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MULTISTABILITY, IN-PHASE AND ANTI-PHASE CHAOS SYNCHRONIZATION IN PERIOD-DOUBLING SYSTEMS

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We consider mechanisms of multistability formation and complete chaos synchronization loss in mutually coupled period-doubling maps. Cases of in-phase and anti-phase synchronization are investigated. Influence of non-identity of partial oscillators is also discussed.

1. Introduction

Phenomenon of complete synchronization of chaos has being intensively investigated for the last time. Majority of authors consider a case of in-phase synchronization when oscillations of subsystems are equal or almost equal to each other in the every moment of time [1, 2]. The other case of complete synchronization is antiphase synchronization when the subsystems oscillate identically but with opposite signs: $x_1(t) = -x_2(t)$. The antiphase synchronization of chaos was considered in the work [3]. The authors investigated «master - slave» synchronization [4], when one subsystem unidirectionally influences on the other one.

Bifurcational mechanisms of both in-phase and antiphase complete chaotic synchronization are in close connection with bifurcations of saddle periodic orbits embedded in the synchronous chaotic attractor. In a system of symmetrically coupled identical oscillators a limit set relating to synchronous oscillations locates in the symmetric subspace ($\mathbf{x}_1 = \mathbf{x}_2$) (for the in-phase synchronization) or in the antisymmetric subspace ($\mathbf{x}_1 = -\mathbf{x}_2$) (for the antiphase synchronization) of the whole phase space of the system, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are vectors of identical dynamical variables of interacting subsystems. If a chaotic set is attracting in the normal to the subspace direction, namely when its largest transversal Lyapunov exponent is negative, the synchronous oscillations are observed in experiment. When the exponent changes its sign to the positive, the chaotic attractor becomes non-attracting in the normal direction and transforms to a chaotic saddle. The synchronous oscillations are not observed in experiment further. However, the case is possible when the largest transversal Lyapunov exponent on the chaotic attractor is negative, but the exponents on some limit sets encapsulated in the attractor are positive. In this case the synchronous regime remains stable but becomes unrobust. Any infinitesimal noise or the parameters mismatch can lead to the «bubbling» of the attractor. Time-series of the oscillations related to motions in the normal to the subspace direction becomes an intermittency process when the phase point moves in vicinity of the symmetric subspace for a long time (laminar phase) and leaves it from time to time (turbulent bursts) [12]. The bubbling of attractor is the first step to the desynchronization of chaos. Then, with changing of the controlling parameters more quantity of encapsulated cycles lose their stability in the normal direction. This enforces the process of bubbling and then the averaged on the attractor largest normal Lyapunov exponent can become positive. As a result, the chaotic set in the symmetric subspace becomes non-attractive. This phenomenon is called the blowout bifurcation [5]. The bubbling of attractor can be followed also by the riddling of its basins when «holes» from the basins of another attractor appear in infinitesimly small vicinity of the attractor . In this case, the presence of small noise or the parameters mismatch leads to leaving of the phase point to the another attractor. Regimes which accompany the process of chaotic synchronization loss in the coupled logistic maps were described in works [13], [14].

In the paper we detaily describe phenomena and mechanisms of in- and anti-phase complete synchronization of chaos in period-doubling maps with symmetric diffusive coupling. The paper is organized as a follows: The section one describes bifurcational mechanisms of destruction of in-phase synchronization and formation of multistability in system of symmetrically coupled cubic maps. The found regularities are compared with that in other systems. In the second section we consider the influence of small parameter's mismatch on mechanisms of synchronization loss in logistic maps. In the third section we hold comparing mechanisms of in-phase and anti-phase synchronization of regular regimes in coupled cubic maps. We propose a method of control for anti-phase chaos synchronization and describe the phenomena which accompany it. The conclusion summarises main results of the paper.

2. Mechanisms of destruction of in-phase synchronization and formation of multistability in coupled cubic maps

Let's consider a system of two identical discrete maps with symmetrical diffusive coupling:

$$x_{n+1} = f(x_n) + \gamma\,(f(y_n) - f(x_n)). \quad (1)$$

$$y_{n+1} = f(y_n) + \gamma\,(f(x_n) - f(y_n)). \quad (2)$$

It is seen, that this system is invariant to the transformation ($x \leftrightarrow y$) and therefore the subspace ($x=y$) is invariant to the operator of the evolution of the system. For investigation of the stability properties of symmetric solutions it is convenient to use «normal» variables: $u = (x+y)/2$, $v = (x-y)/2$.

Adding and subtracting equations (1) and (2) and then linearizing results in the vicinity of the symmetric subspace we get:

$$u_{n+1} = f(u_n). \quad (3)$$

$$v_{n+1} = (1-2\gamma) f'(u_n) v_n\,. \quad (4)$$

The equation (3) describes the dynamics inside the symmetric subspace. It is evidently the equation of the single map. The tangent stability of the synchronous solution is described by the tangent Lyapunov exponent:

$$\Lambda_\tau{}' = \lim_{N\to\infty} (1/N)\, \Sigma_{n=1}^{N} \ln |f'(u_n)|. \quad (5)$$

The equation (4) describes the dynamics in the normal direction to the symmetric subspace in its small vicinity. The transversal stability of the synchronous solution is determined by the transversal Lyapunov exponent:

$$\Lambda_{\perp}^{\ i} = \lim_{N\to\infty} (1/N)\, \Sigma_{n=1}^{N} \ln|\,(1\text{-}2\gamma) f'(u_n)|. \tag{6}$$

Comparing (3) and (4) we see that the tangential and the transversal Lyapunov exponents satisfy the relation:

$$\Lambda_{\perp}^{\ i} = \Lambda_{\tau}^{\ i} + \ln|1\text{-}2\gamma| \tag{7}$$

and hence, for small positive coupling ($0<\gamma<0.5$) the normal Lyapunov exponent is smaller than the tangent one. Any in-phase regular oscillations are normally stable and in-phase chaotic oscillations are stable only at sufficient large coupling. Let the single map be a period-doubling one. In this case in the symmetric subspace a cascade of period-doubling bifurcations leading to formation of synchronous chaos takes place. The resulting chaotic attractor contains infinite number of saddle periodic orbits taken part in its formation. According to (7) every orbit in the cascade can undergo one more period-doubling bifurcation which takes place in the normal direction to the symmetric subspace. As a result the saddle orbits which undergo the bifurcation become repellers and saddle orbits of double periods appear in their neighborhood. Then, the new saddle orbits become stable with further parameter changing. Hence, if first period-doubling bifurcations lead to complicating of synchronous oscillations, the second ones lead both to multistability and to transforming synchronous saddles into repellers.

In the Fig. 1 we built an example of a scheme of bifurcations that begin the formation of multistability. For better clarity we use here the following notation: the first index denotes the period of the orbit (or Cycle), the upper number is the number of steps on which oscillation in y map is delayed from that in x one. Obviously, synchronous oscillations have zero upper index. The bottom index identifies the orbit if there are several ones. Firstly, we observe the situation (Fig. 1, *a*) when there are two saddle orbits: period-one C^0), period-two ($2C^0$) and one stable period-four orbit ($4C^0$) in the symmetrical subspace. Then, the saddle orbit (C^0) undergoes the second period-doubling bifurcation in the normal to the subspace direction. As a result it becomes a repeller and a saddle orbit of period-two $2C^1$ appears outside the diagonal ($x=y$) (Fg. 1, *b*). With further parameters change it undergoes pitchfork bifurcation in the result of which the orbit $2C^1$ becomes stable and two symmetrical orbits $2C_1^{\,s1}$ and $2C_2^{\,s2}$ appears near it (Fig. 1, *c*). Then, similar series of bifurcations occurs with orbits of higher periods (Fig. 1, *d*).

The considered scheme is a typical one. It is observed for a number of different period-doubling oscillators. Here we consider a coupled cubic maps system:

$$f(x) = (a\text{-}1)\, x - ax^3. \tag{8}$$

We investigate bifurcations of the periodic orbits located inside the symmetrical subspace and of periodic orbits appeared from them. The structure of lines of tangential and transversal period-doubling bifurcations on the parameters plane is represented in the Fig. 2.

Horizontal lines l_{01}, l_{02}, l_{03} marks period-doubling bifurcations of periodic orbits C^0, $2C^0$ and $4C^0$ inside the symmetric subspace. On the line l_0 the periodic orbit C^0 undergoes tangent period-doubling bifurcation. As a result, it transforms to saddle and a stable periodic orbit of double period $2C^0$ appears in its neighborhood. Then, on the line l_1 the saddle C^0 undergoes the transversal period-doubling bifurcation. As a result, it loses stability in transversal direction and transforms to repeller. In its neighborhood, outside the symmetrical subspace a period-two saddle orbit $2C^1$ appears. With further parameter change this orbit becomes stable through pitchfork bifurcation on line l_2. The similar bifurcations take place with other periodic orbits in the symmetric subspace (see lines l_{03}, l_3, l_4 and lines l_{04}, l_5). Choosing value of the coupling we can observe different sequences of tangent and transversal bifurcations with the parameter a change. For example, the scheme described in the Fig. 1 corresponds to $\gamma \simeq 0.18$. Further increasing of the

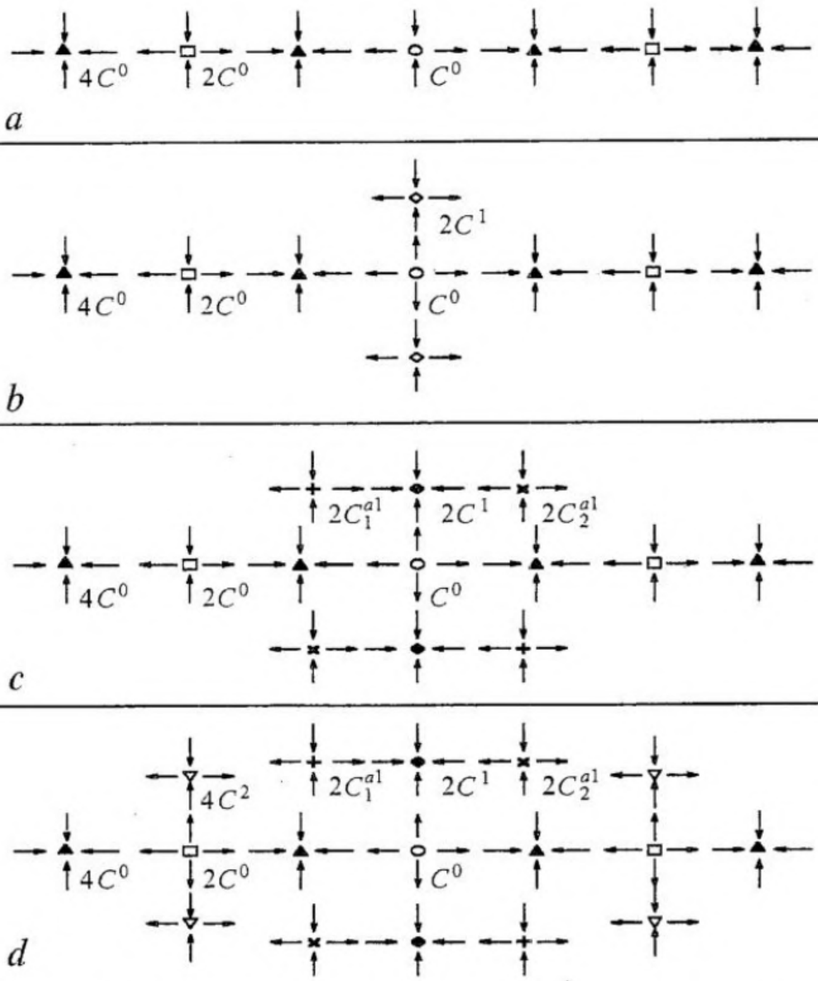

Fig. 1. Scheme of multistability formation beginning in the system of two logistic maps. Orbits $1C^0$ (o) $2C^0$ (□) $4C^0$ (Δ) are located inside the symmetric subspace. Orbits $2C^1$ (◇) $4C^2$ (∇) $2C_1$ (+) and $2C_2$ (×) are outside it

parameter of nonlinearity (line l_{04} in the Fig. 2) leads to transition to synchronous chaos $2^N A^0$ ($2^N A^k$ is 2^N *-band* self-symmetric chaotic attractor originated on the base of the periodic orbit $2^N C^k$). Inside chaotic region band-merging bifurcations and windows of periodicity are observed. On the line l_{05} the system transits to one-band synchronous chaos A^0.

Every transversal period-doubling bifurcation for a periodic orbit $2^N C^0$ located in the symmetric subspace, which is accompanied by its transformation to repeller adds points of local transversal instability to the chaotic attractor $2^N A^0$. From these points phase trajectory leaves the symmetric subspace at transversal perturbations. The regime of synchronous oscillations becomes unrobust. Any small noise and mismatch of the subsystems lead to destroying of the complete synchronization. Time-series of the difference (x_n -y_n) becomes intermittency process (on-off intermittency), when motion in the symmetric subspace is intermittent by bursts from it. As a result a boundary of the synchronous region in the system with any small noise is shifted relatively to one in the system without noise. We have hold numeric investigations on determination of the boundary of the synchronization region. In these investigations we define oscillations as synchronous if the time-series of the subsystems are equal with precision of (ε) during the whole time interval of observations:

$\max|x_n - y_n| < \varepsilon$, $n=1,2,3,\ldots,N_{observ}$.

at chosen values: $\varepsilon = 0.0001$, $N_{observ} = 2000000$ iterations. In the Fig. 2 (o) mark the experimentally determined boundary of the robust synchronization region. From the right side of it stable synchronous chaotic oscillations take place. Under the line l_{05} the system demonstrates many-band synchronous chaotic attractors $2^N A^0$, over it there is one-band synchronous chaotic attractor A^0. Adding small noise with intensity ~ 0.00001 to the system doesn't lead to desynchronization.

From the left side of this boundary and from the right side of the boundary marked by (◇) there is a region of unrobust synchronous chaos. The synchronous regime is observed only in the absence of noise. The transition process to this regime has the form of intermittency. Its duration essentially depends on the chosen initial conditions. Adding very small noise to the system destructs the synchronous regime. The system demonstrates bubbling behaviour. In the region from the left side of the boundary (◇) we observe riddling of the basins of the synchronous chaotic attractor. The basins is riddled with holes that belong to basins of other regular or chaotic attractors.

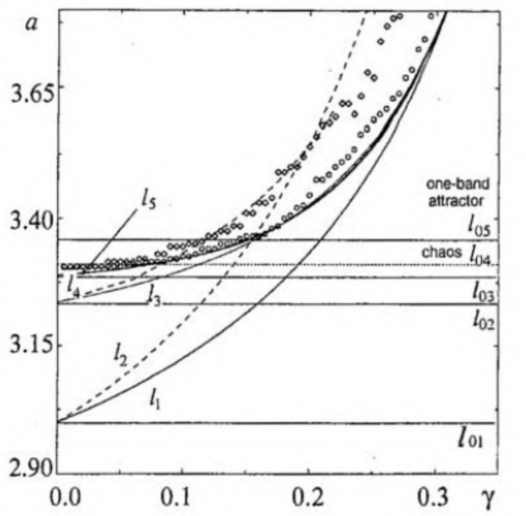

Fig. 2. Location of bifurcational lines on the plane «coupling - nonlinearity» for the system of coupled cubic maps. l_{01}, l_{02}, l_{03} are lines of tangent period-doubling bifurcations of orbits C^0, $2C^0$ and $4C^0$ respectively, l_1, l_3 and l_5 are lines of transversal period-doubling bifurcations of the same orbits, l_2 and l_4 are lines of pitch-fork bifurcations of orbits $2C^1$ and $4C^2$. Line l_{04} marks transition to synchronous chaos; line l_{05} - transition to one-band the synchronous chaotic attractor. Symbol smark destroying of chaotic synchronization: (o) - bubbling process, (◇) - riddled basins

Comparing mechanisms of multistability formation for different systems (coupled cubic maps, logistic maps [10, 11], Hennon maps [18], Chua's self-oscillators [19], Rossler oscillators [20]) we have concluded that:

1. It is common for period-doubling oscillators with diffusive symmetrical coupling including invertible and non-invertible discrete maps and continuous time oscillators.

2. The same bifurcations of the same periodic cycles form the basis of both appearing of new stable oscillatory regimes and destroying of the regime of in-phase chaotic synchronization.

3. The main mechanism of multistability formation and synchronization loss is based on the cascade of transversal period-doubling bifurcations which take place with orbits of main family forming the skeleton of the synchronous chaotic attractor.

4. Bifurcations of synchronous periodic orbits initiate and then enforce the bubbling process of the chaotic attractor. Bifurcations of unsynchronous orbits appeared from synchronous ones lead to riddled basins of the attractor.

3. Synchronization of chaos in weakly non-identical oscillators

Real oscillators are non-identical. Hence, there is a reasonable question: is it possible to apply theory of synchronization based on symmetry properties to real systems. At what conditions idealized pure identical systems will behave similarly to real objects. For investigation of the complete synchronization phenomenon identical interacting systems are usually used as mathematical models. Then obtained in the frameworks of such idealization results are applied to explain behavior of real experimental systems. If the regime of synchronization is stable and the used mathematical model is rough it is observable in real experiments. Intervals of synchronization on the coupling parameter are practically similar for identical and slightly mismatched systems. In this sense the behaviors of the identical and slightly mismatched systems correspond to each other. However, when we investigate more exact effects such as mechanism of the synchronization loss from the point of view of bifurcations of saddle periodic orbits embedded in the chaotic attractor, there are differences in the scenario for identical and weakly non-identical systems. This situation can take place when the symmetry breaking bifurcations take part in the process of synchronization loss. For example this is the pitch-fork bifurcation. From the bifurcation and catastrophe theory it is well-known (see [21, 22]) that the point of this bifurcation is the cusp catastrophe. At slight non-identity between interacting systems the bifurcation is eliminated by the certain ways. Non-identity can qualitatively change behavior of orbits in dependence on a parameter of the system.

We consider this subject on the example of coupled logistic maps with weak non-identity between elements:

$$x_{n+1} = \lambda - x_n^2 + \gamma\,(x_n^2 - y_n^2)$$
$$y_{n+1} = \lambda\delta - y_n^2 + \gamma\,(y_n^2 - x_n^2) \qquad (9)$$

here δ is a detuning parameter. The considered system has no more subspace of symmetry. Hence, we can not define synchronous oscillations as motions inside the surface $x=y$. In this case we must use «experimental» description of oscillating regimes in asymmetric system. In the frameworks of the present description we call the chaotic regime synchronous, if $|x_n - y_n| < \Delta$, at any moment of time n, where Δ is a suitable given value that is small with respect to the intensity of the chaotic oscillation. In our investigations we use the following value of the parameters: $\lambda=1.56$, $\delta=1$ (identical oscillators) and $0.995 \le \delta < 1$ (non-identical oscillators). These values correspond to regime of one-band chaotic attractors in both oscillators at zero coupling.

Let's consider firstly the identical case ($\delta=1$). In the system (9) the synchronization region has a finite interval. The stability loss of the symmetric one-band chaotic attractor A^0 in the transversal direction occurs both at decreasing ($\gamma<0.5$) and at increasing ($\gamma>0.5$) of the coupling coefficient γ. The synchronization loss is induced by bifurcations of saddle orbits $2^N C^0$ which are embedded in the chaotic attractor and form its skeleton. In the both cases of coupling increasing and decreasing the loss of stability begins with a bifurcation of the saddle point $1C^0$, which induces the bubbling transition in the system.

With γ decreasing the saddle point $1C^0$ undergoes the period-doubling bifurcation. In the result it becomes a repeller and the saddle period-2 orbit $2C^1$ appears in its vicinity outside the symmetric subspace. This bifurcation induces the bubbling transition in the system. With further γ decreasing the saddle orbits $2^N C^0$ of higher periods undergo the same bifurcations. This enforces the bubbling phenomenon.

Then the saddle orbit $2C^1$ located outside the symmetric subspace undergoes one more bifurcation. It becomes stable and a pair of period-2 saddle symmetric to each other orbits appear in its vicinity (at inverse parameter changing this bifurcation is the subcritical pitch-fork bifurcation). The bifurcation of the orbit $2C^1$ induces the riddling transition in the system. With further decreasing of the coupling the chaotic attractor gradually «loses» its basins and transforms into a chaotic saddle. The described mechanism fully repeats one of the cubic maps. At the coupling increasing we observe the other mechanism: The point $1C^0$ undergoes the pitch-fork bifurcation. In the result it becomes a repeller and in its vicinity a pair of saddle points C_1 and C_2 symmetric to each other appear. This bifurcation induces the bubbling transition. With further increasing of γ other saddle orbits $2^N C^0$ undergo the period-doubling bifurcations similarly as in the case of coupling decreasing. The riddling phenomenon of the A^0 basins is a result of the bifurcation of the saddle points C_1 and C_2. They become stable and in their vicinities saddle orbits of double period appear (at inverse parameter changing this bifurcation is the subcritical period-doubling bifurcation). In the cases of both the coupling decreasing and the coupling increasing bifurcational scenario of the synchronization loss are very similar. The difference is only in the following. At weak coupling $1C^0$ undergoes the period-doubling bifurcation, but at strong coupling - the pitch-fork bifurcation. Other saddle orbits $2^N C^0$ undergo the period-doubling bifurcations in the both cases. At γ decreasing the process of riddling basins of A^0 begins with the pitch-fork bifurcation of the orbit $2C^1$, but at γ increasing - with the period-doubling bifurcations of C_1 and C_2.

The depended on coupling sequence of bifurcations of periodic orbits which begin the bubbling of the attractor and then the riddling of its basins is built in the Fig. 3, *a*.

Let's now consider the parameter mismatch effect on the bifurcational scenario of the synchronization loss of the system (9) when $\delta \neq 1$. We consider the synchronization loss both at decreasing and increasing of the coefficient of coupling γ. At small value of δ we investigate bifurcations of unstable periodic orbits which lead to breaking of regime of nearly identical chaotic oscillations in the coupled systems.

At γ decreasing a period-doubling bifurcation of the saddle point C^0 induces the transition to the bubbling behavior. After this bifurcation a rebuilding of the phase space structure occurs in the vicinity of the A^0. Namely, inside the quasi-symmetric region the saddle C^0 transforms to repeller and a saddle orbit $2C^1$ appears outside it. Stable manifolds of the saddle $2C^1$ lean on the repeller C^0 and unstable manifolds leave to the quasi-symmetric region. The appearance of such structure changes the character of motions from nearly identical oscillations to the bubbling behavior. At increasing of coupling scenario of the transition to the bubbling behavior is different. With increasing of γ firstly we observe a gradual displacement of the saddle C^0 in the normal direction. It leaves the quasi-symmetric region. Other saddle orbits $2^N C^0$ practically do not change their locations. Then the saddle-repeller bifurcation takes place in the system. In the vicinity of the quasi-symmetric region a repeller C_r^0 and a saddle C_s^0 appear. With further increasing of γ the fixed points diverge. The repeller C_r^0 enters the quasi-symmetric region and the saddle C_s^0 moves away from it. As a result there is the same structure of the phase space in the vicinity of the A^0 as in the case of identical oscillators, but it is formed on the base of other bifurcations. In the quasi-symmetric region there is the repeller C_s^0 on which stable manifolds of the saddles C^0 and C_s^0 lean. Their unstable manifolds leave to the quasi-symmetric region. This phase space structure also leads to the bubbling behavior. Then, with further coupling increasing both saddles C^0 and C_s^0 undergoes subcritical period-doubling bifurcations. As a result they become stable and the trajectory from vicinity of the ex-attractor A^0 transits to one of them. This process is very similar to the case of identical oscillators except the fact that now we do not observe riddled basins more. A trajectory leaves A^0 from any its neighborhood. But the duration of the transition process can be extremely large and essentially depends on the initial values. The scheme of bifurcations that take place in the mismatched system are represented in the Fig. 3, *b*.

Comparing the behaviors of the system (9) at $\delta=0.995$ and at $\delta=1$ we see good qualitative correspondence. The regime of complete synchronization corresponds to nearly identical chaotic oscillations. The bubbling transitions in the symmetric system corresponds appearance of bubbling behavior in the system with mismatch. With further coupling change the same stable orbits appear both in the symmetric and asymmetric systems. In the asymmetric systems there is no riddled basins but one can observe the

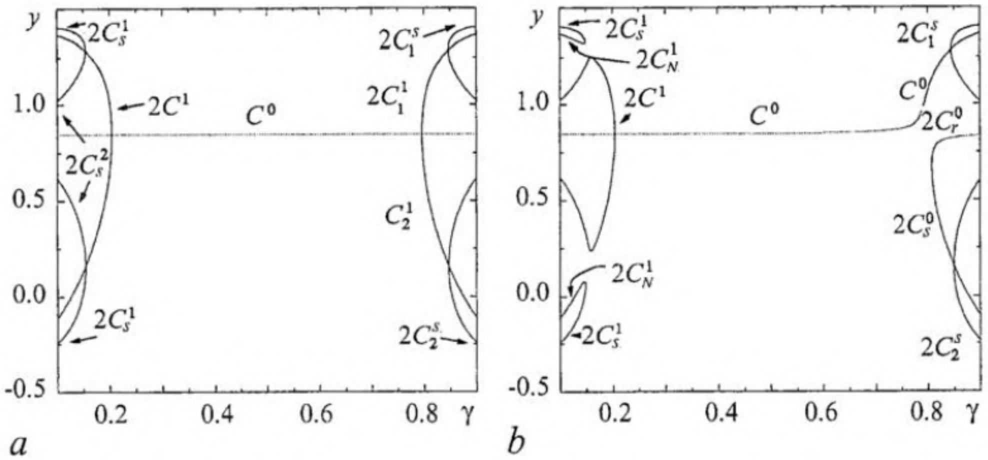

Fig. 3. The scheme of bifurcations on coupling which initiate bubbling and then riddling process in the system of identical (*a*) and weakly non-identical (*b*) logistic maps

sensitive dependence of the transition process time on initial conditions. However, comparing the results quantitatively one need take into account the following: At the coupling decrease the value $|x_n - y_n|$ exceeds the chosen threshold value Δ almost at the same value of γ that corresponds to the bubbling transition in the identical systems. At the coupling increase the corresponding values of γ are very different. This difference of changing of left and right boundaries of the synchronization interval is a result of difference of behavior of unstable periodic orbits embedded in the chaotic attractor which takes place at decreasing and increasing of γ. This difference appears as a result of elimination of the bifurcation conditioned by the symmetry of the system. Thus, if the bubbling transition in the symmetric system is induced by «uneliminated» bifurcation (the period-doubling bifurcation of the saddle C^0 at the coupling decreasing) weak asymmetry does not influence on the bifurcational scenario of the transition to the bubbling behavior. If the bubbling transition is determined by the bifurcation conditioned by the symmetry of the system (the pitch-fork bifurcation of the saddle C^0 at the couplingin creasing) the weak non-identity of the subsystems eliminates it and the bubbling behavior appears according to another scenario.The determined structure of the phase space in the vicinity of A^0 is formated not as a result of the bifurcation of the saddle C^0, but after saddle-repeller bifurcation of birth of new unstable points, namely the repeller C_r^0 and the saddle C_s^0. The completion of the process of the chaos synchronization loss occurs according to different scenario in the symmetric and non-symmetric systems. At the coupling decreasing slight non-identity eliminates the bifurcation of the saddle $2C^1$. Besides it the saddle-node bifurcation of the new stable period-2 orbits $2C_N^1$ and $2C_s^1$ birth take place. Starting from the vicinity of A^0 phase trajectories move to this stable orbit. At the coupling increasing the loss of synchronization in the non-symmetric system is completed by the bifurcation of the saddle C^0. After the bifurcation the point C^0 becomes stable.

4. Antiphase complete synchronization of chaos

In this section we consider another case of complete synchronization of chaos: the antiphase synchronization on example of the coupled cubic maps (1, 2, 8). The single cubic map has a symmetry to transformation of the coordinate:

$$I:\ x \leftrightarrow -x.$$

The system of the coupled maps posses the symmetric property of the single map to the transformation:

$$I:\ \ x \leftrightarrow -x,\quad y \leftrightarrow -y,$$

and due to the symmetric coupling and identity of the subsystem sit also posses symmetry to transformation:

$$R:\ x \leftrightarrow y.$$

Because I and R commutate with each other, their combination is also a symmetric transformation for the system (1, 2, 8):

$$I \circ R:\ x \leftrightarrow -y,\ y \leftrightarrow -x.$$

Consequence of the symmetry of the system to the transformation $R \circ I$ is a possibility of existence there anti-phase oscillations, which are satisfied condition x=-y.

Let's consider the stability properties of the antiphase motions in the coupled maps. In this case we also use normal variables.The equations in the small vicinity of the antisymmetric subspace (x=-y) have the form:

$$u_{n+1} = f'(v_n)u_n \tag{10}$$

$$v_{n+1} = (1\text{-}2\gamma) f(v_n). \tag{11}$$

In this case the dynamics inside the antisymmetric subspace is described by the equation (11). Contrary to the case of in-phase synchronization it depends on the coupling coefficient γ. Stability of an antisymmetric solution to the tangent perturbations is determined by the tangent Lyapunov exponent:

$$\Lambda_\tau^{a} = \lim_{N\to\infty} (1/N)\ \Sigma_{n=1}^{N} \ln|(1\text{-}2\gamma) f'(v_n)|. \tag{12}$$

The equation (10) determines dynamics in the normal direction to the antisymmetric subspace in its vicinity. It has no obvious dependence on the coupling coefficient γ but it depend on it through the variable v_n, which is determined by the eq. (11). The normal Lyapunov exponent which determines transversal stability of the antiphase oscillations has the form:

$$\Lambda_\perp^{a} = \lim_{N\to\infty}(1/N)\ \Sigma_{n=1}^{N} \ln|f'(v_n)|. \tag{13}$$

It is seen that the normal and tangent Lyapunov exponents are connected with each other:

$$\Lambda_\tau^{a} = \Lambda_\perp^{a} + \ln|1\text{-}2\gamma|. \tag{14}$$

This relation is the opposite to the in-phase case. Here $\Lambda_\tau^{a} \le \Lambda_\perp^{a}$ and hence, the every antiphase oscillating regime firstly loses its stability in the normal to the antisymmetric subspace direction and secondly in the tangent direction. Because of relation (14) the antiphase self-synchronization of chaos is impossible in the considered systems. For a chaotic attractor $\Lambda_\tau^{a}>0$ and therefore the normal Lyapunov exponent must be positive. Hence chaotic antiphase oscillations can not be transversally stable.

Oscillating regimes inside the antisymmetric subspace are formed on the base of the fixed points C_{10} and C_{20} which appeared from the trivial fixed point C_{00} = (0; 0). Limit sets formed on the base of these points are identical up to symmetry transformation. Therefore we consider only one family of the regimes (for example, near the point C_{10}).

The saddle fixed point C_{10} appears from the saddle fixed point C_{00} in the result of the symmetry breaking bifurcation. It is unstable to the perturbations directed transversally to the antisymmetric subspace. On the line l_{tp}^{1} (Fig. 4) it becomes stable

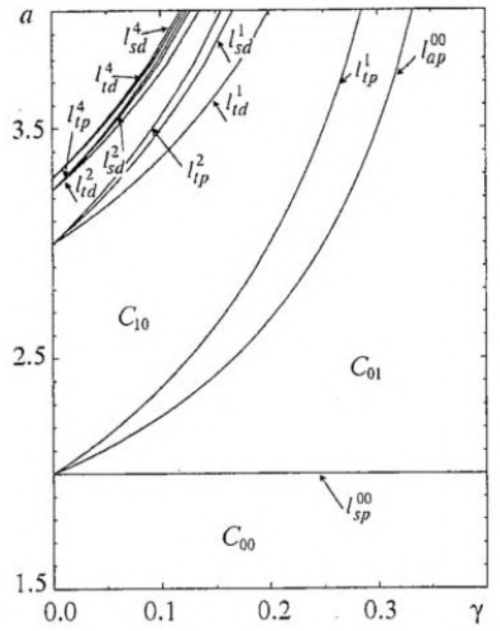

Fig. 4. Bifurcational lines of antisymmetric periodic orbits on the pane of the parameters γ - a

through pitchfork bifurcation. With changing of the parameters a and γ on the base of this fixed point there is a cascade of period doubling bifurcations which leads to formation of a chaotic set inside the antisymmetric subspace. The every orbit undergoes the period-doubling bifurcation twice in the cascade. Firstly, as stable orbit on the first multiplier, secondly as saddle orbit on the second multiplier. As a result of the first period doubling the orbit loses its stability in the normal to the subspace direction. In its vicinity, outside the antisymmetric subspace a stable orbit of double period appears. As a result of the second period doubling the saddle orbit loses stability in the tangent direction and becomes repeller. In its vicinity, inside the antisymmetric subspace a saddle periodic orbit of double period appears. In the Fig. 4 the lines of the first period doubling bifurcations are denoted: l_{td}^1 (for the orbit of the period-one), l_{td}^2 (for the orbit of the period-two), l_{td}^4 (for the orbit of the period-four), and the lines of the second period doubling bifurcations as l_{tsd}^1, l_{tsd}^2, l_{tsd}^4 respectively. Then, with further parameters changing, the appeared saddle antisymmetric orbits become stable through the subcritical pitchfork bifurcations. In the Fig. 4 these lines are denoted as l_{tp}^2 and l_{tp}^4. Therefore, on the parameters plane there is a stable antiphase period-one orbit in the region between the lines l_{tp}^1 and l_{td}^1, a stable antiphase period-two orbit between the lines l_{tp}^2 and l_{td}^2 and a stable antiphase period-four orbit between the lines l_{tp}^4 and l_{td}^4. Bifurcations of orbits of higher periods take place by similar way. Hence, on the plane of the parameters there are bounds of stability for regular antiphase regimes, between which bounds of transversal instability exist.

The considered bifurcational scenarium is very similar to the one for in-phase orbits. However, in the case of in-phase synchronization the bifurcations inside the symmetric subspace precede the bifurcations in the normal direction. Therefore, in the case of antiphase synchronization, contrary to the in-phase synchronization:

• regions of transversal stability are divided by the regions of transversal instability;

• in the symmetrical subspace the transversally stable chaotic attractor is not formed.

Antiphase synchronous system in the diffusively coupled period-doubling maps is impossible. However, for stabilization of antiphase chaotic oscillations one can apply feedback controlling technics. We want to find the controlling function in the form which does not change the form of antiphase oscillations, but changes their stability. Hence, the controlling function $\Psi(x, y)$ must be equal to zero inside the antisymmetric subspace, namely: $\Psi(x, -x) = 0$. In our work we suggest the function in the form: $\Psi(x, y) = r[f(x)+f(y)]$. The controlling term is added to the right side of the first equation of the system (1, 2):

$$x_{n+1} = f(x_n) + \gamma(f(y_n) - f(x_n)) + r(f(x_n) + f(y_n)) \tag{15}$$

$$y_{n+1} = f(y_n) + \gamma(f(x_n) - f(y_n)). \tag{16}$$

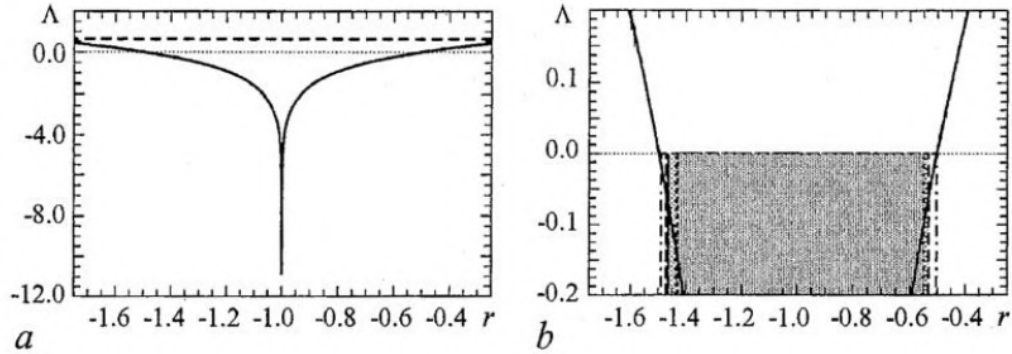

Fig. 5. Dependence of the normal Lyapunov exponent for antiphase chaotic attractor on the controlling parameter r *(a)* at a=3.8, γ =0.04. Dashed line denotes the values of the tangent Lyapunov exponent. In the *(b)* this dependence is presented in larger scale with regions of the antiphase controlled synchronization

The term $r(f(x_n)+f(y_n))$ can be considered as another coupling loop with coupling coefficient r.

The equations in the normal variables for the system with the control have the form (near the antisymmetric subspace):

$$u_{n+1} = (1+r)\, f'(v_n) u_n \tag{17}$$

$$v_{n+1} = (1-2\gamma)\, f(v_n) + rf'(v_n) u_n . \tag{18}$$

In the case of antiphase oscillations u_n =0 and equation (18) transforms to the (11). The normal Lyapunov exponent for the system with the control is:

$$\Lambda_{\perp}{}^{a}{}_{contr} = \lim_{N\to\infty} (1/N)\, \Sigma_{n=1}^{N} \ln|\,(1+r) f'(v_n)| \tag{19}$$

and hence:

$$\Lambda_{\perp}{}^{a}{}_{contr} = \Lambda_{\perp} + \ln|1+r|. \tag{20}$$

We chose parameter r near value -1 to make the normal Lyapunov exponent sufficiently small and hence, the synchronous chaotic regime transversally stable. The Fig. 5, *a* represents the dependences of the normal Lyapunov exponent on the controlling parameter r. Values of the other parameters corresponds to the regime of the developed chaotic attractor: a=3.8, γ =0.04. To transit to regime of antiphase synchronization we use the following procedure: We chose initial conditions from the basins of the chaotic attractor. In the every moment of time we appreciate the distance between the phase point and the antisymmetric subspace: $\rho = |x_n + y_n|$. The distance was compared with the chosen value ε. If $\rho > \varepsilon$, the phase point is far from the subspace and the controlling influence is switched off. The trajectory evolves on the unsynchronous chaotic attractor. When phase point appears near the antisymmetric subspace ($\rho \leq \delta$), the controlling influence is switched on. If the controlling parameter r locates in the interval where $\Lambda_{\perp contr} < 0$ the chaotic set inside the antisymmetric subspace becomes stable to transversal perturbation and the trajectory is attracted to the subspace. After this the controlling influence tends to zero. In our numerical experiments we chose ε =0.01. In the Fig. 6 phase portraits of the oscillations without control (*a*) , with control (*c*, *e*) and correspondent time-series of x_n+y_n (*b*, *d*, *f*) are represented. The original chaotic attractor (Fig. 6, *a*) corresponds to the regime of unsynchronous chaos. The phase trajectory draws the square-like region. With applying small controlling influence the diagonal line x=-y appears on the region (Fig. 6, *c*). The time-series has interval of synchronous behavior (Fig. 6, *d*). With further changing r the interval of synchronous behavior grows and as a result the system transits to fully synchronous oscillations (Fig. 6, *e*, *f*). In this case the resulting chaotic attractor is a one-band attractor located in the antisymmetric subspace. In the Fig. 5, *b* the intervals of the parameter r sufficient for complete synchronization at different intensities of noise are presented. The more dark color corresponds to larger noise. Without noise the interval of the synchronization coincides with the interval of r where the normal Lyapunov exponent is negative. With noise the controlled synchronization region becomes more narrow (Fig 5, *b*).

As we have demonstrated before the process of the in-phase synchronization loss is accompanied by the bubbling phenomenon and riddled basins. It is reasonable question: Do these phenomena exist in the case of antiphase synchronization loss? To answer this question we consider the evolution of the chaotic attractor with changing of the coefficient r. In the numerical experiments we chose initial values near the

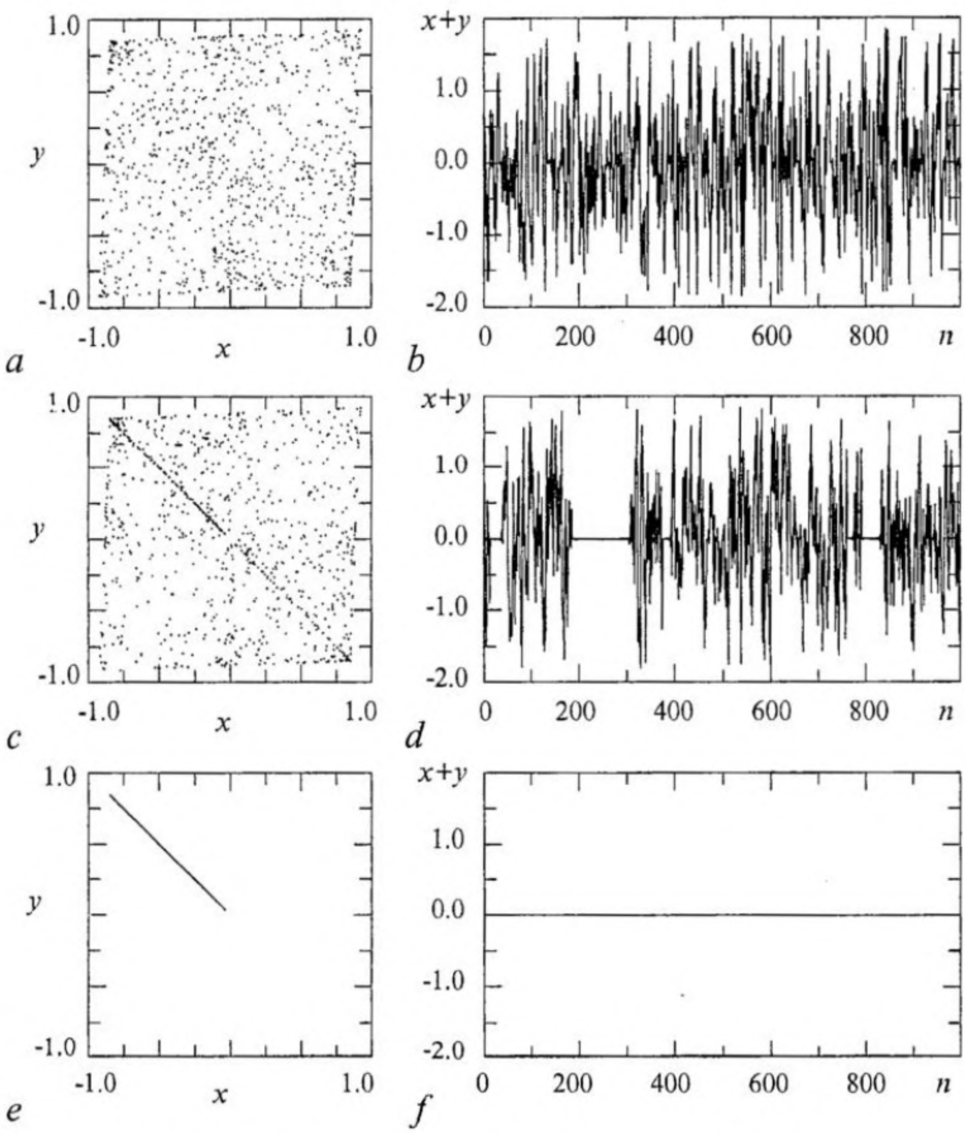

Fig. 6. The phase portraits and time-series of the oscillating regimes without control (*a*, *b*), with partial control (*c*, *d*) and in the regime of the complete antiphase synchronization (*e*, *f*)

antisymmetric subspace. The controlling influence is switched on during the whole time of observation (not depending on nearness ρ of the phase point to the subspace). At $-1.46<r<-0.525$ the chaotic attractor inside the antisymmetric subspace is stable to transversal perturbations. The synchronous regime is robust. Adding noise of small intensity (~ 0.00001) doesn't lead to visible changing in the systems behavior. With increasing of the controlling parameter at $r>-0.525$ a bubbling attractor is observed in the system. The chaotic attractor remains stable to transversal perturbations but the time of transient process becomes extremely large (hundreds of thousands iterations) and it sensibly depends on the initial values. Adding noise of small intensity leads to essential rebuilding of the phase portrait of the oscillations. The attractor gets finite transversal size. Phase point begins to visit neighborhoods of the both fixed points C_{10} and C_{20}. The corresponding time-series of $x_n + y_n$ is the on-off intermittency process [12]. The Fig. 7 demonstrates phase portraits of the attractor without noise (*a*) and with small noise (*c*). In

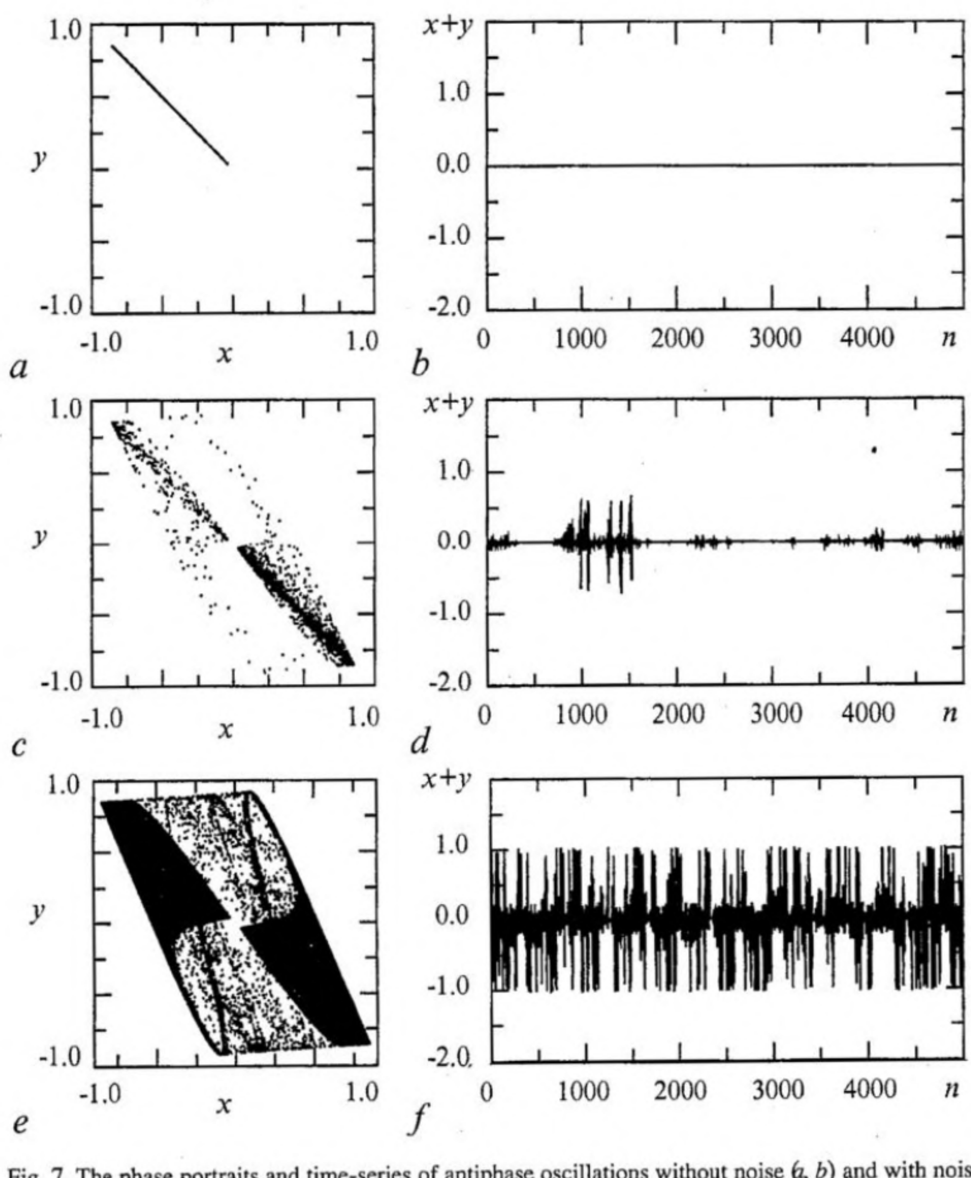

Fig. 7. The phase portraits and time-series of antiphase oscillations without noise (*a*, *b*) and with noise (*c*, *d*). In (*e*, *f*) there is chaotic oscillations resulted from the blowout bifurcation

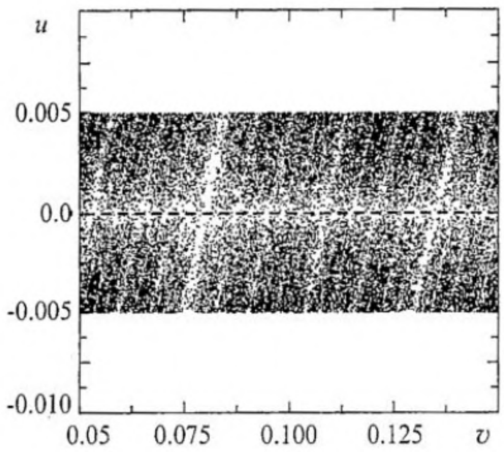

Fig. 8. A part of the basins of the chaotic attractor inside the antisymmetric subspace (white color) at r =-1.485. The black color denotes regions relating to the basins of the attractor in the infinite

the (b, d) there are corresponding time-series. With small noise phase point moves along the antisymmetric subspace for a long time. Then, it is short burst apart from the subspace, after which the phase point return to the vicinity of the antisymmetric subspace. The averaged frequency of the bursts increase with increasing of the parameter r. Finally, at r =-0.406 the blowout bifurcation [5] takes place when the chaotic attractor is not already stable in the normal direction and it transforms to the chaotic saddle. The synchronous oscillations are not observed further in the system both with noise and without it. The phase portrait of oscillations looks like the bubbling attractor in presence of noise (Fig. 7, e).

With decreasing r from the synchronization region, at r <-1.46 we observed the transition from the robust stable synchronous regime to the bubbling behavior. Then, at r <-1.472 the basins of the synchronous attractor is riddled by holes of the basins of the attractor in the infinity.

In the Fig. 8 we present a fragment of the basins of the chaotic attractor in the antisymmetric subspace (white color) with holes from the basins of the infinity attractor (black color) wedged in it. This basins is represented in the normal coordinates u and v, the antisymmetric subspace is marked by the dashed line. The results were obtained for the parameters values: a =3.8, γ =0.04, r=-1.485.

Comparing in-phase and anti-phase synchronization of chaos we demonstrate that bifurcational mechanism inside the antisymmetric subspace is similar to the one in the symmetric subspace except the order of bifurcations taking place tangently and normally to the subspace. The anti-phase self-synchronization in similar systems is possible only for regular regimes. The chaotic synchronization can be achieved with applying methods of chaos control. We demonstrate that the process of loss of this type of synchronization can be similar to the case of in-phase synchronization. It demonstrates bubbling behavior, riddled basins and blowout bifurcation.

Conclusion

We consider in- and anti-phase complete synchronization of chaos in dissipativelly coupled periodic-doubling oscillators. For in-phase synchronization cases of purely identical and slightly mismatched subsystems are investigated.

We demonstrate that the processes of loss of in-phase and anti-phase synchronization are very similar. The both ones go through stages of bubbling attractor and riddled basins. Structures of bifurcations inside symmetric and antisymmetric subspaces are similar to each other except the order of their bifurcations. In the both cases every periodic orbit undergoes two period-doubling bifurcations: in tangent and normal direction to the subspace. In the case of symmetric subspace the tangent bifurcations proceed the transversal ones. In the case of antisymmetric subspace the transversal bifurcations proceed the tangent ones. Two period-doubling cascades lead to formation of multistability. The bifurcations inside the symmetric subspace form the synchronous chaotic attractor. The similar bifurcations in antisymmetric subspace form synchronous chaotic saddle which can be stabilized in normal direction by using control of chaos technic. In the both cases the transversal period-doubling bifurcations lead to (*a*) loss of transversal stability and (*b*) forming new periodic regimes outside the corresponding subspace.

Comparison of identical and slightly mismatched systems demonstrates that in the both cases the loss of in-phase synchronization goes through similar steps with small differences: bubbling attractor and riddled basins for identical systems and bubbling behavior and essential dependence of the duration of transition process to another attractor for mismatched ones. In the both cases we observe similar structure of the phase space near the symmetric subspace. However, this structure can be following of different bifurcational mechanisms: pitch-fork bifurcation in the case of identical oscillators and saddle-repeller bifurcation in the mismatched ones.

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Мультистабильность, синфазная и противофазная синхронизация в системах с бифуркациями удвоения периода

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В статье рассматриваются механизмы образования мультистабильности и потери полной синхронизации хаоса в диффузионно связанных отображениях с бифуркациями удвоения периода. Рассматриваются случаи синфазной и противофазной синхронизации. Для синфазной синхронизации исследуется влияние неидентичности между осцилляторами на механизм потери синхронизации хаоса.

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