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STATISTICAL PROPERTIES OF DETERMINISTIC AND NOISY CHAOTIC SYSTEMS

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This work represents a survey of the results that were recently obtained in the research group supervised by Prof. Dr. Vadim S. Anishchenko and published in a series of scientific papers. The presented results are referred to statistical description of dynamical chaos and to the effect of noise on different types of chaotic attractors. We consider peculiarities of the relaxation of an invariant probability measure in systems with chaotic attractors of different types and perform the correlation and spectral analysis of chaotic self-sustained oscillations.

1. Introduction

Dynamical chaos, like a random process, requires a statistical description. When chaotic systems are studied in computer or physical expriments, probability characteristics, such as a stationary probability distribution on an attractor, correlation functions, power spectra and others, are usually calculated or measured. Chaotic oscillations that correspond to different types of chaotic attractors in the phase space of dynamical systems are characterized by various statistical properties as well as by a different degree of sensitivity of the statistical characteristics to the influnce of noise.

From a viewpoint of the rigorous theory, hyperbolic chaos is often called «true» chaos and is characterized by a homogeneous and topologically stable structure [1-4]. However, strange chaotic attractors of dissipative systems are not, as a rule, robust hyperbolic sets. They are rather referred to as a nearly hyperbolic attractors, e.g., the Lorenz attractor. Nearly hyperbolic (quasi-hyperbolic) attractors include some nonrobust orbits, e.g. separatrix loops, but their appearances and disappearances often do not affect the observed characteristics of chaos, such as a phase portrait, the power spectrum, Lyapunov exponents and others. Dynamical systems in a chaotic regime may give rise to an invariant measure which does not depend on an initial distribution and fully reflects the statistical properties of the attractor. The existence of an invariant measure has been theoretically proven for hyperbolic and nearly hyperbolic systems [5-10].

However, the most of chaotic attractors which we deal with in numeric simulation and real experiments are nonhyperbolic [11-13]. The problem of the existence of an invariant measure on a nonhyperbolic chaotic attractor involves serious difficulties because it is generally impossible to obtain a stationary probability distribution being independent of an initial distribution. A nonhyperbolic attractor is a maximal attractor of the dynamical system and encloses a countable set of both regular and chaotic attracting subsets [11,12]. When δ-correlated Gaussian noise is added to the system, an invariant measure on such attractors exists too [14]. In the nonhyperbolic case the behavior of phase trajectories is significantly affected by noise [15-18] while it changes only slightly in systems with hyperbolic and nearly hyperbolic chaos [15,16,19,20].

A statistical description of noisy nonhyperbolic chaotic attractors is an important and still unsolved problem of the dynamical systems theory. One of the topical problems in this direction is to study the relaxation to stationary distributions in time. There are a number of fundamental questions which have as yet unclear answers. What is a real relaxation time of the system to a stationary distribution? Which factors define this time? Which characteristics can quantify the relaxation time to the stationary measure? What is the role of the noise statistics and the noise intensity in regularities of the relaxation to the stationary distribution? Is there any connection between the relaxation process and the system dynamics? These problems are studied in [21,22] with the methods of computer simulation.

The relaxation to a stationary distribution is described by the evolutionary operator that can be represented by the Fokker-Planck operator or the Frobenius-Perron operator. The eigenvalues and eigenfunctions of the evolutionary operator determine the rate and character of the relaxation process and characteristics of mixing, which are related to the relaxation to an invariant probability measure. However, if the dynamical system is high-dimensional ($N \geq 3$), the nonstationary solution of the Fokker-Planck equation is difficult enough to find even numerically. Therefore, the method of stochastic differential equations was used in the studies described in [21,22].

The presence of mixing causes autocorrelation functions to decay to zero for large times (correlation splitting). This means that the system states separated by a sufficiently large time interval become statistically independent [6,8,23-25]. From the property of mixing it follows that a dynamical system is ergodic. Additionally, for chaotic dynamical systems the splitting of correlations in time is connected with an instability of chaotic trajectories and with the system property to produce entropy [6,8,23-27]. In spite of their significant importance, correlation properties of chaotic processes have been studied insufficiently. It is widely believed that autocorrelation functions of chaotic systems exponentially decrease at a rate being defined by the Kolmogorov entropy [23]. The Kolmogorov entropy, H_K, in turn is bounded from above by the sum of positive Lyapunov exponents [8,27,28]. But this estimation is true only for some special cases.

It has been proven for some classes of discrete maps (expanding and Anosov ones), which exhibit a mixing invariant measure, that the decay of correlations with time is bounded from above by an exponential function [9,29-31]. There are different estimations of the rate of this exponential decay which are not always connected with Lyapunov exponents [32-34]. For continuous-time systems, there are no theoretical results at all for estimating the rate of correlation splitting [35].

The studies of specific chaotic systems testify to a complicated behavior of correlation functions, which is defined not only by positive Lyapunov exponents but also by different characteristics and peculiarities of the system chaotic dynamics [32,34,36].

In the papers [37-39] the correlation and spectral properties of chaotic oscillations are studied for several types of chaotic attractors which can be observed in autonomous differential systems with three-dimensional phase space. Classical models of nonlinear dynamics such as the Rössler oscillator [40], the Lorenz system [41], and the Anishchenko-Astakhov oscillator that represents a mathematical model of a real radiotechnical device [42] were chosen for the studies. In the cited papers an attempt was taken to answer several fundamental questions. Which peculiarities of the system's chaotic dynamics can define the rate of correlation splitting and the basic spectral line width? How does noise affect the spectral and correlation characteristics of chaos? Basing on the results of numerical simulation, we would like to show that in the context of correlation properties, different types of chaotic self-sustained oscillations can be associated with basic models of stochastic processes such as harmonic noise and a telegraph signal.

The aim of this work is to present a brief review of the recent results reported in [21,22,37-39]. The presented results concern some probabilistic aspects of chaotic dynamics such as peculiarities of the relaxation to a stationary probability distiribution, the rate of mixing and the correlation and spectral analysis of chaotic regimes of different types. A special attention is paid to the effect of noise on the statistical properties of chaotic dynamics.

2. Relaxation to a stationary probability distribution of chaotic attractors in the presence of noise

2.1. Models and numerical methods. We will study chaotic attractors of well-known model systems such as the Rössler oscillator [40]

$$\begin{aligned} \dot{x} &= -y - z + (2D)^{1/2}\xi(t), \\ \dot{y} &= x + ay, \\ \dot{z} &= b - z(m-x), \end{aligned} \tag{1}$$

and the Lorenz system [41]

$$\begin{aligned} \dot{x} &= -\sigma(x-y) + (2D)^{1/2}\xi(t), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -\beta z + xy. \end{aligned} \tag{2}$$

In both models $\xi(t)$ is a normal white noise source with the mean value $\langle\xi(t)\rangle\equiv 0$ and correlation $\langle\xi(t)\xi(t+\tau)\rangle\equiv\delta(\tau)$, where $\delta()$ is Dirac's function. The parameter D denotes the noise intensity. For the Rössler system we fix a=0.2 and b=0.2 and vary the control parameter m in the interval [4.25, 13.0]. In the Lorenz system we choose two different regimes, namely, a quasi-hyperbolic attractor (σ=10, $\beta={}^{8}/_{3}$, and r=28) and a nonhyperbolic attractor (σ=10, β=8/3, and r=210).

We integrate Eqs (1) and (2) using a fourth-order Runge-Kutta routine with noise sources taken into account. Chaotic attractors of systems (1) and (2) have been studied in detail and are typical examples of quasi-hyperbolic and nonhyperbolic chaos [43,44]. Thus, results obtained for Eqs (1) and (2) can be generalized to a wide class of dynamical systems.

To examine the relaxation to a stationary distribution in these systems, we analyze how points situated at an initial time in a cube of small size δ around an arbitrary point of the trajectory belonging to an attractor of the system evolve with time. We take δ=0.09 for the size of this cube and fill it uniformly with n=9000 points. As time goes on, these points in the phase space are distributed throughout the whole attractor. To characterize the convergence to the stationary distribution we follow the temporal evolution of this set of points and calculate the ensemble average

$$\overline{x}(t) = \int p(x,t)x\,dx \simeq 1/n\sum_{i=1}^{n} x_i(t). \tag{3}$$

Here, x is one of the system dynamical variables, and $p(x,t)$ is the probability density of the variable x at the time t which corresponds to the chosen initial distribution. It is known that the phase trajectory of system (2) visits neighborhoods of two saddle-foci. In this case, when calculating $\bar{x}(t)$ one may first sum separately over points having falled in the neighborhood of each saddle-focus, and then combine the obtained results. However, the mean value appears to approach zero in a short time interval and its further evolution is badly detected. To follow the relaxation in (2) we compute the mean value when points in the neighborhood of only one saddle-focus are taken into account. In this case the relaxation to this quantity goes more slowly in time. Then we calculate the function $\gamma(t_k)$:

$$\gamma(t_k) = |\bar{x}_m(t_{k+1}) - \bar{x}_m(t_k)|, \tag{4}$$

where $\bar{x}_m(t_k)$ and $\bar{x}_m(t_{k+1})$ are successive extrema of $\bar{x}(t)$. Thus, $\gamma(t_k)$ characterizes the amplitude of the mean value oscillations. In the expression (4) t_k and t_{k+1} are successive time moments corresponding to the extrema of $\bar{x}$. The temporal behavior of $\gamma(t_k)$ allows to judge the character and the rate of relaxation to the probability measure on the attractor.

We also calculate the maximal Lyapunov exponent (LE) λ_1 of a chaotic trajectory on an attractor. Besides, we also compute the normalized autocorrelation function (ACF) of steady-state oscillations $x(t)$:

$$\Psi(\tau) = \psi(\tau)/\psi(0), \quad \psi(\tau) = \langle x(t)x(t+\tau)\rangle - \langle x(t)\rangle \langle x(t+\tau)\rangle. \tag{5}$$

The brackets $\langle ... \rangle$ denote time averaging.

To make some figures more informative and compact, instead of $\gamma(t_k)$ and $\Psi(\tau)$ we plot (where it is necessary) their envelopes $\gamma_0(t_k)$ and $\Psi_0(\tau)$, respectively.

2.2. Relaxation to a stationary distribution in the Rössler system: Mechanism of the effect of noise on the rate of mixing. A chaotic attractor realizing in the Rössler system (1) at fixed a=b=0.2 and in the parameter m interval [4.25, 8.5] serves as a well-known example of a spiral attractor. The phase trajectory on the spiral attractor rotates with a high regularity around one or several saddle-foci. The autocorrelation function is oscillating and the power spectrum exhibits narrow-band peaks corresponding to the mean rotation frequency, its harmonics and subharmonics. By virtue of these properties spiral chaos is called phase-coherent [43,45-47].

The chaotic attractor of system (1) is qualitatively changing as the parameter m increases. In the interval $8.5<m<13.0$ there occurs a nonhyperbolic attractor of noncoherent type, called funnel attractor [42,46]. Phase trajectories on the funnel attractor make complicated loops around a saddle-focus and thus, demonstrate a nonregular rotation behavior. Consequently, the autocorrelation function of noncoherent chaos decreases much rapidly than that in the coherent case, and the power spectrum does not already contain sharp peaks.

The calculations performed for $m\in[4.25, 7.5]$ (spiral chaos) and for $m\in[8.5, 13.0]$ (noncoherent chaos) allow to assume that an invariant probability measure exists for the parameter values considered. All the effects being observed for each type of attractor in the system (1) are qualitatively preserved when the parameter m is varied. In our numeric simulation we fix m=6.1 for the spiral attractor and m=13.0 for the funnel attractor.

Figure 1 shows the typical behavior of γ_0(t) for both the spiral and the funnel attractor of the Rössler system. We find that the noise significantly influences the rate of mixing in the regime of spiral attractor in the Rössler system. The relaxation time is strongly decreasing for increasing noise intensity (see Fig. 1, a).

We find a quite different situation for the funnel attractor. Noncoherent chaos is

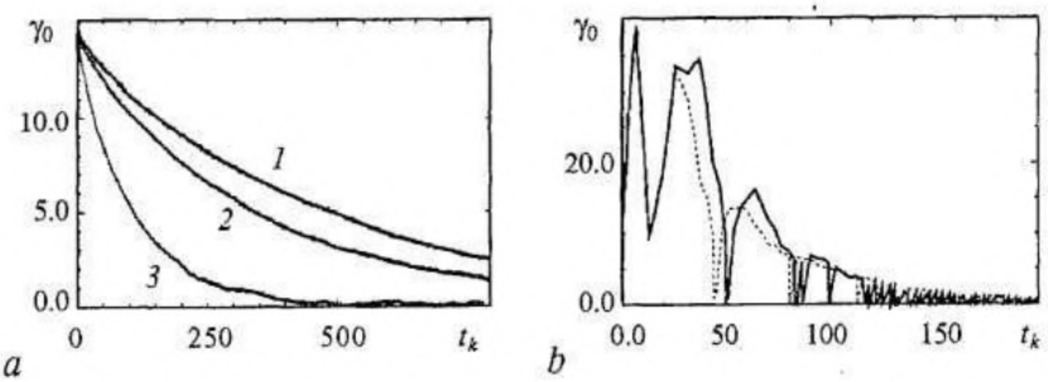


Fig. 1. $\gamma_0(t_k)$ for attractors in the Rössler system (1). (*a*) For the spiral attractor ($a=b=0.2$, $m=6.1$) at $D=0$ (curve *1*), $D=0.001$ (curve *2*), and $D=0.1$ (curve *3*); (*b*) for the funnel attractor ($a=b=0.2$, $m=13$) at $D=0$ (solid line) and $D=0.01$ (dotted line)

practically insensitive to noise perturbations. Behavior of $\gamma_0(t_k)$ does not significantly change when noise is added to Eqs (1) (see Fig. 1, *b*). At the same time, it is well known that noncoherent chaos exhibits a close similarity to random processes. This fact can be verified, e.g. by means of the autocorrelation function $\Psi(\tau)$ for the spiral and the funnel

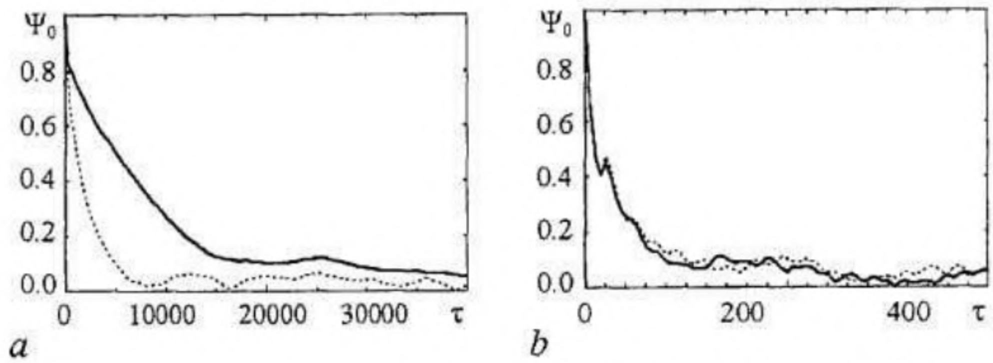


Fig. 2. Envelopes of the normalized autocorrelation function $\Psi_0(\tau)$ for attractors in (1). (*a*) At $m=6.1$ and for $D=0$ (solid line) and $D=0.01$ (dotted line); (*b*) at $m=13$ for $D=0$ (solid line) and $D=0.01$ (dotted line)

attractors in system (1) (Fig. 2). Our numerical experiments show that the correlation times are essentially different for these two chaotic regimes: without noise they differ by two orders. On the one hand, in the case of coherent chaos the correlation time decreases dramatically in the presence of noise (Fig. 2, *a*). On the other hand, the autocorrelation function for the funnel attractor in the deterministic case practically coincides with that in the presence of noise (Fig. 2, *b*). Hence, noncoherent chaos, which is nonhyperbolic, demonstrates some property of hyperbolic chaos, i.e. «dynamical stochasticity» turns out to be much stronger than that imposed from an external (additive) one [6]. This experimental result is interesting and requires a more detailed consideration. It is also worth noting another finding of our simulations. We have found that the positive LE for both the spiral chaos and the funnel chaos is weakly sensitive to fluctuations (see Fig. 3), and

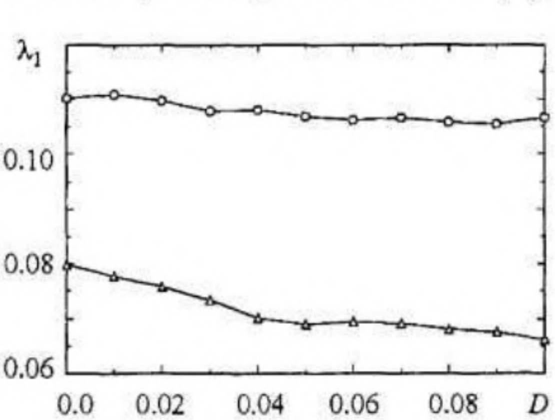


Fig. 3. For the Rössler system, λ_1 on the spiral (triangles) and the funnel (circles) attractor as functions of the noise intensity D

rather grows not much with increasing noise intensity, whereas in certain cases the correlation time changes considerably under the influence of noise. Thus, in the regime of spiral chaos the rate of mixing is not uniquely determined by the largest LE but depends strongly on the noise intensity.

We suppose that the essential effect of noise on relaxation to the stationary distribution in the regime of spiral chaos may be associated with peculiarities of the phase trajectory dynamics in the neighborhood of an unstable equilibrium state. Since the trajectory rotates almost regularly on the spiral attractor, the relaxation process appears to be very long. The addition of noise to the system destroys the relative regularity of the trajectory and, consequently, the rate of mixing significantly increases.

It is known that for chaotic oscillations one can introduce the notion of instantaneous amplitude and phase [47]. The instantaneous phase characterizes the rotation of a trajectory around a saddle-focus. System (1) is of such type because the trajectory in the (x-y) projection rotates around the unique saddle-focus located very near to the origin. Let us introduce the substitution of variables

$$x(t) = A(t)\cos\Phi(t), \quad y(t) = A(t)\sin\Phi(t), \tag{6}$$

that defines the amplitude $A(t)$ and the total phase $\Phi(t)$ of the chaotic oscillations. Then the instantaneous phase $\Phi(t)$ can be calculated as follows:

$$\Phi(t) = \arctan(y(t)/x(t)) + \pi n(t), \tag{7}$$

where $n(t)$=0,1,2,... is the number of intersections of the phase trajectory with the plane x=0.

The component of mixing along the flow of trajectories is related with the divergence of the instantaneous phase values and thus, is determined by the temporal behavior of the phases. The instantaneous phase of an ensemble of initially close trajectories on the spiral attractors remain very close to each other over a long period of time, although the points in the secant plane are spread over the whole attractor section. In this case the relaxation to a stationary probability distribution on the whole attractor of a flow system will be much longer than that in the Poincaré map. The violation of regular rotation of trajectories is characteristic for the funnel attractor and leads to a nonmonotonic dependence of the intantaneous phase on time. The phase trajectory creates complicated loops at nonequal time intervals that causes the value of the current phase to slightly decrease. This results in a rapid divergence of the phase values of neighboring trajectories. The influence of noise on spiral chaos leads to similar effects. Figure 4, *a* shows the temporal dependences of the variance σ_Φ^2 of the instantaneous

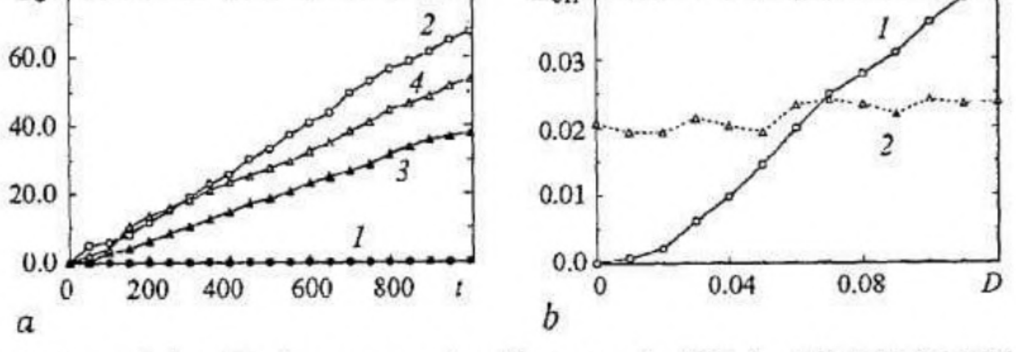


Fig. 4. Characteristics of the instantaneous phase divergence of neighboring trajectories for spiral chaos (m=6.1) and funnel chaos (m=13) in Eqs (1). (*a*) Temporal dependences of the variance of the intantaneous phase σ_Φ^2 for spiral chaos at D=0 (curve *1*), D=0.1 (curve *2*), and for noncoherent chaos at D=0 (curve *3*), D=0.1 (curve *4*). (*b*) The effective diffusion coefficient B_{eff} as a function of the noise intensity D for spiral (curve *1*) and noncoherent (curve *2*) chaos

phase on an ensemble of initially close trajectories for both the spiral and the funnel attractor of system (1). We observe that in both the noisy and the noise-free case the variation grows almost linearly on the time intervals being considered. The fact that the temporal dependence of the instantaneous phase variance of the chaotic oscillations in the Rössler system is a linear function was assumed in [$-\pi;\pi$]. Nevertheless, this suggestion was confirmed neither theoretically nor numerically or experimentally. In the case of spiral chaos without noise (curve *1*), the value of σ_Φ^2 is small (on the given time interval it does not exceed the variation of the uniform phase distribution on the interval [$-\pi;\pi$]) and increases much slower than in the other cases considered. The linear growth of the variation allows to estimate the divergence of the intantaneous phases by using the effective diffusion coefficient:

$$B_{\text{eff}} = {}^1/_2\, d\sigma_\Phi^2(t)/dt. \tag{8}$$

Figure 4, *b* illustrates the dependences of B_{eff} of the instantaneous phase of chaotic oscillations on the noise intensity for both the spiral and the funnel attractor in the Rössler system (1). It is seen that in both cases B_{eff} grows with increasing D but for spiral chaos this growth is more significant. This result strongly testifies that B_{eff} is a very effective characteristic for diagnosing the statistical properties of a chaotic attractor in the presence of fluctuations.

2.3. Relaxation to a probability measure in the Lorenz systems. Well-known quasi-hyperbolic attractors in three-dimensional continuous-time systems, such as the Lorenz attractor, the Morioka-Shimizu attractor [48], are attractors of the switching type. The phase trajectory switches chaotically from the neighborhood of one saddle equilibrium state to the neighborhood of another one. Such switchings are accompanied by chaotic phase changes even without noise. In this case the addition of noise does not change considerably the phase dynamics and, consequently, does not influence the rate of relaxation to the stationary distribution.

Figure 5 shows the behavior of $\gamma_0(t_k)$ for both quasi-hyperbolic and nonhyperbolic chaotic attractors of the system (2) with and without noise added. We find that noise does not significantly influence the relaxation rate for the Lorenz attractor (Fig. 5, *a*). However, we observe a quite different situation for the nonhyperbolic attractor. There the rate of relaxation is strongly affected by noise (Fig. 5, *b*).

Now we are going to check whether the other characteristics of the mixing rate, such as the LE and the correlation time, will also depend on noise perturbations. For the same chaotic attractors in the Lorenz system we compute the largest LE λ_1 and estimate the normalized autocorrelation function $\Psi(\tau)$, $\tau=t_2-t_1$, of the dynamical variable $x(t)$ for different noise intensities D. We find that for both types of chaotic attractors the LE does

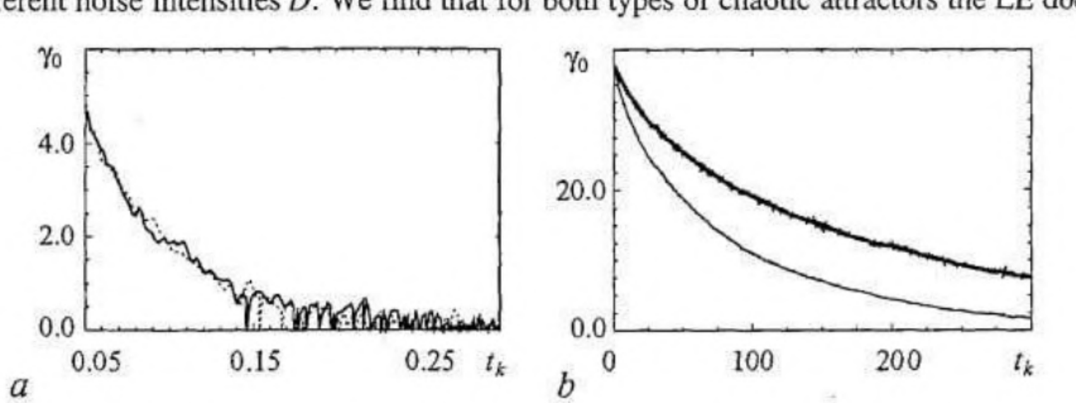


Fig. 5. $\gamma_0(t_k)$ for chaotic attractors in the Lorenz system (2). (*a*) For r=28 and D=0 (solid line), and D=0.01 (dotted line); (*b*) for r=210 and D=0 (thick line), and for r=210 and D=0.01 (thin line). Other parameters are σ=10, β=8/3

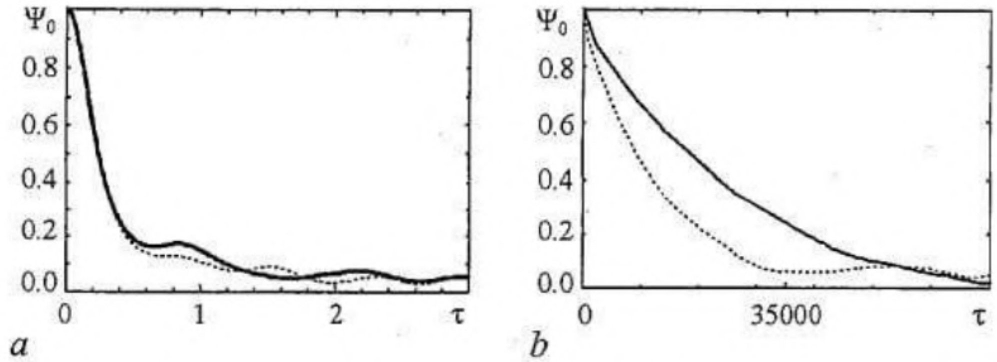


Fig. 6. Envelopes of the normalized autocorrelation function $\Psi_0(\tau)$ for attractors in system (2). (*a*) r=28 and D=0 (solid line), and D=0.01 (dotted line); (*b*) r=210, D=0 (solid line), and D=0.01 (dotted line)

not depend within the calculation accuracy on the noise intensity. The autocorrelation function of the quasi-hyperbolic attractor is practically not affected by noise (see curves *1* and *2* in Fig. 6, *a*). However, in the regime of a nonhyperbolic attractor it decreases more rapidly in the presence of noise (see curves in Fig. 6, *b*).

3. Correlation and spectral analysis of dynamical chaos

Let us now examine correlation and spectral properties of different types of chaotic oscillations in more details. Experience of the studies of dynamical chaos in three-dimensional differential systems shows that two classical models of random processes can be used to describe the correlation and spectral properties of a certain class of chaotic systems. They are the models of harmonic noise and a telegraph signal.

As we will demonstrate below, the model of harmonic noise represents sufficiently well correlation characteristics of spiral chaos, while the model of telegraph signal is quite suitable for studying statistical properties of attractors of the switching type, such as attractors in the Lorenz system [41] and in the Chua circuit [49].

In the following we summarize the main characteristics of the above mentioned classical models of random processes.

Harmonic noise is a stationary random process with zero mean. It is represented as follows [50-52]:

$$x(t) = R_0[1+\alpha(t)]\cos[\omega_0 t+\phi(t)], \tag{9}$$

where R_0 and ω_0 are constant (average) values of the amplitude and frequency of oscillations, respectively; $\alpha(t)$ and $\phi(t)$ are random functions that characterize amplitude and phase fluctuations, respectively. The process $\alpha(t)$ is assumed to be stationary.

Several simplifying assumptions which are most often used are as follows: (i) the amplitude and phase fluctuations are statistically independent, and (ii) the phase fluctuations $\phi(t)$ represent a Wiener process with a diffusion coefficient B. Under the assumptions made, the ACF of the process (9) can be written as follows [50-52]:

$$\psi(\tau) = {}^1\!/_2\, R_0^2[1+K_\alpha(\tau)]\exp(-B|\tau|)\cos\omega_0\tau, \tag{10}$$

where $K_\alpha(\tau)$ is the covariation function of reduced amplitude functions $\alpha(t)$[1]. Using the Wiener-Khinchin theorem one can derive the corresponding expressions for the spectral power density.

[1]Prefactor $R_0^2[1+K_\alpha(\tau)]$ is the covariation function $K_A(\tau)$ of the random amplitude $A(t)=R_0[1+\alpha(t)]$. This notion is most convenient to use in our further studies.

Generalized telegraph signal. This process describes random switchings between two possible states $x(t)=\pm a$. Two main kinds of telegraph signal are usually considered, namely, random and quasi-random telegraph signals [52,53]. A random telegraph signal is characterized by a Poissonian distribution of switching moments t_k. The latter leads to the fact that the impulse duration θ has the exponential distribution:

$$\rho(\theta) = n_1 \exp(-n_1\theta), \quad \theta \geq 0, \tag{11}$$

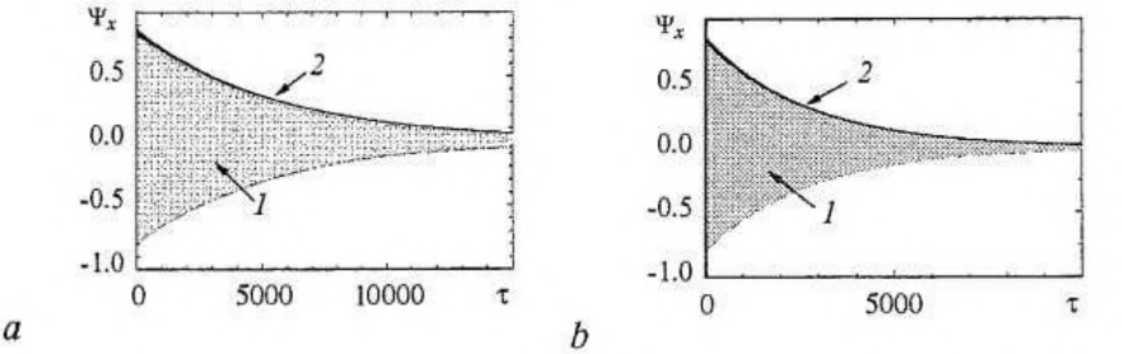
where n_1 is the mean switching frequency. The ACF of such a process can be represented as follows:

$$\psi(\tau) = a^2 \exp(-2n_1|\tau|). \tag{12}$$

Another type of telegraph signal (a quasi-random telegraph signal) corresponds to random switchings between the two states $x(t)=\pm a$, which can occur only in discrete time moments $t_n=n\xi_0+\alpha$, $n=1,2,3,\ldots$, where ξ_0=const and α is a random quantity. If the probability of switching events is equal to $^1/_2$, then the ACF of this process is given by the following expression:

$$\begin{aligned} \psi(\tau) &= a^2(1-|\tau|/\xi_0), \quad \text{if } |\tau| < \xi_0; \\ \psi(\tau) &= 0, \quad \text{if } |\tau| \geq \xi_0. \end{aligned} \tag{13}$$

3.1. Correlation and spectral analysis of spiral chaos. From a physical viewpoint, chaotic attractors of the spiral type possess the properties of a noisy limit cycle. However, spiral attractors are realized in fully deterministic systems, i.e., without external fluctuations. Consider the regime of spiral chaos in the Rössler system (1) for $a=b=0.2$ and $m=6.5$. Let us introduce the instantaneous amplitude $A(t)$ and phase $\Phi(t)$ according to the relations (6). We calculate the normalized autocorrelation function of the chaotic oscillations $x(t)$ (grey dots region *1*, Fig. 7), the covariance function of the amplitude $K_A(t)$ and the effective phase diffusion coefficient B_{eff}. Figure 7 shows the results for $\Psi_x(\tau)$ in the system (1) both without noise and in the presence of noise. The ACF decays almost exponentially both without noise (Fig. 7, *a*) and in the presence of



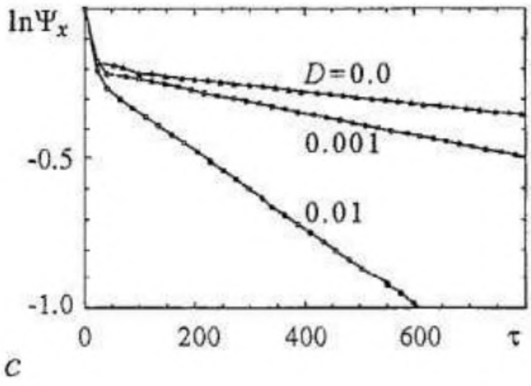


Fig. 7. Normalized ACF of the $x(t)$ oscillations in system (1) for m=6.5 (grey dots region *1*) and its approximation by (2) (black dots *2*) for D=0 (*a*) and $D=10^{-3}$ (*b*). The envelopes of ACF in a linear-logarithmic scale for different D (*c*)

noise (Fig. 7, *b*). Additionally, as seen from Fig. 7, *c*, for $\tau<20$ there is an interval on which the correlations decrease much faster.

Using Eq. (10) we can approximate the envelope of the calculated ACF $\Psi_x(\tau)$. To do this, we substitute the numerically computed characteristics $K_A(\tau)$ and $B=B_{eff}$ into an expression for the normalized envelope $\Gamma(\tau)$:

$$\Gamma(\tau) = K_A(\tau)/K_A(0)\exp(-B_{eff}|\tau|). \quad (14)$$

The calculation results for $\Gamma(\tau)$ are shown in Fig. 7, *a*, *b* by black dots (curves *2*). It is seen that the behavior of the envelope of $\Psi_x(\tau)$ is described well by Eq. (14). Note that taking into account the multiplier $K_A(\tau)/K_A(0)$ enables us to obtain a good approximation for all times τ. This means that the amplitude fluctuations play a significant role on short time intervals, while the slow process of the correlation decay is mainly determined by the phase diffusion. Thus, we can observe a surprisingly good agreement between the numerical results for the spiral chaos and the data for the classical model of harmonic noise. At the same time, it is quite difficult to explain rigorously the reason of such a good agreement. Firstly, the relationship (10) was obtained by assuming the amplitude and phase values to be statistically independent. However, this approach cannot be applied to a chaotic regime. Secondly, when deriving (10) we used the fact that the phase fluctuations are described by a Wiener process. In the case of chaotic oscillations, $\Phi(t)$ is a more complicated process and its statistical properties are unknown. It is especially important to note that the findings presented in Fig. 7, *a* were obtained in the regime of purely deterministic chaos, i.e. without noise in the system.

We have shown that for $\tau>\tau_{cor}$ the envelope of the ACF for the chaotic oscillations can be approximated by the exponential law $\exp(-B_{eff}|\tau|)$. Then according to the Wiener-Khinchin theorem, the spectral peak at the average frequency ω_0 must have a Lorenzian shape and its width is defined by the effective phase diffusion coefficient B_{eff}:

$$S(\omega) = CB_{eff}/[B^2_{eff}+(\omega-\omega_0)^2], \quad C = \text{const}. \quad (15)$$

The calculation results presented in Fig. 8 justify this statement. The basic spectral peak is approximated by using (15) and this fits quite well with the numerical results for the power spectrum of the $x(t)$ oscillations. We note that the findings shown in Figs 7 and 8 for the noise intensity $D=10^{-3}$ have also been verified for different values of D, $0<D<10^{-2}$, as well as for the range of parameter m values which correspond to the regime of spiral chaos. Our findings for the approximation of the ACF and the shape of the basic spectral peak are completely confirmed by our investigations of spiral attractors in other dynamical systems.

The spectral and correlation properties of spiral chaos were also explored in a physical experiment with the Anishchenko-Astakhov oscillator [42,43]. The performance of such kind of experiment is important as the stochastic equations of the oscillator are approximate only and cannot take into account all sources of natural fluctuations that are really operating in the electronic scheme of the oscillator. Experimental results are presented in Fig. 9 and completely confirm all the data obtained numerically.

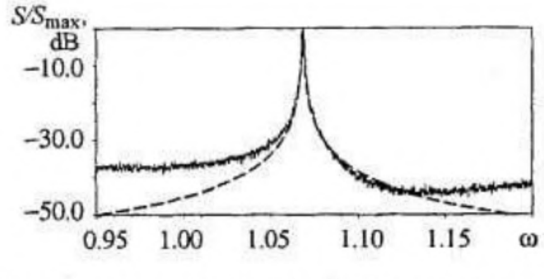


Fig. 8. A part of the normalized power spectrum of $x(t)$ oscillations in system (1) for $a=b=0.2$, and $m=6.5$ (solid line) and its approximation by Eq. (15) (dashed line) for the noise intensity $D=10^{-3}$

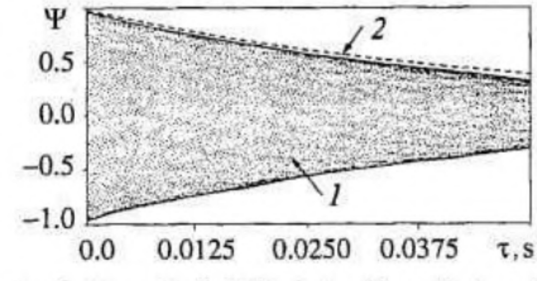


Fig. 9. Normalized ACF of the $x(t)$ oscillations in the Anishchenko-Astakhov oscillator(region 1) and its exponential approximation $\exp(-B_{eff}|\tau|)$ (curve 2) (physical experiment). The phase diffusion coefficient B_{eff} was calculated from experimental data independently on the ACF

3.2. Correlation characteristics of the Lorenz attractor. In the previous section we have used the effective phase diffusion coefficient to describe the correlation properties of the Rössler system and the Anishchenko-Astakhov oscillator. However, such an approach cannot be applied to approximate autocorrelation functions of chaotic oscillations of a switching type. Some chaotic attractors demonstrating a rather complex structure can contain certain regions which are separated by manifolds of saddle points and cycles. Transitions (switchings) between these regions can occur provided that certain conditions are fulfulled [54]. Such oscillations can be observed, for example, in the Lorenz system [41]. Let us consider the Lorenz system in the regime of the quasi-hyperbolic attractor for $r=28$, $\sigma=10$, and $b=8/3$.

In the phase space of the Lorenz system there are two saddle-foci that are symmetrical about the z-axis and are separated by the stable manifold of a saddle point in the origin. This stable manifold has a complex structure that allows the trajectories to switch between the saddle-foci in specific paths [11,54] (see Fig. 10). Unwinding about

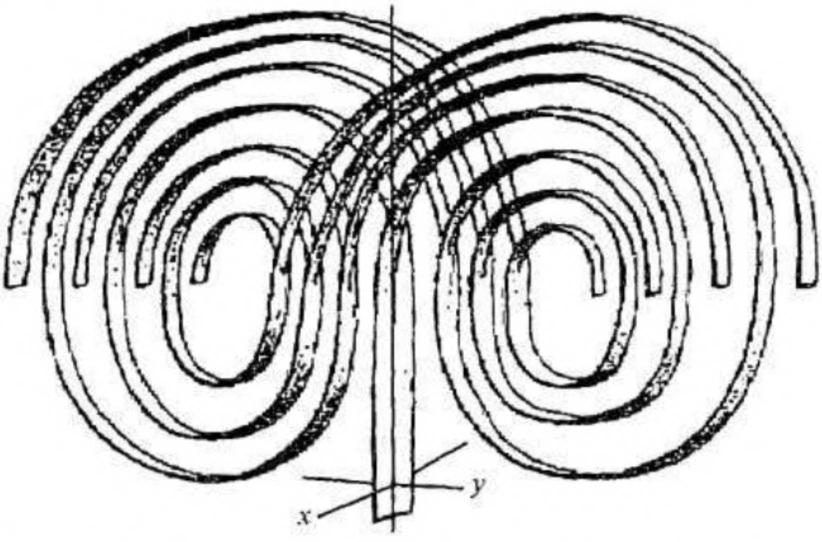


Fig. 10. Qualitative illustration of the structure of manifolds in the Lorenz system

one of the saddle-foci the trajectory approaches the stable manifold and then can jump to the other saddle-focus with a certain probability. The rotation about the saddle-foci does not contribute considerably to the decay of the ACF, while the frequency of «random» switchings essentially affects the rate of the ACF decay. Consider the time series of the x coordinate of the Lorenz system, that is shown in Fig. 11. If one introduces a symbolic dynamics, i.e., one excludes the rotation about the saddle-foci, one obtains a telegraph-like signal. Figure 12 shows the ACF of the $x(t)$ oscillations for the Lorenz attractor and the ACF of the corresponding telegraph signal. Comparing these two figures we can state that the time of the correlation decay and the behavior of the ACF on this time scale are predominantly determined by switchings, whereas the rotation about the saddle-foci makes a minor contribution to the ACF decay on large times. It is worth noting that the ACF decreases linearly on short times. This fact

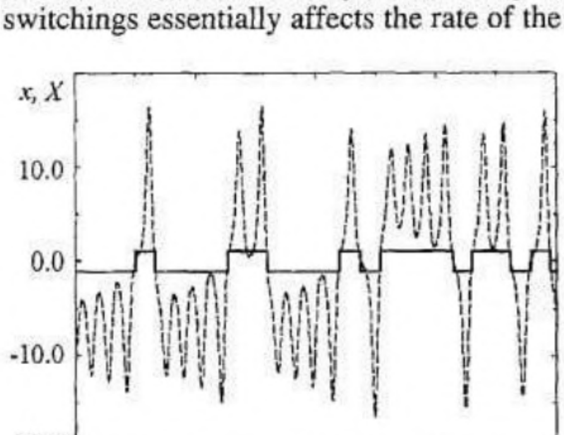


Fig. 11. Telegraph signal (solid curve) obtained for the $x(t)$ oscillations (dashed curve) of the Lorenz system at $\sigma=10$, $\beta=8/3$, and $r=28$

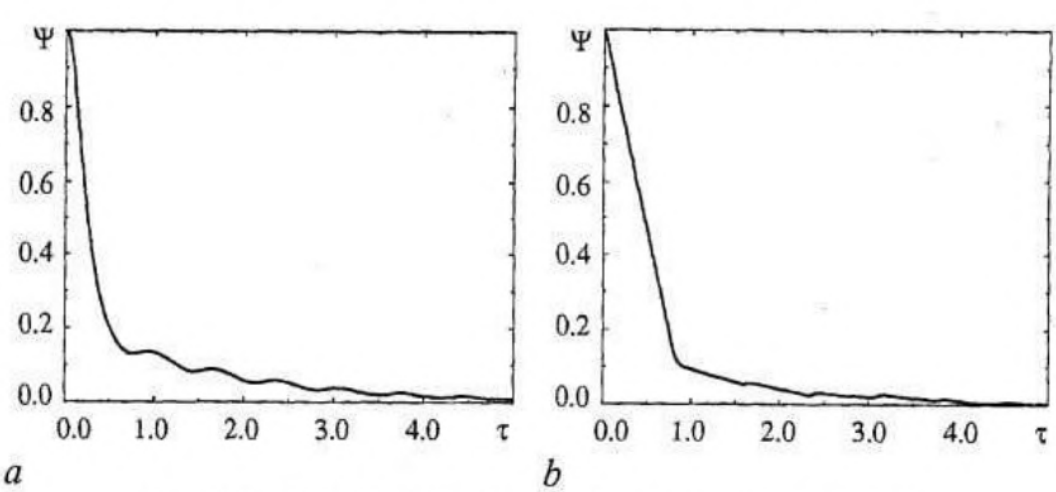


Fig. 12. The ACF of the $x(t)$ oscillations (*a*) and of the telegraph signal (*b*)

is remarkable as the linear decaying of the ACF corresponds to a discrete equidistant residence time probability distribution in the form of δ-peaks. Additionally, the probability of switchings between the two states is equal to $^1/_2$ [52,53].

Figure 13 shows the residence time distribution calculated for the telegraph signal resulting from switchings in the Lorenz system. As can be seen from Fig. 13, *a*, the residence time distribution in the two attractor regions really has a structure that is quite similar to an equidistant discrete distribution. At the same time the peaks are characterized by a finite width. Figure 13, *b* represents the probability distribution of switchings which occur at multiples of ξ_0, where ξ_0 is the minimal residence time in one of the states. This dependence shows that the probability of transition at time ξ_0 is close to $^1/_2$. The discrete character of switchings can be explained by peculiarities of the structure of the manifolds of the Lorenz system (see Fig. 10). In the vicinity of the origin x=0, y=0 the manifolds split into two leaves. This leads to the fact that probability of switchings between two states in one revolution about the fixed point is approximately equal to $^1/_2$. This particular aspect of the dynamics ensures that the ACF of the $x(t)$ and $y(t)$ oscillations on the Lorenz attractor has the form defined by expression (13). However, the finite width of the peaks in the distribution and deviations from the probability $^1/_2$ can lead to an ACF that decays to a certain finite, nonvanishing value.

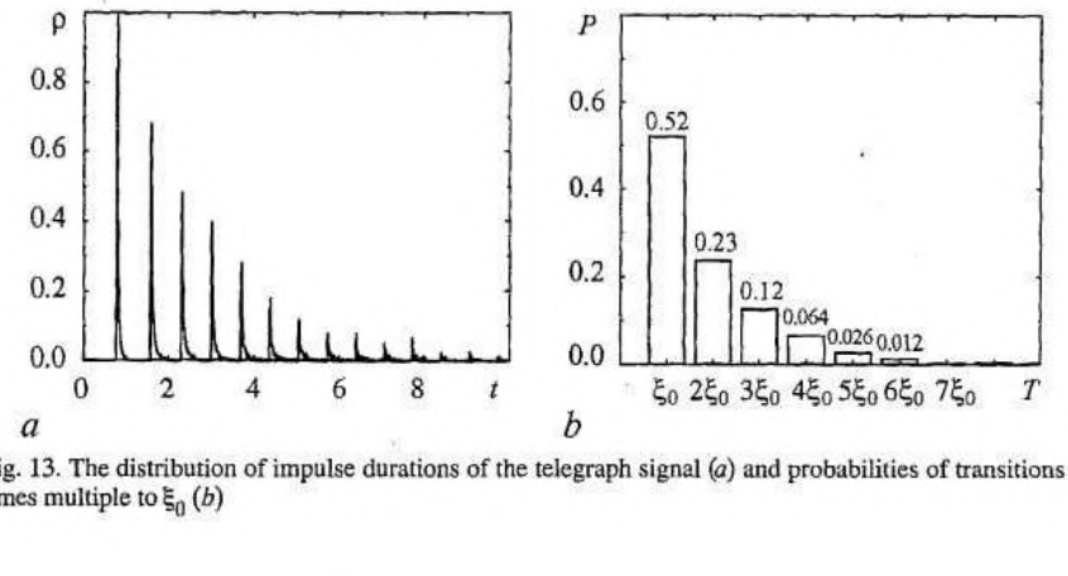


Fig. 13. The distribution of impulse durations of the telegraph signal (*a*) and probabilities of transitions at times multiple to ξ_0 (*b*)

4. Conclusion

In our studies we have shown that there is a group of nonhyperbolic attractors of spiral type for which noise strongly influences the characteristics of the relaxation to a stationary distribution as well as the correlation time and practically does not change the positive Lyapunov exponent.

The rate of mixing on nonhyperbolic attractors in R^3 is determined not only by the positive Lyapunov exponent but also depends on the instantaneous phase dynamics of chaotic oscillations. In the regime of spiral chaos noise causing phase changes can essentially accelerate the relaxation to a stationary distribution.

For chaotic attractors with a nonregular behavior of the instantaneous phase the rate of mixing cannot be considerably affected by noise. This statement is true for nonhyperbolic attractors of funnel type and for the attractors of switching type, for example, for the quasi-hyperbolic Lorenz attractor.

We have shown in our numerical simulation that the spiral chaos retains to a great extent the spectral and correlation properties of quasi-harmonic oscillations. With this, the rate of correlation splitting in a differential system depends on short times on both the instantaneous amplitude and the instantaneous phase diffusion. The width of the basic peak in the power spectrum of the spiral chaos is correspondingly defined by B_{eff} and oscillations of the instantaneous amplitude determine the level of the spectrum background. The effective phase diffusion coefficient in a noise-free system is defined by its chaotic dynamics but is not directly related to the positive Lyapunov exponent.

Our studies of statistical properties of the Lorenz attractor have demonstrated that the properties of the ACF is mainly defined by a random switching process and slightly depends on the rotation about the saddle-foci. The classical model of telegraph signal enables one to describe the behavior of $\psi(\tau)$ for the Lorenz attractor by using the expression (13). In particular, this expression approximates quite well a linear decay of the ACF from 1.0 to 0.2 that allows to estimate theoretically the correlation time. The power spectrum of the Lorenz attractor both in a flow and in the Poincaré map was studied in [36] by applying the symbolic dynamics methods. Already in this paper it has been established that the power spectrum is not a Lorenzian. Our results obtained by using the model of telegraph signal are in a good agreement with the findings presented in [36].

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СТАТИСТИЧЕСКИЕ СВОЙСТВА ДЕТЕРМИНИРОВАННЫХ И ЗАШУМЛЕННЫХ ХАОТИЧЕСКИХ СИСТЕМ

В.С. Анищенко, Т.Е. Вадивасова, Г.И. Стрелкова, Г.А. Окрокверцхов

Данная работа представляет собой обзор результатов, недавно полученных в группе исследователей, возглавляемой профессором В.С. Анищенко, и опубликованных в ряде научных статей. Представляемые результаты относятся к статистическому описанию динамического хаоса и влиянию шума на различные типы хаотических аттракторов. Рассматриваются особенности релаксации инвариантной вероятностной меры в системах с хаотическими аттракторами различных типов, проводится корреляционный и спектральный анализ хаотических автоколебаний.

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