Izv. VUZ «AND», vol.10, № 3, 2002

MULTI-PARAMETER PICTURE OF TRANSITION TO CHAOS

A.P. Kuznetsov, L.V. Turukina, A.V. Savin, I.R. Sataev,
J.V. Sedova, S.V. Milovanov

In this paper we outline several research directions linked with multi-parameter analysis of complex dynamics of nonlinear systems. In particular, we discuss examples of realistic models of multi-parameter systems, critical phenomena at the chaos threshold, correspondence of features of differential equations and maps etc.

Introduction

The commonly recognized conception of a scenario of transition to chaos suggests that this term designates some sequence of bifurcations observing under variation of one control parameter. For example, in a driven nonlinear oscillator increase of the amplitude of the external force is accompanied by a period-doubling cascade with subsequent transition to chaos. However, in a system with two or more control parameters we should imagine a picture on parameter plane or in parameter space. (For example, in the case of the mentioned oscillator it is natural to use a plane of amplitude versus frequency of the external force.) A study of arrangement of the parameter space implies revealing of typical bifurcations, regularities of their coexistence and subordination, characteristic forms of phase portraits at representative points of the parameter space, plotting bifurcation trees associated with definite paths in the parameter space, consideration of plots of Lyapunov exponents etc.

Moreover, it is natural to develop and generalize the concept of scenario of transition to chaos in application to multi-parameter systems. Let us imagine some three-parameter nonlinear system demonstrating transition to chaos via the period-doubling cascade. It is clear that in general the period-doubling bifurcations in such system will occur at some curved surfaces, and they will accumulate to the limit, *the Feigenbaum critical surface*, which corresponds to the border of chaos. It may be expected that in some cases this surface may have an edge, some *critical curve*. In a neighborhood of this curve some special regularities of coexistence of bifurcations and scaling laws should be observed distinct from those of Feigenbaum. In turn, on the critical curve some critical points may occur, etc. The corresponding classification and discussion may be found in reviews [1-5].

Multi-parameter approach to a study of nonlinear systems is proven to be productive and has induced a number of research directions, which will be discussed in the present article.

1. Two-parameter analysis of physical systems

First of all, we must outline the search for physical systems with complex dynamics, for which the multi-parameter analysis is of importance. Very often researches working in nonlinear dynamics tend to use rather formal models like logistic map, Henon map, Arnold's cat map, etc. An alternative is constructing maps describing dynamics of physical systems from «first principles», that is, from their fundamental evolution equations (Newton equations, Maxwell equations, etc). For physically motivated maps dynamical variables and parameters have usually a clear sense, and this circumstance increases its value and significance. For example, to account noise in such models we turn to physical argumentation and mechanisms, while in abstract models the fluctuations are introduced rather in formal and artificial manner.

Periodically kicked nonlinear oscillator. As a first example let us consider the Duffing oscillator excited by a periodic sequence of δ-pulses

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + \beta x^3 = \Sigma\, C\delta(t - nT). \tag{1}$$

Here x is a coordinate of the oscillator, γ - is coefficient of decay, ω_0 is frequency of free linear oscillations, T is a period of pulses, C is their amplitude. For intervals between the kicks one can derive an approximate analytical solution using the method of slow amplitudes. It yields a 2D map:

$$z_{n+1} = A + Bz_n \exp(i\,(|z_n|^2 + \psi)), \tag{2}$$

where z_n is complex amplitude just before the n-th kick, and dimensionless parameters A, B and ψ are expressed via parameters of the original oscillator as follows:

$$A = (C/\omega_0)[(3\beta T/8\omega_0)(1-e^{-\gamma T})/\gamma T]^{1/2}, \quad B = e^{-\gamma T/2}, \quad \psi = \omega_0 T. \tag{3}$$

System of Ikeda. It appears that the map (2) also describes dynamics of the optical system considered by Ikeda et al. It is a circular optical resonator containing medium with phase nonlinearity and excited by laser beam [6]. In this case parameter A represents a dimensionless intensity of the incident light, and B characterizes dissipation in the resonator. Fig. 1, *a* shows a chart of dynamical regimes in the parameter plane (A, B) for Ψ=0.

The gray tones designate domains of definite period of motion generated by the map. Fig. 1, *b* is a magnified fragment of the chart, it presents a very wide-spread pattern of parameter-space structure called the «crossroad area».

In Fig. 1, *a* together with the chart we present several phase portraits of the Ikeda map. There is some domain in the parameter plane where the phase portrait tends to turn to a circle; it indicates that the description in terms of 1D map becomes appropriate. The explicit form of this map may be derived from (2) [7, 8] and reads

$$x_{n+1} = \lambda \cos x_n + \varphi. \tag{4}$$

Here $x=\lambda\xi+\varphi$, $\mathrm{Re}\tilde{z}_n=\xi_n$. New parameters λ and φ are expressed via parameters of the original map as

$$\lambda = 2A^2B, \quad \varphi = A^2 + \psi. \tag{5}$$

We see that the kicked nonlinear oscillator allows description in terms of differential equations and in terms of analytically derived (approximate) 2D and 1D mappings.

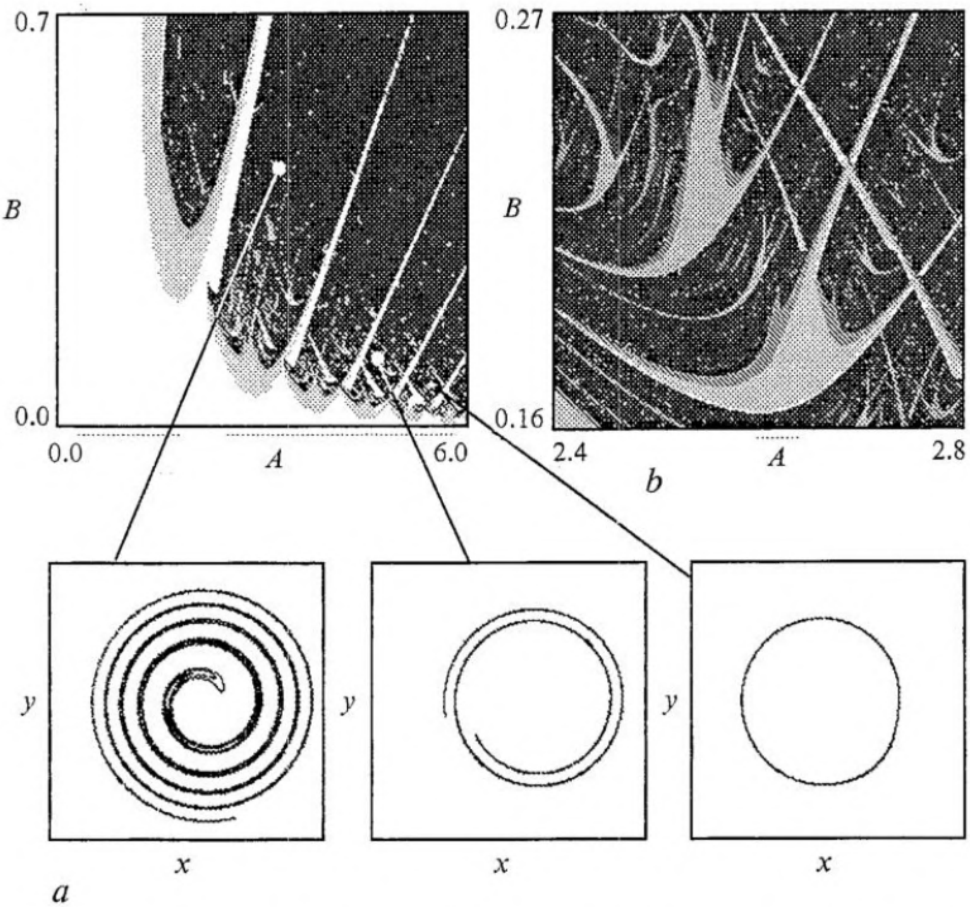

Fig. 1. *a* - Parameter plane for the Ikeda map (2) and typical phase portraits in select points; *b* - magnified fragment of parameter plane

Gravitational machine of Zaslavsky (bouncing ball on a vibration table). Originally, the idea of gravitational machine was formulated in context of astrophysics and celestial mechanics and consists in a use of alternating gravitational field (e.g. of a double star) for acceleration of a spaceship or a celestial body [9]. The model suggested by Zaslavsky is a ball of mass m bouncing up and falling down under the gravitational force on a horizontal plate, oscillating in the vertical direction. A traditional simplification used in a course of derivation of the basic dynamical equation (map) consists in neglecting displacement of coordinate of the plate in the moment of impact. It seems a very natural assumption, as the amplitude is small enough. The resulting map is rather simple and looks like [9]

$$\begin{aligned} v_{n+1} &= (1-\varepsilon)v_n + k\sin\phi_n, \\ \phi_{n+1} &= \phi_n + v_{n+1}, \quad (\mathrm{mod}\,2\pi). \end{aligned} \tag{6}$$

Here dimensionless variables and parameters are introduced: v_{n+1} is a velocity of the ball just after an impact, $\varphi_n=\omega t_n$ is dimensionless time of the impact, $k=2(2-\varepsilon)V_0\omega/g$ is amplitude of the oscillating velocity of the plate, ε is a coefficient characterizing a fraction of energy loss in an impact of the ball with nonmoving plate. In particular, this approximate mapping was discussed in books of Moon [10], Lichtenberg and Liberman [11], Guckenheimer and Holms [12] as one of classic examples of chaotic systems.

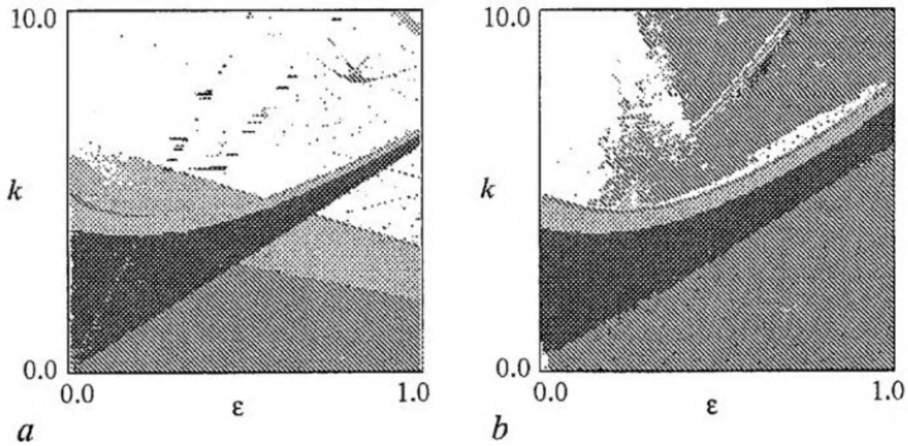

Fig. 2 Parameter planes: *a* - approximate (6) and *b* - exact (7) maps of bouncing ball

Alternatively, one can undertake more accurate analysis and obtain a map without the restriction in respect to the amplitude of the plate vibrations [13]. It is an implicit relation

$$\begin{aligned} v_{n+1} &= -(1-\varepsilon)v_n - k\sin\phi_n + 2(\phi_{n+1} - \phi_n), \\ [(1-\varepsilon)v_n + k\sin\phi_n](\phi_{n+1} - \phi_n) - (\phi_{n+1} - \phi_n)^2 &= [k/(2-\varepsilon)](\cos\phi_n - \cos\phi_{n+1}). \end{aligned} \tag{7}$$

The charts of dynamical regimes both for the approximate and accurate models, the maps (6) and (7), are shown in Fig.2.

Relativistic electron beam interacting with a backward electromagnetic wave. As known, in the system of electron beam interacting with a backward wave a generation of the electromagnetic oscillations is possible. A very rough approach to description of the dynamics is based on assumption that the electromagnetic field effects the electron beam only in a narrow spatial domain near the input edge of the device, and the beam radiates energy into the backward wave in a narrow domain near the opposite end. In other words, we assume that the interaction takes place only in two gaps (δ-functions). In this case the dynamical equations can be reduced to a 1D map. In the relativistic case it was obtained and studied in Ref. [14]:

$$\begin{aligned} A_{n+1} &= F(A_n), \\ F(A) &= (L/2\pi)\,|\int_0^{2\pi} \exp(-i\,(\alpha + (L/\nu)[1+(L\nu A/4)\cos\alpha]^{-2}(-L/\nu))d\alpha|. \end{aligned} \tag{8}$$

Here A_n is a dimensionless amplitude of the wave at the input of the electron beam, L is a dimensionless length of the interaction space (proportional to the cube of the beam current), ν is the relativistic parameter. The chart of dynamical regimes on the parameter plane L, ν exhibits an obvious resemblance with the chart for the kicked oscillator and for the Ikeda model (Fig. 3, upper panel). The bottom panel in Fig. 3 shows magnified fragment of the parameter plane and demonstrates formation of crossroad area type on a base of the period-3 cycle. Also, several graphs of the map at some representative points are shown.

Free-electron laser in regime of mode selection. Using approximation analogous to that discussed above, it is possible to derive a map for description of interaction of two modes (a basic mode and a parasitic one) in a free-electron laser [15]

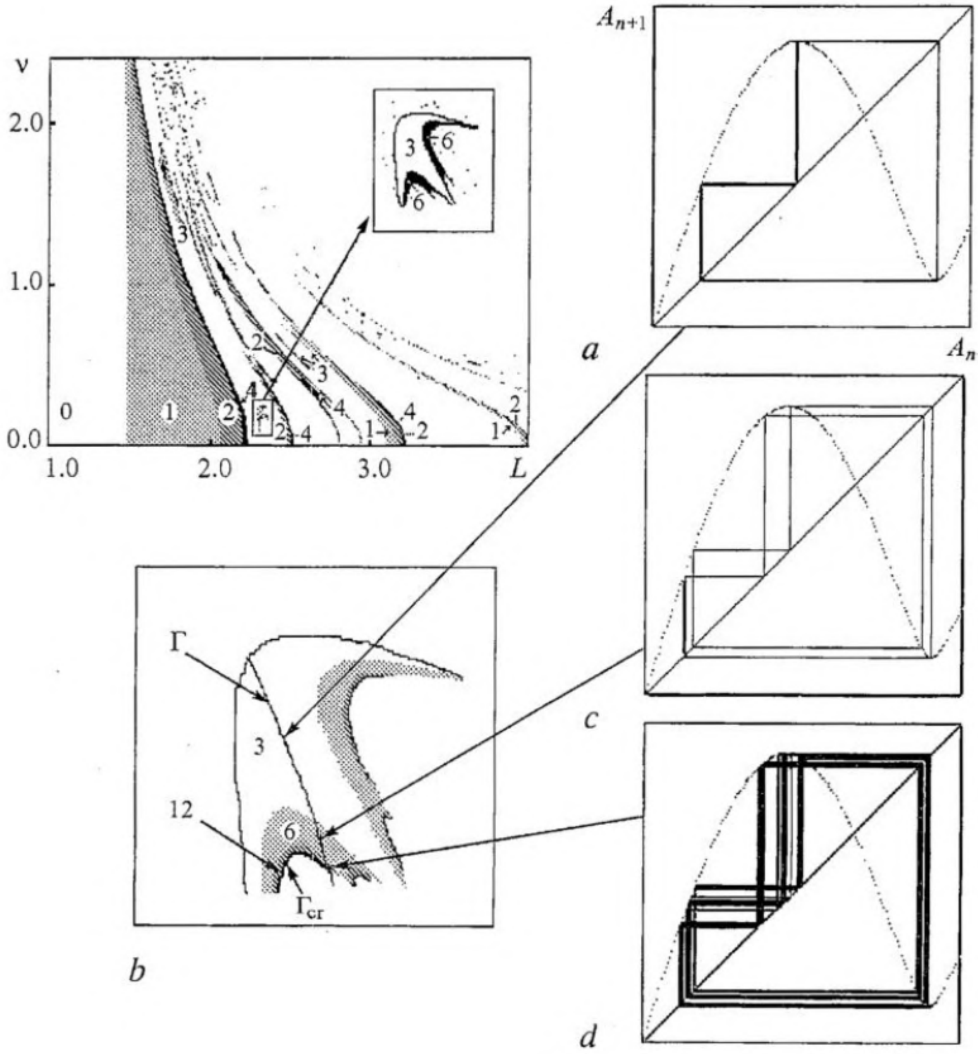

Fig. 3. At the top of figure - parameter plane of the map (8) for the relativistic electron beam interacting with a backward electromagnetic wave. At the foot of figure (on the left) (b) - its fragment. Line Γ, which corresponds to mapping the maximum onto minimum, and Feigenbaum critical line Γ_{cr} are shown. Figures *a*, *c*, *d* - iteration diagrams in typical points. Figure *d* corresponds to tricritical point. Lines Γ и Γ_{cr} are converged, and Feigenbaum critical line is terminated

$$R^{-1}x_{n+1} - x_n = L^3 | J_0(y_n) J_1(x_n) |,$$
$$R^{-1}y_{n+1} - y_n = L^3 | J_0(x_n) J_1(y_n) |, \tag{9}$$

where x_n and y_n are dimensionless amplitudes of the basic and parasitic modes, respectively, J_0 and J_1 are the Bessel functions, L is the normalized length of the interaction space, R is a combination of reflection coefficients at the input and output edges of the system. The arrangement of the parameter plane for the model (9) was studied and discussed in Refs [16, 17].

2. On an effectiveness of the analytical methods in nonlinear dynamics

As we see, the analytical derivation of a map is possible usually with use of some assumptions and physical approximations. How effective they are? Due to concepts of universality (Feigenbaum and other authors) investigators tend to regard as a habitual fact that the realistic systems should demonstrate the same phenomena as simple formal models, like logistic or cubic maps. When we construct a map analytically, we assume definite assumptions, which may seem very natural and justified. However, speaking about regimes of strong nonlinearity and high sensitivity in respect to initial conditions, we must be extremely careful: quite logical approximations may appear to be unsuccessful or to have a disappointingly restricted domain of application.

In Ref. [13] an example of such situation was demonstrated for the gravitational machine of Zaslavsky. Fig. 4, *a* shows attractors of the approximate and exact mappings (left and right panels, respectively) for several parameter values. Observe that they are absolutely different although the selected parameters are from domain traditionally used by researchers. Analogous situation occurs in a conservative case (Fig. 4, *b*).

We conclude that investigation of models of realistic physical systems in context of nonlinear dynamics requires careful handling and attentive revealing conditions of validness of the accepted assumptions. The last may turn to be a separate voluminous

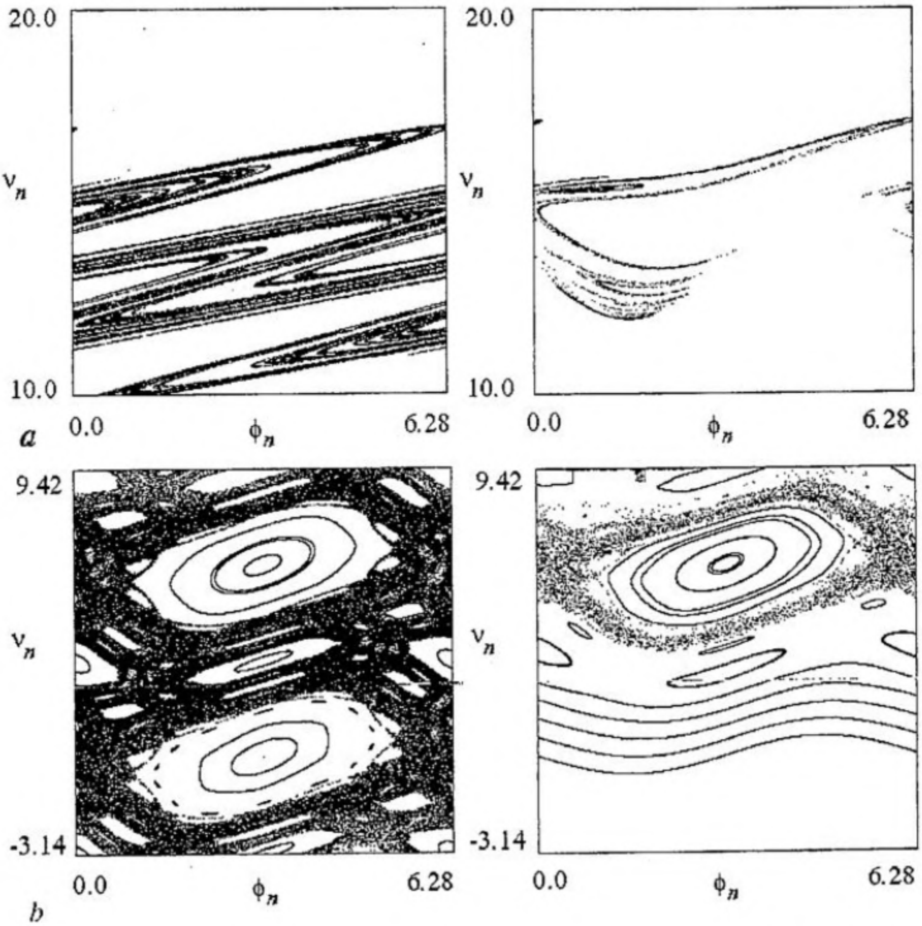

Fig. 4. At the top of figure (*a*) - phase portraits for approximate (on the left) and exact (on the right) dissipative map of bouncing ball. Parameter values are ε=0.4, k=6.5. At the foot of figure (*b*) - phase portraits in the conservative case parameter values are ε=0, k=1.2

research. For example, such a study of the original differential system (1), the 2D map (2), and 1D map (4) was presented in Ref. [7, 8].

3. Critical points in the parameter plane

One of the simplest critical phenomena mentioned in the Introduction is the so-called tricritical dynamics. This is a phenomenon that may occur in two-parameter analysis of bimodal (with two extrema) one-dimensional maps. In parameter plane the tricritical point appears as a terminal point of the Feigenbaum critical line. Also a line comes to this point associated in the parameter plane with a condition that one quadratic extremum is mapped precisely to another. On this line, obviously, the twice iterated map has a quartic extremum. Hence, moving along this line we will observe period-doubling cascade, but the scaling constant will be δ_T=7.284686... and it is distinct from the well-known Feigenbaum value δ_F=4.6692016...

As an example, let us consider Fig. 3. In the parameter plane of the system of relativistic electron beam and electromagnetic wave one can see the Feigenbaum critical line and the line of existence of the quartic extremum in the twice iterated map. These two lines intersect at the tricritical point. Fig. 3, *d* shows typical iteration diagrams at these lines and precisely at the tricritical point. Detailed two-parameter analysis and discussion of scaling properties of the system may be found in Ref. [14].

Usually, a tricritical point is not unique, but there exists an infinite number of such points forming a complex nontrivial set in the parameter plane. To study and classify them, MacKay and Van Zeijtz suggested a specific procedure of constructing «a binary tree of superstable orbits» [5]. In Fig. 5 this tree is plotted on the parameter plane of the map (4), which describes approximately dynamics of the kicked nonlinear oscillator. In a sense, the «crown» of the tree organizes the complex form of the chaos border in the parameter plane. Detailed study of the tricritical dynamics in this system and illustrations of scaling may be found in Refs [7, 8, 18].

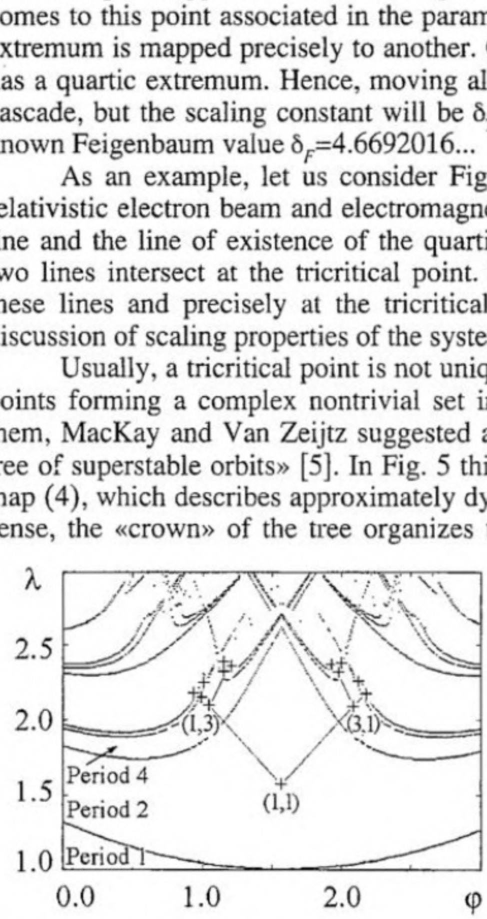

Fig. 5. Binary tree of superstable orbits in the parameter plane of «cosine map» (4) the branch of the tree are lines, which corresponds to mapping the maximum onto minimum after some iterations. The type of doubly superstable cycles associated with the branching points at the tree are indicated in brackets

It is worth noting that the maps for the gravitational machine (6) and (7) demonstrate one more type of critical behavior. In these maps, Feigenbaum's critical line in the parameter plane comes to the point *H*, where the dissipation vanishes. This point may be found as limit of period-doubling cascade in the Hamiltonian system (along the line ε=0). A neighborhood of the critical point *H* on the parameter plane is characterized by a two-parameter scaling, with universal constants δ_1= 8.721097... and δ_2=2.

4. Comparative description of complex dynamics in terms of mappings and differential equations

Correspondence between description of physical systems with three-dimensional phase space in terms of differential equations and of 2D maps is, in a sense, perfect. Indeed, the 2D map may be thought as obtained from the Poincare cross-section

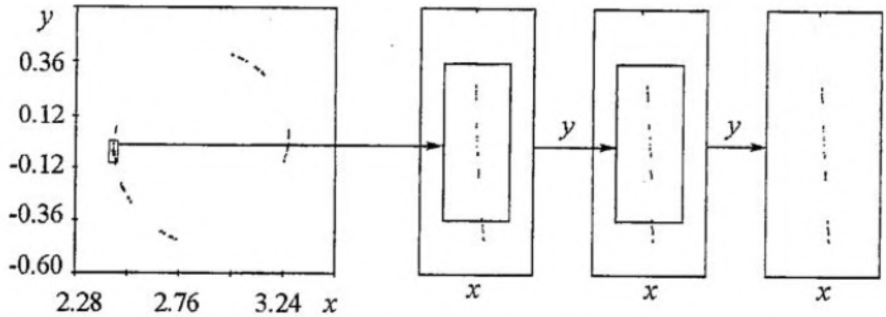

Fig. 6. Scaling properties on the three level of resolution on the attractor for the Ikeda map (2) at the pseudo - tricritical point A=2.8988007984, B=0.1445571961. We magnify each fragment by α= =-1.69030297...

construction. However, passage from a 2D map to a 1D map may lead to principal differences in subtle details of dynamics. In particular, it relates to critical behavior of the maps at the chaos threshold, as mentioned in the Introduction.

In Ref. [18] the related matters were discussed by means of comparison of the Ikeda map (2) and of its 1D approximation (4). As we know, both Feigenbaum's and tricritical dynamics allow description in terms of the renormalization group. On this basis, one could expect that the exact 2D maps and the approximate 1D maps will relate to the same universality class and demonstrate identical scaling regularities. The only notable difference would be in some displacement of the Feigenbaum critical lines and tricritical points due to the approximate nature of the 1D map. However, the attempt to find the tricritical points in the 2D Ikeda map appears to be unsuccessful! It occurs that the tricritical behavior in the 2D map survives only in a sense of somewhat an *intermediate asymptotics*, perhaps over a sufficiently large number of period doublings. In the last case it is convenient to speak on the so-called *«pseudo-tricritical» points* In computations performed in Ref. [18] they have been located in the parameter plane of the Ikeda map, and tricritical scaling has been illustrated in the attractor structure at several levels of its resolution (see Fig. 6). However, the tricritical scaling behavior at those points is destroyed inevitably at some level of the resolution. The larger parameter of dissipation B, the less number of levels at which the tricritical scaling is valid.

So, in the picture of scenarios of transition to chaos a number of phenomena exists that can not be extended from 1D maps onto 2D maps, and further, onto the differential equations.

5. Universality and scaling in presence of noise

As known, in early 80-th Crutchfield et al. have revealed the property of universality and scaling for Feigenbaum's period doubling scenario in presence of noise [19]. The appropriate version of the renormalization group analysis was developed, and a new universal constant was estimated, $\gamma(2)$=6.6190365: to observe one more level of the period doubling one has to decrease the noise magnitude by this factor. Due to the universality intrinsic to the critical behavior, the regularities are in a high degree insensitive in respect to correlation properties of noise and to details of the form of the distribution function.

Naturally, analogous problem concerning the effect of noise arises in respect of the critical behaviors relating to other classes of universality, which may appear in the multi-parameter analysis of the transition to chaos.

First, we may consider an effect of noise onto 1D maps with extrema of different degree χ: $x_{n+1}=1-\lambda|x_n|^\chi+\varepsilon\xi_n$, where ξ_n is a random sequence. The scaling property consists in the following: to observe each one new level of the fractal structure of the attractor at the critical point, it is necessary to decrease the noise amplitude ε by a definite factor γ that depends (in a universal manner) on the degree χ [20].

As noted, in two-parameter analysis of smooth 1D maps situations can be met, when the twice iterated map has an extremum of degree 4 and the period-doubling accumulation lead to the tricritical point. Analogously, in three-parameter analysis of 1D maps additional critical situations may appear, when a twofold iteration yields a map with extremum of the 6-th degree, or a threefold iteration gives rise to an extremum of the 8-th degree. These situations correspond to certain critical points of codimension three [4]. In all these cases the constants responsible for scaling properties in respect to noise differ from that of Crutchfield et alt. The respective numbers have been computed in Ref. [20] and summarized in the Table. Also charts of the Lyapunov exponent on the parameter plane of the noisy maps were plotted, and illustrations of the scaling properties in presence of noise were given.

Table

Universal constants responsible for scaling in respect to noise for different types of criticality in 1D maps

Type of criticality	χ	$\gamma(\chi)$
Feigenbaum	2	6.6190365
Tricritical	4	8.2439109
Type S («six power»)	6	10.037886
Type E («eight power»)	8	11.523865

Also we have revealed and illustrated in numerical experiments the scaling properties associated with the effect of noise at the so-called *bicritical point B*. This type of criticality occurs in a system of two unidirectionally coupled period-doubling subsystems as we bring both of them to the threshold of chaos by tuning their control parameters [21, 22]. The model equations are of the following form:

$$x_{n+1} = 1 - \lambda x_n^2 + \kappa\xi_n, \quad y_{n+1} = 1 - Ay_n^2 - Bx_n^2 + \varepsilon\eta_n, \tag{10}$$

where ξ_n and η_n are random sequences effecting the master and slave subsystems, respectively. An essential qualitative and quantitative difference was noted between the respond of the system to the noise added either into the first, or into the second subsystem. The universal constants responsible for the scaling in respect to the noise intensity are γ=6.619036 and ν=2.713695, respectively. (The first one, naturally, coincides with the constant of Crutchfield et al.) Fig.7 illustrates scaling regularities on the charts of the Lyapunov exponents.

6. Complex dynamics of nonlinear oscillators and catastrophe theory

Concept of the multi-parameter analysis allows formulation of a novel and original view onto a study of complex dynamics of nonlinear oscillators. Traditionally, researc-

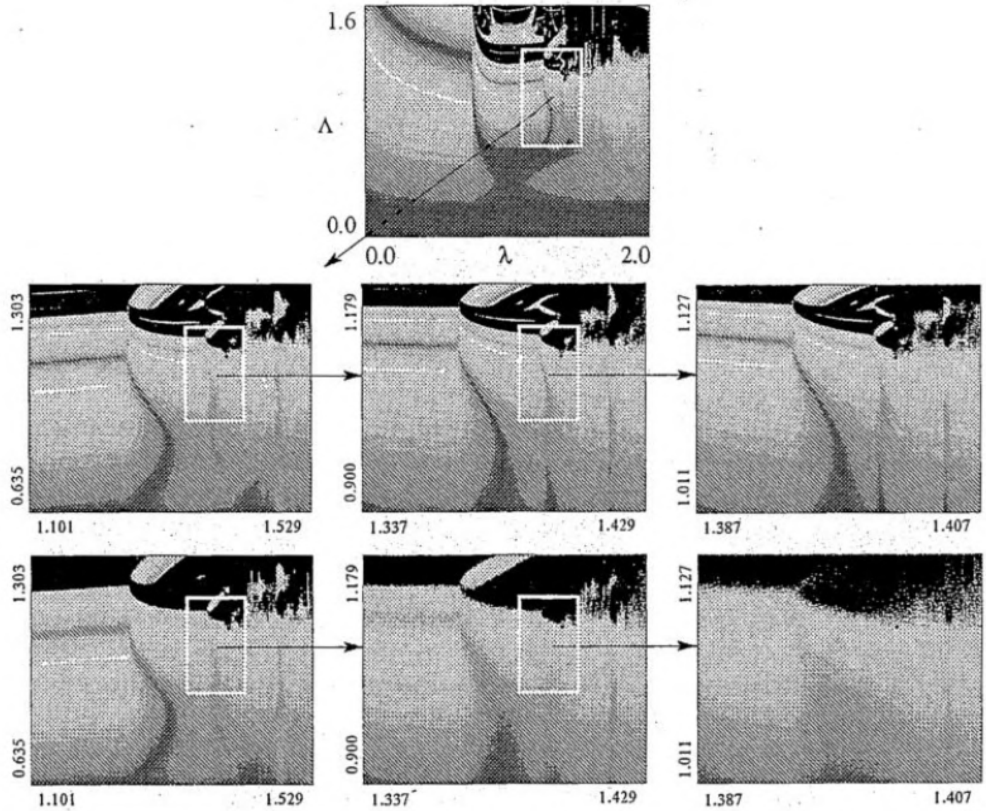

Fig. 7. Charts of the largest Lyapunov exponent in the parameter plane for two unidirectionally coupled logistic maps with noise (10) (the grey tones designate magnitude of the Lyapunov exponent). Values of noise amplitude are k=0, ε=0.005. Top row of the fragments are demonstrated scaling properties. We magnify fragments by 4.6692016 along λ and by 2.392724 along A in the vicinity of the bicritical point λ=1.4011552, A=1.0900943 and rescale value of noise by 2.713695. Lower row of fragments are demonstrated role of the noise in case if its amplitude is not rescale

hers fix a form of the potential relief for the oscillator and regard amplitude and frequency of the external force, or the dissipation coefficient, as variable control parameters. We suggest an alternative approach based on ideas of the catastrophe theory. Let us assume that a function that defines the potential relief is controlled by one, two, or more parameters. Classification of such functions is delivered by the catastrophe theory and, in particular, by the Rene Thom classification theorem. This approach gives an opportunity to formulate and study a sequence of situations of increasing codimension corresponding to potentials associated with canonical catastrophe theory forms: fold, cusp, swallow tail, etc. For example, a potential function $U(x) = -bx - (1/2)ax^2 - (1/3)x^3$ depending on two parameters a and b corresponds to the fold catastrophe. It generates a family of nonlinear oscillators which are governed (in presence of dissipation and of periodic external force) by the following equation:

$$\ddot{x} + \gamma\dot{x} + b + ax + x^3 = B\cos\omega t. \qquad (11)$$

The main object of attention is now an analysis of dynamics of the system in dependence on parameters a and b responsible for configuration of the potential relief. Concrete examples of nonlinear oscillators considered in literature may be regarded now as particular representatives in a frame of the suggested generalized scheme. A study of complex dynamics for forced dissipative oscillators with potential functions associated with different elementary catastrophes may be found in Ref. [23].

7. Non-invertible 2D maps

Mappings constructed in a course of the Poincare cross-section procedure appear to be invertible because the differential equations themselves allow continuation of solutions back and forth in time. Nevertheless, noninvertible 2D maps are of interest too. They may arise in a straightforward way in some physical problems [24]. The simplest example, however, is a pair of coupled logistic maps (coupled period-doubling systems):

$$x_{n+1} = 1 - \lambda x_n^2 - C y_n^2,$$
$$y_{n+1} = 1 - A y_n^2 - B x_n^2. \qquad (12)$$

Here λ and A are control parameters of two subsystems, C and D are coupling constants, and the coupling is supposed to be quadratic.

Fig. 8, *a* presents examples of charts of dynamical regimes on parameter plane (λ, A). One can see there bifurcation points of codimension 2, where the bifurcation lines of Andronov-Hopf (birth of quasiperiodic regimes) and those of period-doubling meet together. The sequence of the codimension 2 bifurcation points converge to somewhat new critical point called the *FQ critical point* [4]. (FQ stands for «Feigenbaum» and «Quasiperiodicity».) A magnified fragment of the chart (Fig. 8, *b*) demonstrates that the system is characterized by an unusual form of the synchronization tongues. More detailed study shows their nontrivial metamorphoses. For instance, Fig. 8, *c* and *d* present examples of tongues of ring-like form. Yet more fascinating picture can be obtained for coupled 2D maps, say Hénon maps.

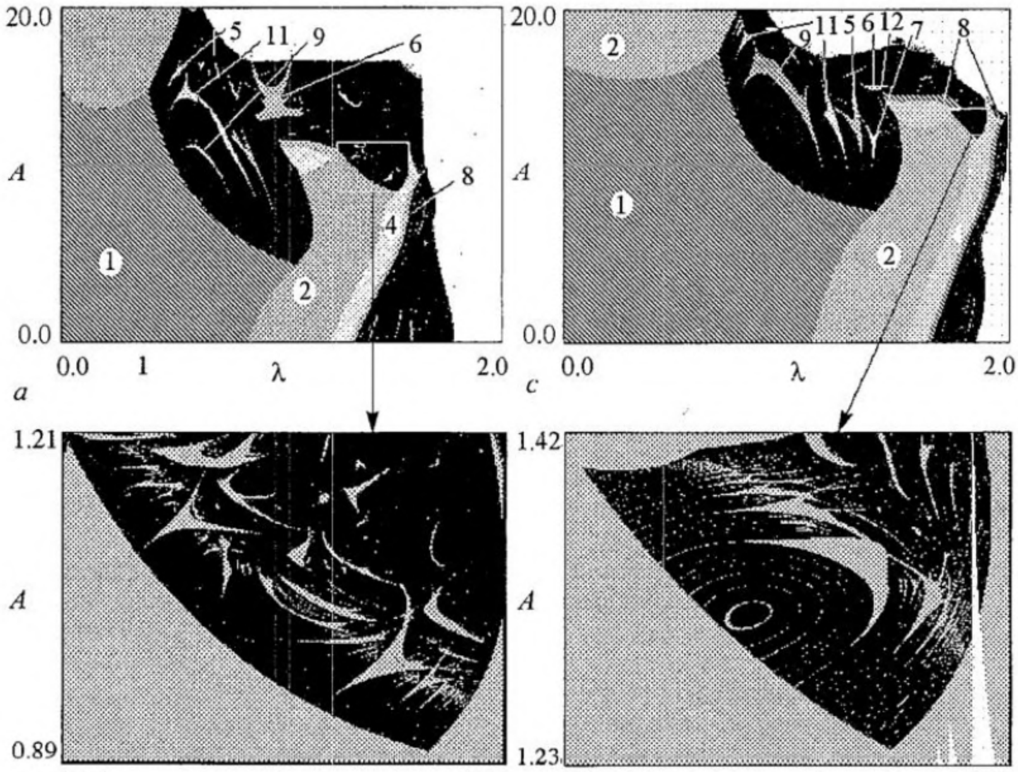

Fig. 8. Parameter plane of the system of two coupled logistic maps (12) (*a*) and its magnified fragment (*b*). Parameter plane of the coupled Henon maps (*c*) and its magnified fragment (*d*)

Interesting features of synchronization of noninvertible maps challenge us to elaborate a «universal» 2D map, which would demonstrate all basic bifurcations typical for two-dimensional phase space. To construct such a map, let us start with a note that a fixed point of any 2D map has a stability domain in the parameter plane of trace S and determinant J of the Jacobian matrix represented by a triangle with borders $\{J=1, S-J=1, S+J=-1\}$. Now, we construct the model to have S and J as natural parameters of the linearized part of the map, and add arbitrarily some quadratic nonlinearity:

$$x_{n+1} = Sx_n - y_n - (x_n^2 + y_n^2),$$
$$y_{n+1} = Jx_n - (1/5)(x_n^2 + y_n^2). \quad (13)$$

Chart of dynamical regimes for this «universal» map is shown in Fig. 9. One can see a set of synchronization tongues with sharp edges at the upper side of the triangle, representing the Andronov-Hopf bifurcation line. Observe that the arrangement of the tongues differ from that intrinsic, for example, to the circle map and to the ring map. In particular, bifurcations occur, for which two multipliers are equal to 1 and -1, respectively. These codimension-2 bifurcation points may accumulate to somewhat new critical point called *the critical points of C-type* [4].

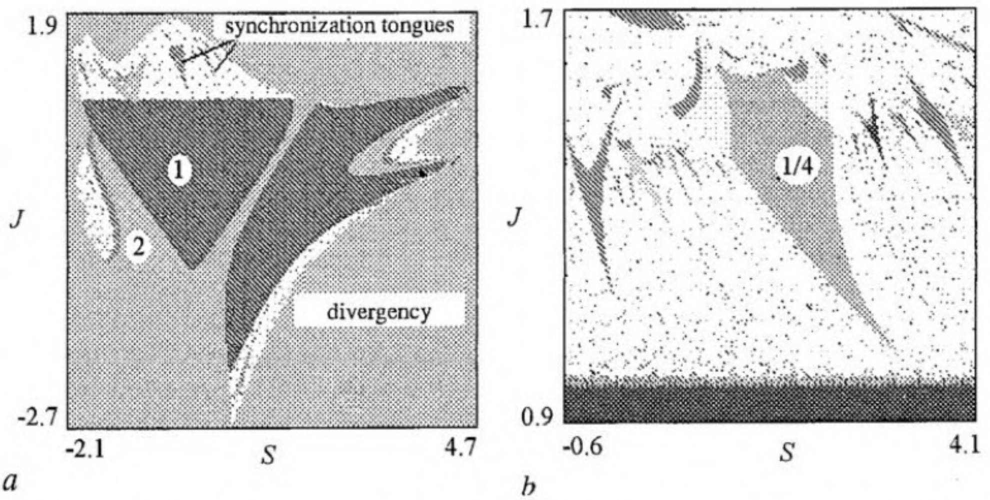

Fig. 9. Parameter plane S, J «universal» 2D map (13). One can see «triangle of stability», which is formed by lines of tangent bifurcation, period doubling bifurcation and Neimark bifurcation. Inside of synchronization tongues there are the points of C-type, which corresponded to accumulation of the points, in which lines of the period doubling are terminated in the bounds of synchronization tongue

8. Synchronization and bifurcations of cycles

The idea of multi-parameter study opens new possibilities in research of the phenomenon of synchronization. Indeed, let a non-autonomous system be characterized by one or several control parameters, and undergoes some bifurcations under their variation. Then, the metamorphoses of synchronization regimes should be studied naturally in a parameter space of dimension increased by 2. (We add amplitude A and frequency ω of the external force to a number of internal parameters of the system.) The simplest example is a situation of a unique control parameter of the autonomous system λ; let us suppose that its variation gives rise to a period-doubling bifurcation cascade. Then, in a cross-section of the parameter space by a plane (ω, r) the period-doubling bifurcation lines are terminated at edges of the synchronization tongues, these are

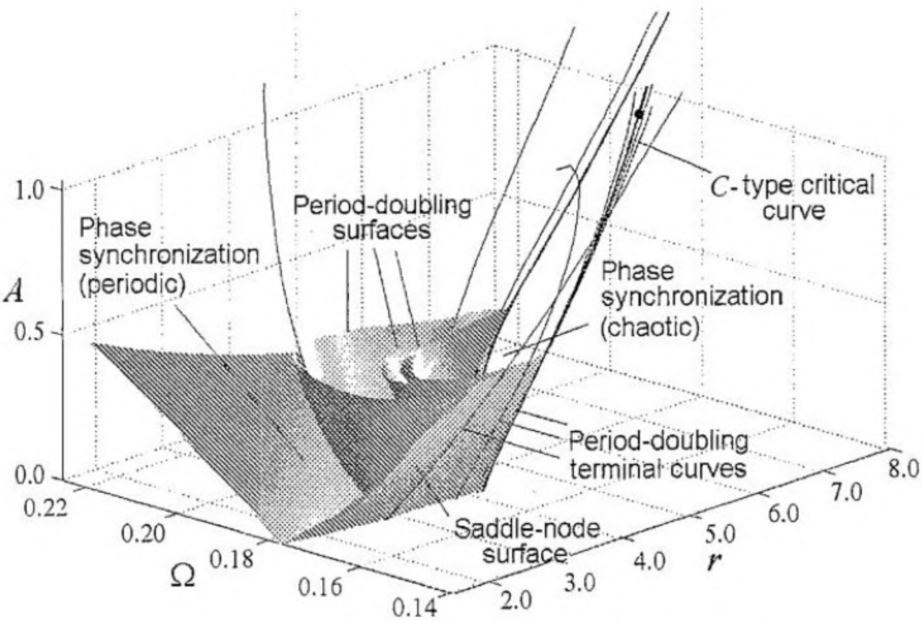

Fig. 10. Synchronization tongue for the forced Rössler system (14) and its interior arrangement in the parameter space; ω is frequency, A is amplitude of the external force, r is the parameter, controlled period doubling. One can see surfaces of tangent and period-doubling bifurcations. These surfaces are intersected along the lines, which are accumulated to critical lines of C-type as the number of period-doubling increase

codimension-2 bifurcation points. In 3-dimensional parameter space (ω,r,A) they correspond to codimension-2 bifurcation lines accumulated towards some critical line. The respective picture for the forced Rössler system [24]

$$\dot{x} = -y - z + A\sin 2\pi\Omega t, \quad \dot{y} = x + ay, \quad \dot{z} = b + z(x - r), \tag{14}$$

is reproduced in Fig. 10. The critical line obtained as a limit of codimension-2 bifurcation lines corresponds to the criticality of C-type [4] mentioned in the previous section. In Ref. [24] the respective scaling properties intrinsic to this criticality are revealed and discussed.

Next, let us consider a self-oscillator with hard excitation under external periodic force governed by an equation

$$\ddot{x} + (\lambda - x^2 + \alpha x^4)\dot{x} + x + \beta x^3 = b\sin\omega t. \tag{15}$$

Note that in autonomous case variation of a control parameter gives rise to bifurcation of collision of a stable and unstable limit cycles in this system.

Using a method of slow amplitudes one can derive the following reduced equation:

$$\dot{R} = -R + R^3 - kR^5 - \varepsilon\cos\varphi, \quad \dot{\varphi} = -\Delta + 3\beta R^2 + (\varepsilon/R)\sin\varphi. \tag{16}$$

Here ε is a dimensionless amplitude of the external force, Δ is a dimensionless deviation of the external frequency from the frequency of self-oscillations, parameter k controls mutual location of the stable and unstable limit cycles, $k=\gamma\lambda$. At $k=0.25$ they collide and disappear.

Bifurcation analysis in the parameter space of the model (15) reveals a fascinating picture. At small values of k there exist two synchronization tongues, one corresponds to a stable, and another to an unstable regime. With increase of k one observes their

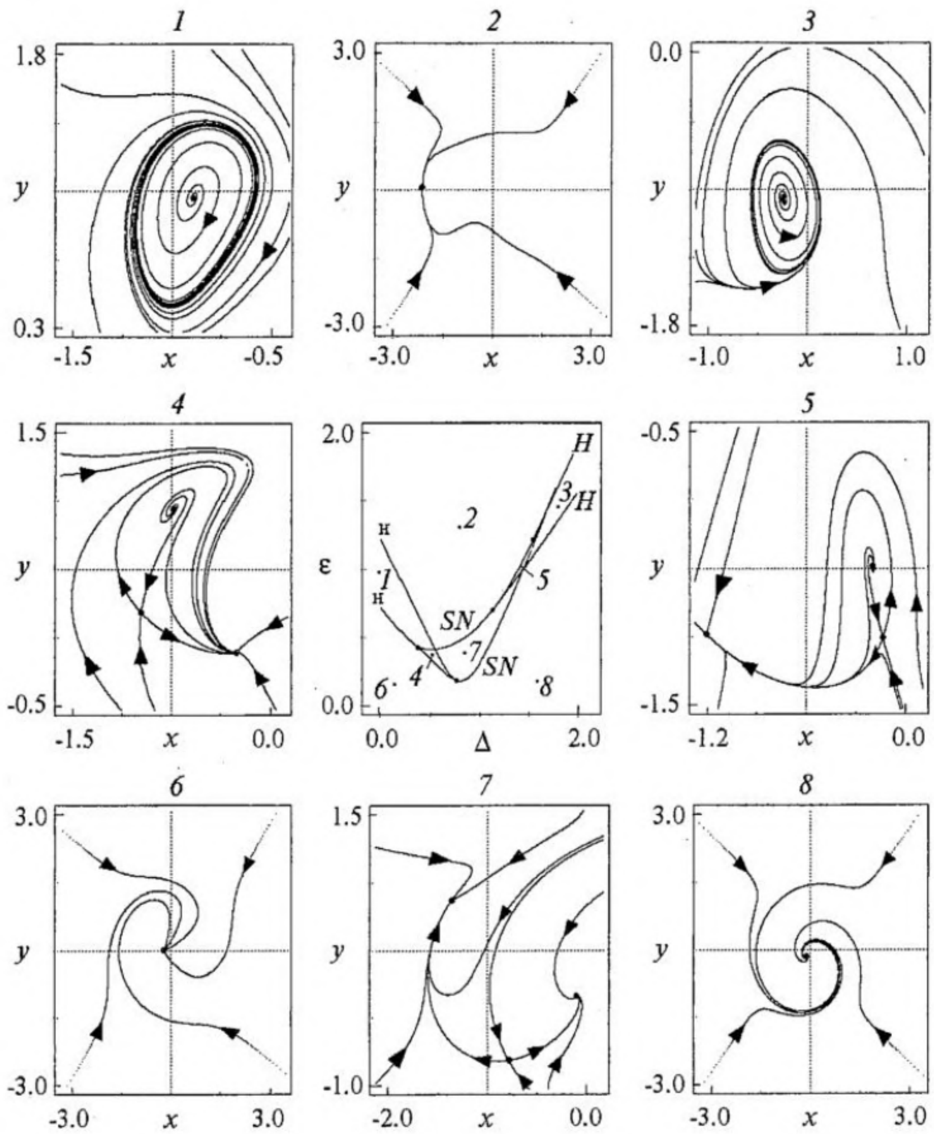

Fig. 11. Parameter plane (ε is a dimensionless amplitude and Δ is dimensionless deviation of the frequency) for system (15) after collision of the limit cycles in the autonomous system. k=0.29

unification via bifurcation situations of codimension 3. In particular, a known catastrophe of «swallow tail» [25] is observed. At the bifurcation of the autonomous system the synchronization tongue loses touch with the frequency axis and forms a singularity known in the catastrophe theory called «lips» [25]. In this situation the quasiperiodic regimes at small amplitudes of the external force disappear. The respective picture is shown on the central panel of Fig. 11. Also the Andronov-Hopf bifurcation lines are shown there, which have common points with the tongue edge (the Bogdanov-Takens points). It may be seen that above the bifurcation threshold of the autonomous system the quasiperiodic regimes are possible yet at large amplitudes of driving. They are observed in a region between the Andronov-Hopf bifurcation lines. Only at k=0.3 one more codimension-3 bifurcation occurs, when the pairs of the Andronov-Hopf bifurcation lines collide and disappear.

The authors thank S.P. Kuznetsov for discussions. The work was supported by CRDF and Ministry of Education of Russian Federation (grant REC-006), by Russian Foundation of Basic Research (grants 00-02-17509 and 02-02-06469) and Russian Foundation of assistance to science fo our country.

References

1. *A.P. Kuznetsov, S.P. Kuznetsov.* Critical dynamics of the lattices coupled map at the onset of chaos (review). Izvestiya VUZ. - Radiophysics. Vol. 34, 1991, № 10 - 12. P. 1079 - 1115.

2. *A.P. Kuznetsov, S.P. Kuznetsov.* Critical dynamics for 1D maps. Part I. Feigenbaum's scenario. Izvestiya VUZ. Applied Nonlinear Dynamics, 1993. Vol. 1, № 1. P. 15 - 32.

3. *A.P. Kuznetsov, S.P. Kuznetsov, I. R. Sataev.* Critical dynamics for 1D maps. Part II. Two-parameter transition to chaos. Izvestiya VUZ. Applied Nonlinear Dynamics, 1993. Vol. 1, № 3 - 4. P. 17 - 35.

4. *A.P. Kuznetsov, S.P. Kuznetsov, I.R. Sataev.* A variety of period-doubling universality classes in multi-parameter analysis of transition to chaos. Physica D. Vol. 109, 1997. P. 91 - 112.

5. *A.P. Kuznetsov, S. P. Kuznetsov, I. R. Sataev.* Codimension and Typicity in a Context of Description of Transition to Chaos via Period - Doubling in Dissipative Dynamical Systems. Regular and Chaotic Dynamic. 1997. Vol. 2, № 3 - 4. P. 90 - 105.

6. *Ikeda K., Daido H., Akimoto O.* Optical turbulence: chaotic behavior of transmitted light from a ring cavity. Phys. Rev. Lett. 1980. Vol. 45. P. 709

7. *A.P. Kuznetsov, L.V. Turukina.* Dynamical systems of different classes as models of the kicked nonlinear oscillator. Izvestiya VUZ. Applied Nonlinear Dynamics. Vol. 8, 2000, № 2. P. 31 - 42.

8. *A.P. Kuznetsov, L.V. Turukina, E. Mosekilde.* Dynamical systems of different classes as models of the kicked nonlinear oscillator. International Journal of Bifurcation and Chaos. Vol. 11, 2001, № 4. P. 1065 - 1078.

9. *Tufillaro N.B., Albano A.M.* Chaotic Dynamics of a Bouncing Ball. Am. J. Phys. Vol. 54, 1986, №. 10. P. 939 - 944.

10. *Moon F.* Chaotic vibrations. 1987, Wiley and Sons.

11. *Lieberman M., Lichtenberg A.* Stochastic and Adiabatic Behavior of Particles Accelerated by Periodic Forces. Phys. Rev. Vol. 5, 1972. P. 1852 - 1866.

12. *Guckenheimer J., Holmes P.* Nonlinear Oscillations, Dynamical Systems, and bifurcations of Vector Fields. 1997, Springer.

13. *A.P. Kuznetsov, A.P. Shirokov.* Comparative analysis of the approximate and precise mapping for «bouncing ball» . Izvestiya VUZ. Applied Nonlinear Dynamics. Vol. 8, 2000, № 5. P. 72 - 81.

14. *A.P. Kuznetsov, A.P. Shirokov.* Discrete model of a backward - wave tube. Izvestiya VUZ. Applied Nonlinear Dynamics, 1997. Vol. 5, №. 6. P. 76 - 84

15. *V.L. Bratman, A.V. Savilov.* Scenario of transition to the multi - frequency regime in the FEL - oscillator with a low-Q microwave system. Izvestiya VUZ. Applied Nonlinear Dynamics, 1994. Vol. 2, №. 6. P. 27 - 39.

16. *A.P. Kuznetsov, A.P. Shirokov.* Complex dynamics of two - mode low - dimension model of the free electron laser. Izvestiya VUZ. Applied Nonlinear Dynamics, 1999, № 1. P. 3 - 11.

17. *Kuznetsov A.P., Shirokov A.P.* Analysis of the scenarios of transition to chaos in the discrete two - mode model of free electron laser. Letters in the J. of Technical Physics, 1999. Vol. 25, №. 12. P. 17 - 21.

18. *A.P. Kuznetsov, S.P. Kuznetsov, L.V. Turukina, E. Mosekilde.* Two-parameter analysis of the scaling behavior at the onset of chaos: Tricritical and pseudo-tricritical points. Physica A300, № 3 - 4, 2001. P. 367 - 385.

19. *Crutchfield J.P., Farmer J.D., Huberman B.A.* Fluctuations and simple chaotic dynamics. Phys. Rep., 1982. Vol. 92 №. 2. P. 45 - 82

20. *A.P. Kuznetsov, J.V. Kapustina.* Scaling properties at transition to chaos in model maps in the presence of noise. Izvestiya VUZ. Applied Nonlinear Dynamics. Vol. 8, 2000, № 6. P.78 - 87.

21. *J.V. Gulyaev, J.V. Kapustina, A.P. Kuznetsov, S.P. Kuznetsov.* On scaling properties in unidirectionally coupled period-doubling systems in the presence of noise. Letters in the J. of Technical Physics, 2001. Vol. 27, № 22. P. 58 - 65.

22. *Julia V. Kapustina, Alexandr P. Kuznetsov, Sergey P. Kuznetsov, and Erik Mosekilde.* Scaling properties of bicritical dynamics in unidirectionally coupled period-doubling systems in the presence of noise. Physical Review E. Vol. 64, № 6, 2001, 066207 (12 pages).

23. *A.P. Kuznetsov, A.Yu. Potapova.* Features of the complex dynamics of the nonlinear oscillations with Thom's catastrophes. Izvestiya VUZ. Applied Nonlinear Dynamics. Vol. 8, 2000, № 6. P. 94 - 120.

24. *S.P. Kuznetsov, I.R. Sataev.* Universality and scaling for the breakup of phase synchronization at the onset of chaos in a periodically driven Rössler oscillator. Phys.Rev.E. Vol. 64, №. 4, 2001, 046214 (7 pages).

25. *Arnold V. I.* Catastrophe theory. Moscow: Science, 1990. 128 p.

Saratov State University
Institute of Radio-Engineering and
Electronics of RAS, Saratov Branch

Received 18.05.2002

УДК 517.9

МНОГОПАРАМЕТРИЧЕСКАЯ КАРТИНА ПЕРЕХОДА К ХАОСУ

А.П. Кузнецов, Л.В.Тюрюкина, А.В. Савин, И.Р. Сатаев,
Ю.В. Седова, С.В. Милованов

В работе представлен ряд направлений исследований сложной динамики нелинейных систем, связанных с многопараметрическим исследованием. В частности, обсуждаются примеры реалистичных моделей многопараметрических систем, критические явления на пороге хаоса, сопоставление свойств дифференциальных систем и отображений и др.

Alexander P. Kuznetsov was born in 1957. He is a Doctor of Sciences, Head Researcher of the Laboratory of Theoretical Nonlinear Dynamics of Saratov Branch of IRE RAS, Head of the Chair of Dynamical Systems of Nonlinear Processes Department of Saratov State University, Professor of Nonlinear Processes Department. A.P.Kuznetsov specializes in nonlinear dynamics, dynamical chaos and critical phenomena theory. He is Soros Professor (2000, 2001), scientific leader of the student's laboratory «Theoretical Nonlinear Dynamics» at SSU and SB of IRE RAS, member of Editorial board of the Journal «Empire of Mathematics». Sphere of his scientific interest includes applications of bifurcation and catastrophe theory, development of the concept of the multi-parameter criticality which is a generalization of the idea of scenario of transition to chaos onto multi-parameter nonlinear systems, investigation of features of signals with hierarchical organization («the fractal signals»). A.P.Kuznetsov is an author of more than 100 published works in Russian and International scientific press, an author of several teaching courses for the Nonlinear Processes Department of SSU and five popular scientific books and textbooks.

Ludmila V. Turukina was born in 1977. She is a Postgraduate student of the Nonlinear Processes Department of Saratov State University, a Laureate of Fellowship of RF President for students and post-graduate students (1999-2000, 2001-2002), Soros Student (1998, 1999, 2000), Soros Graduate Student (2001). Her scientific interest includes study of dynamical chaos, critical phenomena at the transition to chaos, comparative analysis of dynamical systems description with the help of mathematical models of different classes. She is an author of 17 publications in Russian and International scientific press. L.V. Turukina took part in 23 scientific conferences, including 10 international conferences. She made several working visits to the Technical University of Denmark and to the Potsdam University (Germany).

Alexey V. Savin was born in 1980 in Saratov, graduated from the Physical Department of Saratov State University. Now A. Savin is post-graduate student. His main scientific interests are the numerical investigations of model of nonlinear dynamics systems, particularly, coupled systems. A. Savin participated in 4 international conferences and has one publication.

Igor R. Sataev was born in 1959, graduated from the Moscow Physical-Technical Institute (1982). He is a Candidate of Sciences (PhD), Senior scientific researcher at the SB of IRE RAS. His basic scientific interests include numerical methods for solving renormalization group equations, numerical simulations of dynamics at the chaos threshold, critical phenomena in nonlinear systems. Sataev I.R. is an author of more than 30 publications in Russian and International scientific press.

Julia V. Sedova (Kapustina) was born in 1979. She is a Postgraduate student of the Nonlinear Processes Department of Saratov State University. Her basic scientific interest includes a noise influence on dynamical systems. She is a Soros Student (1999, 2000, 2001), the winner of the nominal prize fellowship of Saratov administration (1999-2000, 2000-2001 academic years). She made working visit to the Potsdam University, Germany (2002). She is an author of 4 scientific publications in Russian and International scientific press.

Sergey V. Milovanov was born in 1980. He is a fifth-year student of the Nonlinear Processes Department of Saratov State University. His basic scientific interests connect with topological methods in nonlinear dynamics, complex dynamics of non-autonomous systems and two-parameter maps. S. Milovanov took part in 10 scientific conferences, including 2 international conferences. Soros Student (2000, 2001).