «ring-shaped» unstable sets [4], which are different from smooth unstable invariant curves of the system (1). When passing the dashed curve L in fig. 1, *a*, the ring-shaped unstable set of periodic orbits appears due to phase-dependent saddle-node bifurcation (SNB). At the level of approximation k=7 such set is composed of F_7=13 small rings (see fig. 2, *a*). These rings are formed by stable (black) and saddle (gray) F_7-periodic orbits. Originally, the rates of stable and unstable periodic orbits in the ring-shaped set are equal. However, as the parameter ε slightly increases, the chaotic attractor appears via period-doubling of stable periodic orbits, and then it disappears due to collision with the saddle F_7-periodic orbit, see fig. 2, *b*. In the last figure one can see that the rate of stable periodic orbits in the ring-shaped set has contracted, and the unstable (saddle) orbits have become dominant. Therefore we refer such set to as the ring-shaped *unstable* set of orbits. At sufficiently large values of ε, the shape of the rings changes and becomes more complicated (see fig. 2, *c*; more details on the ring-shaped unstable sets are given in ref. [4]).

As we approach the border of intermittent transition on the diagram, the ring-shaped unstable set come closer to the attracting invariant curve (fig. 2, *c*). Then, at some critical value $\varepsilon=\varepsilon_k^{(1)}$ [$\varepsilon_7^{(1)}\simeq$ 0.458706], the phase-dependent SNB between invariant curve and the saddle component of the ring-shaped unstable set occurs, as it is shown in fig. 2, *d*. This bifurcation destroys the invariant curve. From this moment, the attractor of the system becomes nonsmooth, since its approximation contains bifurcations and chaotic regimes. As the parameter ε increases further, the chaotic attractor, which was originally associated with the ring-shaped unstable set, undergoes interior widening at next critical value $\varepsilon=\varepsilon_k^{(2)}$[$\varepsilon_7^{(2)}\simeq$0.459639], and the intermittent chaotic component of the approximation appears, as it is shown in fig. 2, *e*. Thus, in terms of RAs the intermittent transition to SNA consists of two stages: the phase-dependent SNB and widening of the chaotic component of approximation of the SNA. However, in the quasiperiodic limit (at $k\to\infty$) the distance $\Delta\varepsilon_k[=\varepsilon_k^{(1)}-\varepsilon_k^{(2)}]$ between two transition points tends to zero as $|\Delta\varepsilon_k|\sim F_k^{-\alpha}$, where $\alpha\simeq$0.8, and the both values $\varepsilon_k^{(1,2)}$ converge to ε^*.

2. The new mechanisms of basin-boundary and interior crises

Collision of the attractor with unstable orbit lying on the basin boundary causes crisis, which destroys the attractor [5]. On the other hand, collision with unstable orbit inside the basin gives rise to abrupt widening of the attractor known as «interior» crisis. Previously, the mechanisms of basin-boundary and interior crises due to interaction with periodic and quasiperiodic orbits [5] were known. We found new mechanisms of crises of SNAs and CAs in the quasiperiodically forced maps due to collision with the ring-shaped unstable set. The dynamical transitions **a-c** near the «tongue» of quasiperiodic regimes (fig. 1, *a*) are associated with interior crises of the attractors: when a smooth attractor, SNA or CA collides with the ring-shaped unstable set lying inside the basin, then intermittency or widening crisis occurs. On the other hand, routes **d**, **e**, and **f** in the «tongue» correspond to the basin-boundary crises of a smooth attractor, SNA and CA, respectively.

For illustration, we consider the mechanism of basin-boundary crisis corresponding

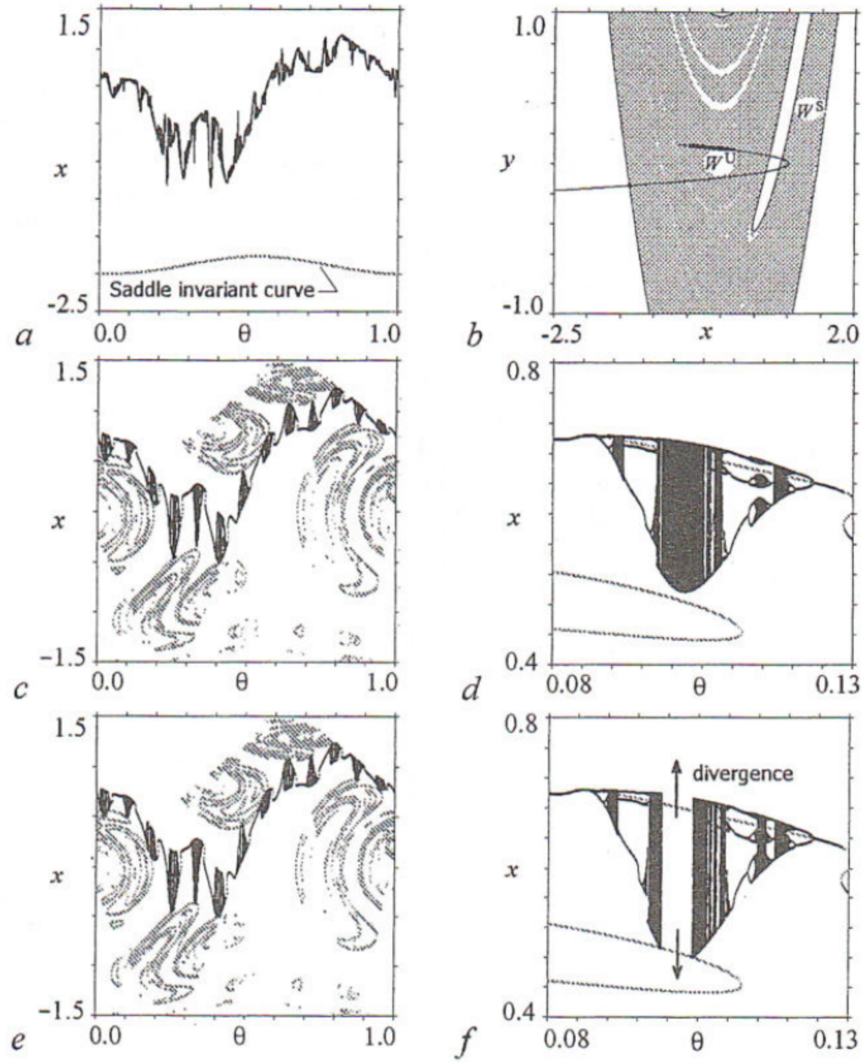

Fig. 3. *a* - The SNA on the threshold of crisis (route **e**) at paramter values a=1.0925, ε=0.41; *b* - section of the phase space by plane θ_0=0.41 at the same parameter values; black dots distributed along W^U denote projection of SNA from interval $\theta \in [\theta_0 - 0.01, \theta_0 + 0.01]$ on the section. The rational approximation of level k=7 for *c* - the SNA and the ring-shaped unstable set on the threshold of crisis at a=1.067, ε=0.41, and *e* - the «remnant» of SNA after crisis at a=1.069, ε=0.41. The figures *d* and *f* represent the enlarged fragments of *c* and *e*, respectively

to route **e**. The SNA (fig. 3, *a*) is disposed along the unstable manifold W^U, while the stable manifold W^S determines its basin (fig. 3, *b*). Due to homoclinic intersection of W^S and W^U, the SNA has fractal-like basin boundary. In terms of RAs, the crisis occurs due to collision of the chaotic component of approximation of SNA with the ring-shaped unstable set (see fig. 3, *c-f*). After such collision, the «gap» in approximation opens, where the trajectories exhibit divergence.

3. Conclusion

We studied dynamical transitions associated with intermittent behavior in the quasiperiodically forced Henon map, using the method of rational approximations. It was shown, that intermittent transition to SNA, as well as the interior and basin-boundary crises of strange attractors, typically occur due to collision of the attractor with a new kind of invariant «ring-shaped» unstable sets. These sets, which in the quasiperiodic limit apparently correspond to a fractal set of chaotic saddles, play a central role in dynamical transitions in the quasiperiodically forced systems.

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О ПЕРЕХОДАХ ЧЕРЕЗ ПЕРЕМЕЖАЕМОСТЬ В КВАЗИПЕРИОДИЧЕСКИ ВОЗБУЖДАЕМЫХ ОТОБРАЖЕНИЯХ

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Исследуются механизмы динамических переходов, сопровождающихся перемежающимся поведением, в квазипериодически возбуждаемом отображении Эно. На основе метода рациональной аппроксимации показано, что переход от гладкой притягивающей инвариантной кривой к странному нехаотическому аттрактору через перемежаемость происходит благодаря фазозависимой седло-узловой бифуркации между инвариантной кривой и неустойчивым (седловым) инвариантным множеством, именуемым в соответствии со своей топологией

«кольцеобразным множеством неустойчивых орбит». Исследованы бифуркационные механизмы кризисов столкновения с границей бассейна и внутренних кризисов для странных нехаотических и хаотических аттракторов модельной системы. Показано, что столкновение аттрактора с кольцеобразным множеством неустойчивых орбит может вызвать внутренний кризис или разрушение аттрактора в зависимости от устройства его бассейна притяжения.

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