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CLUSTER AND GLOBAL SYNCHRONIZATION IN A QUASI-HARMONIC SELF-OSCILLATORY CHAIN IN THE PRESENCE OF NOISE

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We study numerically effects of noise on synchronization phenomena in a chain of Van der Pol oscillators. A structure of frequency clusters in the non-homogeneous noisy chain is analyzed. We generalize the notion of effective synchronization to the case of a spatially extended system. The effect of amplitude relations on the phase dynamics is also explored. The possibility of realizing external synchronization of the homogeneous chain was considered. We clear up a role of two components of coupling (diffusive and one-direct coupling) and a noise sources in relation to the global synchronization.

1. Introduction

The phenomenon of synchronization plays an important role in the behavior of ensembles of interacting nonlinear oscillators. This effect provides the basis for self-organization of ensembles' dynamics and is associated with a variety of phenomena, such as multistability, growth restriction of the Kolmogorov entropy and attractor dimension, spatio-temporal structure formation, etc. The theory of synchronization, originally proposed for quasi-harmonic oscillations [1-4], was generalized to a wide range of systems including chaotic [5-11] and stochastic [11-15] ones.

Phase synchronization in ensembles of locally and globally coupled interacting periodic oscillators has been studied for a long time but these investigations still attract a growing interest of many researchers [4, 16-29]. Ensembles of periodic oscillators have found wide applications in mathematical modeling of physical [30-33], chemical [4, 16], and biological [34-38] processes.

Even the simplest quasi-harmonic oscillators coupled in a large ensemble generate a lot of complicated nonlinear effects such as a phase and frequency synchronization [20, 23, 24, 28, 29, 35, 40], an oscillatory death [21, 22, 27, 29, 40, 41], frozen states [28], formation of a collective chaotic behavior [27, 33, 39] e.t.c. All these effects are the manifestations of the phase - frequency synchronization phenomena.

It is known that fluctuations are inevitably present in real ensembles and a parameter mismatch (random or definitely specified) of partial systems also takes place. Effects of noise and parameter mismatch on phase locking in an ensemble of oscillators are considered in [4, 17, 18, 20-22, 24, 25, 29, 31, 40, 42, 43]. The presence of a linear gradient of native unperturbed frequencies along the medium consisting of locally coupled oscillators leads to the formation of so-called frequency clusters of synchronization [20, 29, 40].

Recently, numerous works have appeared devoted to the study of ensembles of chaotic oscillators [30, 44-52]. It has been shown that the synchronization effect also plays an important role in the dynamics of chaotic ensembles. They demonstrate a number of phenomena which appear to be quite similar to those occurred in ensembles of periodic oscillators. Particularly, effects of phase-locking and cluster phase synchronization have been found in ensembles of chaotic oscillators [50-52]. This fact testifies that the effect of synchronization is generic for a variety of oscillatory systems.

However, even in the case of quasi-harmonic oscillatory ensembles it remains a number of unresolved problems, which devote a special attention. One part of these problems concerns the cluster synchronization in a chain of non-identical oscillators. How much can cluster synchronization be stable to the influence of fluctuations? Is it possible to generalize the notion of effective synchronization of self-sustained oscillations in the presence of noise [54, 55] to spatially extended systems? It is interesting to elucidate how significant may it be if a variation of instantaneous amplitude values of oscillators is taken into consideration? Can the behavior of an ensemble be qualitatively described by the phase equations only? The other part of the problems is connected with the global external synchronization of a chain (i.e. the synchronization of all oscillators of the chain at the external frequency force). What are the conditions of global synchronization of the chain by a harmonic external force applied to the first oscillator? How does the type of coupling between oscillators influence on the global synchronization effect? Is the global synchronization possible in the presence of noise?

Some of these problems were considered in [56, 57]. In the present work we try to answer the above stated questions and this is the main objective of this paper.

The paper is organized as follows. In Sec. I we study the effects of cluster synchronization in a chain of non-identical Van der Pol oscillators with diffusive coupling. The effect of noise on clusters structure is analyzed. In Sec. II we explore the peculiarities of the behavior of the chain of diffusively coupled non-identical Van der Pol oscillators described by the phase equations only. Sec.III is devoted to the external synchronization of a chain of identical Van der Pol oscillators with a harmonic force applied to the first element of the chain. The role of two coupling components (diffusive and one-direct) is discussed. The global synchronization of the chain in the presence of noise is studied. And finally, we give our conclusions in Sec. IV.

2. Effect of noise on cluster synchronization in a chain of non-identical Van der Pol oscillators

The model to study is a chain of Van der Pol oscillators, being similar to that considered in [29, 40] and including additive noise on the chain elements. The chain is described by a system of equations which, in a truncated form, are as follows:

$$\dot{\rho}_j = r(1 - \rho_j^2)\rho_j + g(\rho_{j-1}\cos(\phi_j - \phi_{j-1}) + \rho_{j+1}\cos(\phi_{j+1} - \phi_j) - 2\rho_j) + D/\rho_j + (2D)^{1/2}\xi_j(t), \tag{1}$$

$$\dot{\phi}_j = \omega_j + g(\rho_{j+1}/\rho_j \sin(\phi_{j+1} - \phi_j) - \rho_{j-1}/\rho_j \sin(\phi_j - \phi_{j-1})) + (2D)^{1/2}\eta_j(t)/\rho_j,$$

$$j = 1, 2, 3, \ldots, m,$$

where j is the number of an oscillator, representing a discrete spatial coordinate, ρ_j and ϕ_j are the amplitude and the phase of oscillations of the jth oscillator, respectively. $\xi_j(t)$ and

$\eta_j(t)$ are assumed to be identical uncorrelated Gaussian white noise sources with zero means and with the same intensity D^*.

The boundary conditions were chosen to correspond to a free-ended chain, i.e., $\rho_0=\rho_1$, $\phi_0=\phi_1$, $\rho_{m+1}=\rho_m$, $\phi_{m+1}=\phi_m$. The initial conditions for the oscillators are chosen to be close to homogeneous ones with a small random dispersion within $\delta=0.1$.

The model (1) has the following parameters: r is the excitation parameter (in computation, we fix $r=0.5$), ω_j is the unperturbed frequency of the jth oscillator, i.e., oscillation frequency without coupling and external forcing, g is the parameter of diffusive coupling of nearest-neighbor oscillators. For the model (1) we suppose a case of linear dependences of the unperturbed frequencies on spatial coordinate j, i.e., $\omega_j=\omega_1+(j-1)\Delta$, where Δ is the frequency mismatch of two neighboring oscillators. The peculiarities of the chain dynamics do not depend on the choice of the frequency origin. Therefore, we can set $\omega_1=1$.

We study numerically the chain (1) with $m=100$ elements using a fourth-order Runge-Kutta routine. In the course of numerical experiments, we analyze the dynamics of each element, estimate the variation of phases ϕ_j during a large enough time T and compute the average (perturbed) frequencies $\tilde{\omega}_j$ of the partial oscillators:

$$\tilde{\omega}_j = \langle \dot{\phi}_j(t) \rangle = \lim_{T\to\infty} [(\phi_j(t_0+T) - \phi_j(t_0))/T]. \qquad (2)$$

The angle brackets mean time averaging.

Equality of the mean frequencies $\tilde{\omega}_j$ and $\tilde{\omega}_{j+1}$ corresponds to the limitation of the phase difference of the oscillators $\theta_j=\phi_{j+1}(t)-\phi_j(t)$ in time:

$$\lim_{t\to\infty} |\theta_j(t)| < M, \quad \text{where } 0 \neq M \neq \infty. \qquad (3)$$

The condition (3) is the generalization of the phase locking definition [9]. So defined phase locking notion may be applied not only to harmonic oscillations but also to nonperiodic selfsustained oscillations. The group of oscillators ($j=k_1,\dots,k_2$) synchronized in the sense (3) is named as a frequency cluster. The instant values of the frequencies $\dot{\phi}_j(t)$ of partial oscillators belonging to the same cluster may be different but their time averaged values $\tilde{\omega}_j$ must be equal.

For the chain (1) with a linear frequency gradient along the spatial coordinate j, one can observe frequency cluster formation in a certain range of coupling parameter g values [29, 40]. The partial oscillators exhibit the quasi-periodic oscillations $x_j(t)=\rho_j(t)\cos\phi_j(t)$ and $y_j(t)=\rho_j(t)\sin\phi_j(t)$, and the number of independent frequencies is determined by the number of synchronized clusters. Fig. 1, *a*, *b* illustrate (x_j,y_j) projections of oscillations in the regime of cluster synchronization, which are characteristic for the center and the boundary of a cluster, respectively. If we consider the oscillators within the same cluster, then a representative point rotates around the origin, $x_j=0$, $y_j=0$, on the average, with the same frequency and a bounded phase shift. Oscillators belonging to different clusters have distinct rotation frequencies. Consequently, the form of phase projections (x_j,x_k) is qualitatively different when jth and kth oscillators belong to one cluster (Fig. 1, *c*) and to different clusters (Fig. 1, *d*).

Now we are going to elucidate how the noise influences the cluster synchronization. We fix $\Delta=0.002$ and compute the distribution of perturbed frequencies

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* In fact, in numerical experiments the same pseudo-random number generator was used having a Gaussian distribution. Successive values produced by the generator may be treated as practically independent. To make sure that the noise disturbances are uncorrelated, the noise source added to each subsequent element of the chain was shifted with respect to the previous one by five iterations of the pseudo-random number generator.

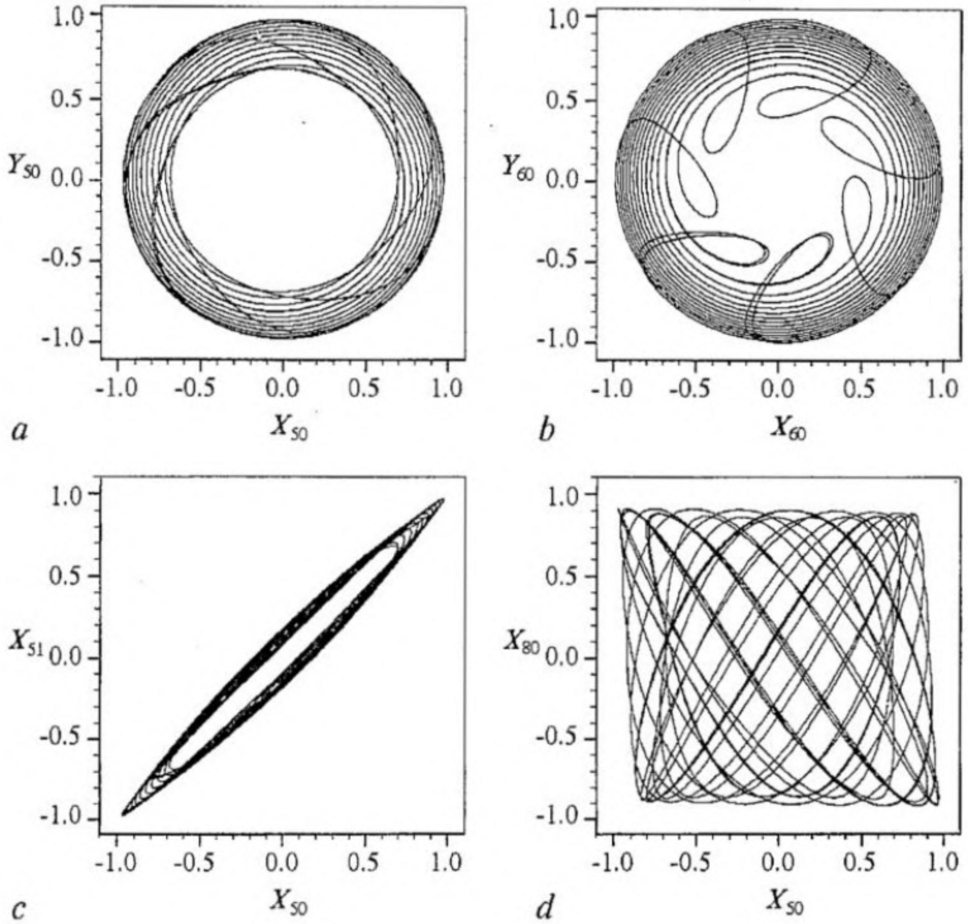

Fig. 1. Phase projections of oscillations of partial oscillators in the regime of cluster synchronization for Δ=0.002 and g=3.8

ϖ_j of oscillators along the chain without and in the presence of noise. The calculation results, shown in Fig. 2, *a* (I), *b* (I) for two different values of the coupling parameter g, clearly demonstrate the effect of cluster synchronization in the noise-free chain (D=0) and completely correspond to the analogous results presented in [29, 40].

Now consider the case when all oscillators are subjected to noisy perturbations. Fig. 2, *a* (II), *b* (II) and *a* (III), *b* (III) presents the distributions of the perturbed frequencies for two different noise intensities D=0.00001 and D=0.001, respectively. It is clearly seen that for both coupling parameters, the clusters of synchronization are destroyed as the noise intensity increases. If the noise is weak, the clusters' boundaries are only smoothing slightly (graphs II). Both smoothing and gradual destruction of the clusters begin with the chain center. For sufficiently large noise (graphs III), all middle clusters are completely destroyed. However, our computations have shown that the first and the last clusters appear to be highly stable to noisy disturbances and only a very strong noise is needed to destroy them.

The effect of noise on cluster synchronization can be more clearly understood by considering how the phase differences $\theta_j(t)$ of neighboring oscillators, located near the clusters boundary, change with time without and in the presence of noise. Without noise,

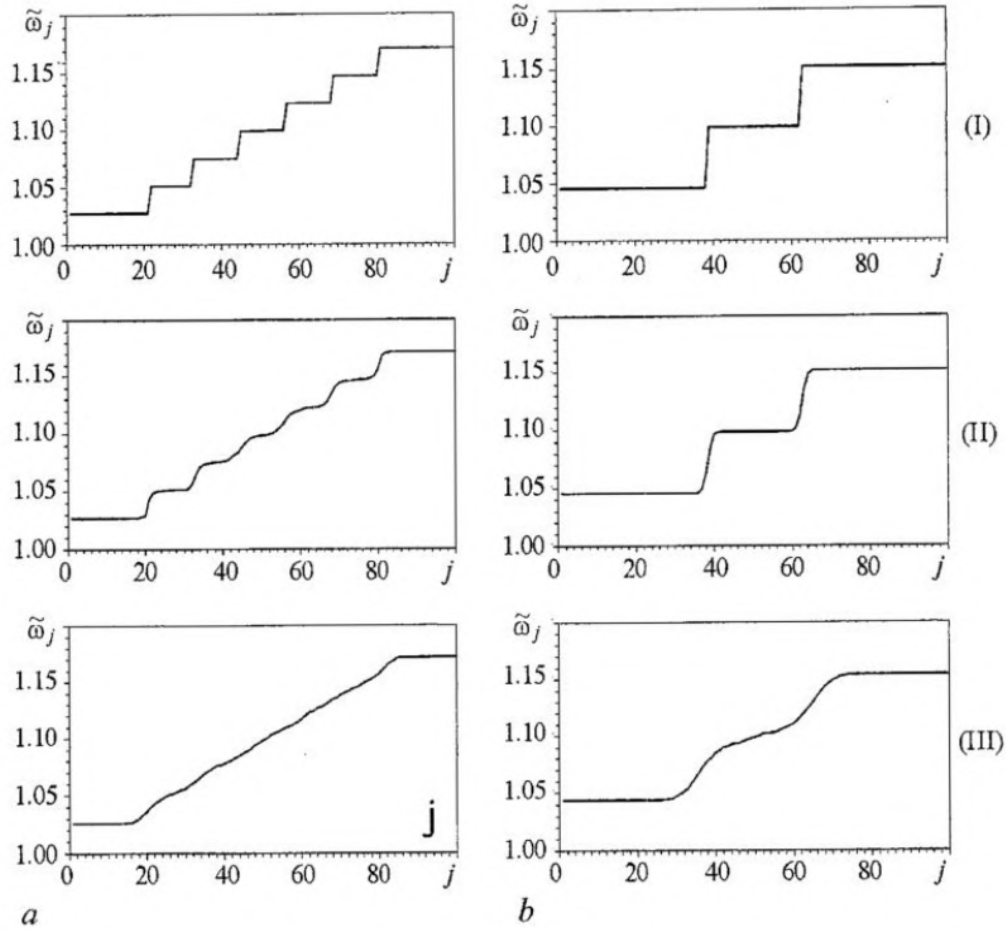

Fig. 2. Distributions of the perturbed oscillator frequencies for Δ=0.002 and for different strengths of coupling: (a) g=0.55; (b) g=3.8. Dependences (I), (II) and (III) are obtained for the chain in the presence of noise with intensity D=0, D=0.00001, and D=0.001, respectively

the phase difference of oscillators belonging to different clusters increases, on the average, linearly as the time goes on. At the same time, the phase difference remains strictly bounded if the oscillators considered belong to the same cluster. When the noise is added, the phase difference of any neighboring oscillators grows indefinitely with time but this growth is linear for none of j. The average growth rate of phase difference is different for different j. This fact allows one to find certain segments of the chain, for which this rate is low. Hence, we can identify clusters of effective synchronization in the presence of noise [55].

The clusters' boundaries in the presence of noise can be estimated by using the effective diffusion coefficient D_{eff} of the phase difference of neighboring oscillators [54]. D_{eff} defines the average rate with which the variance $\sigma_{\theta_j}^2(t)$ of phase difference θ_j increases in time. Its mean value can be calculated as follows:

$$D_{\text{eff}}(j) = \lim_{t\to\infty} \tfrac{1}{2}\left((\sigma^2_{\theta_j}(t) - \sigma^2_{\theta_j}(t_0))/(t - t_0)\right),$$
$$\sigma^2_{\theta_j}(t) = \langle\theta_j^2(t)\rangle - \langle\theta_j(t)\rangle^2. \tag{4}$$

We compute the effective diffusion coefficient versus the spatial coordinate within one cluster ($39 \le j \le 62$) for three different values of noise intensity D. Numerical results are presented in Fig. 3. They testify a gradual destruction of the clusters boundaries as the noise intensity increases. One can note that the dependence $D_{eff}(j)$ is quite similar (taking into account that j is a discrete variable) to a well-known dependence of the diffusion coefficient of the phase difference between a self-sustained system and an external forcing versus detuning. The clusters boundaries of effective synchronization can be defined by specifying some tolerable level of the diffusion coefficient D_{eff}^{max}. In this case oscillators for which $D_{eff} \leq D_{eff}^{max}$ can be considered as belonging to the same cluster. Such a determination of clusters boundaries is enough arbitrary since the value of D_{eff}^{max} can be given in different ways depending on a particular task. However, in any case the length of a cluster decreases with increasing noise intensity. For example, given D_{eff}^{max}=0.001, the boundaries of the cluster shown in Fig. 3 for D=0.0001 correspond to the 44th and the 56th oscillators.

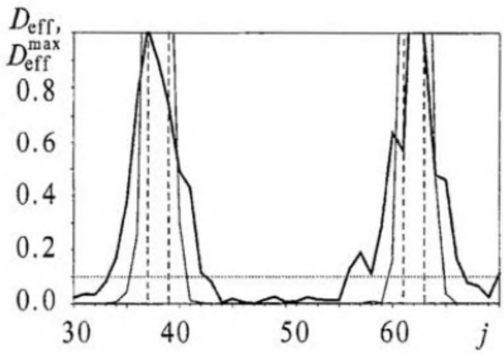

Fig. 3. Effective diffusion coefficient D_{eff} as a function of spatial coordinate j for $D=10^{-8}$ (*thin dashed line*), $D=10^{-5}$ (*thin solid line*), and $D=10^{-4}$ (*thick solid line*). The horizontal dotted line marks the level of D_{eff}^{max}, defining the clusters boundaries. The detuning and the coupling strength are Δ=0.002 and g=3.8

3. Phase dynamics approach

In the previous section we have numerically studied the chain of Van der Pol oscillators, which is described by the system of truncated equations (1) where amplitude and phase dynamics are combined. However, in many cases only phase equations are often used assuming amplitudes to be equal and constant in time. Such an approach allows one to qualitatively describe effects of frequency and phase locking and to simplify numerical simulation. Besides, in some cases the problem can be solved analytically using the phase equations only [4, 16, 20, 24, 25, 27]. Nevertheless, the dynamics of an ensemble may be distorted and some effects may be lost such as, for example, «oscillator death» [21, 22, 29, 53], if the amplitude dynamics is excluded from consideration. In particular, as emphasized in [29, 40], amplitude effects may influence the cluster structure formation. To reveal such an effect, we analyze first cluster synchronization in the enforced chain described by the phase equations only and then compare it with relevant results obtained for the full system of truncated equations (1). The system of phase equations can easily be derived from (1) by setting ρ_j=1 for any j. This means that the amplitudes of all oscillators are taken to be equal to their unperturbed value. The system of phase equations reads:

$$\dot{\phi}_j = \omega_1 + (j-1)\,\Delta + g(\sin(\phi_{j+1} - \phi_j) - \sin(\phi_j - \phi_{j-1})) + (2D)^{1/2}\eta_j(t), \quad j = 1,2,\ldots,m. \qquad (5)$$

The boundary conditions corresponding to free ends are: $\phi_0=\phi_1$, $\phi_{m+1}=\phi_m$. The detuning is fixed as Δ=0.002. The frequency distributions calculated from (5) are shown in Fig. 4 for different strengths of coupling. The first three plots correspond to the noise-free case. In Fig. 4, *a* illustrating the frequency distribution for g=0.55 only two clusters can be observed being formed at the boundaries of the chain. The analogous distribution, presented in Fig. 1, *a* (I) for the full system (1), reflects a more rich synchronization picture. With increasing strength of the coupling the middle clusters also appear (Fig. 4, *b*, *c*) but their structure is somewhat different from that formed when integrating the system (1). As seen from Fig. 4, *b*, *c*, the extreme clusters are extended, while the middle

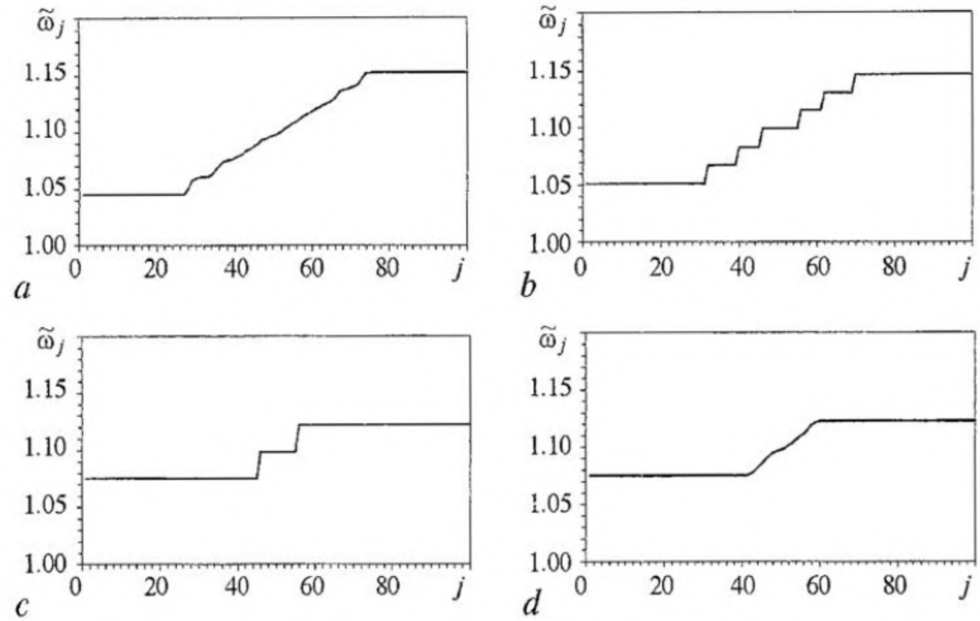

Fig. 4. Distributions of the perturbed frequencies in the chain, described by the phase equations (5), for Δ=0.002 and for different strengths of the coupling: *a*) g=0.55; *b*) g=0.7; *c*) g=1.5 without noise, and *d*) g=1.5 in the presence of noise with intensity D=0.00001

ones become more shorter. The height of clusters' steps, i.e., the difference between the frequencies of neighboring clusters, is less than that for system (1) and decreases rapidly as the strength of coupling increases. Thus, the region of cluster synchronization significantly shrinks when only phase dynamics is taken into consideration. Moreover, in this case the cluster structure appears to be more sensitive to noise perturbations. This is illustrated in Fig. 4, *d* when a weak noise of intensity D=0.00001 is added to the system (5). As follows from the figure, the noise causes the middle clusters to be destroyed.

4. External synchronization of a chain of identical Van der Pol oscillators

To study effects of external synchronization in a chain of quasi-harmonic self-sustained oscillators we use the following model:

$$\begin{aligned}
\dot{\rho}_j &= 0.5(1-\rho_j^2)\,\rho_j + g_1(\rho_{j-1}\cos(\phi_j-\phi_{j-1}) + \rho_{j+1}\cos(\phi_{j+1}-\phi_j) - 2\rho_j) + \\
&+ g_2(\rho_{j-1}\cos(\phi_j-\phi_{j-1}) - \rho_j) + D/\rho_j + F_j(t), \\
\dot{\phi}_j &= \omega_j + (g_1/\rho_j)\,(\rho_{j+1}\sin(\phi_{j+1}-\phi_j) - \rho_{j-1}\sin(\phi_j-\phi_{j-1})) - \\
&- g_2(\rho_{j-1}/\rho_j)\sin(\varphi_j - \varphi_{j-1}) + P_j(t),\ j = 1,2,3...m.
\end{aligned} \tag{6}$$

Here the same signs are used as in the model (1). The functions $F_j(t)$ and $P_j(t)$, j=1,2,…,m, describe forces which are applied to the oscillators of the chain. These forces include independent sources of Gaussian δ-correlated noise $\xi_j(t)$ and $\eta_j(t)$ for all oscillators of the chain, and also the harmonic force applied only to the first oscillator. So, $F_j(t)$ and $P_j(t)$ are

$$F_1(t) = C\sin(\omega_{ex}t - \phi_1) + (2D)^{1/2}\xi_1(t),$$
$$P_1(t) = (-C/\rho_1)\cos(\omega_{ex}t - \phi_1) + (2D)^{1/2}\eta_1(t)/\rho_1,$$
$$F_j(t) = (2D)^{1/2}\xi_j(t), \quad P_j(t) = (2D)^{1/2}\eta_j(t)/\rho_j, \quad j = 2,3\ldots m,$$

where D is the intensity of noise sources, C and ω_{ex} are the amplitude and the frequency of the external force, respectively. The boundary conditions are chosen as: $\rho_0=\rho_1$, $\phi_0=\phi_1$, $\rho_{m+1}=\rho_m$, $\phi_{m+1}=\phi_m$. The initial conditions for the oscillators are chosen to be close to homogeneous ones ($\rho_j(0)=0.5$, $\phi_j(0)=0.1$) with a small random dispersion within $\delta=0.1$.

The parameters of the chain are the unperturbed frequencies of partial oscillators ω_j, the external force parameters (amplitude C and frequency ω_{ex}), the coupling parameters g_1 and g_2, and also the noise intensity D. In the model (6) two types of coupling are used: diffusive coupling and one-direction coupling. So, g_1 is the diffusion coefficient, and g_2 characterizes a propagation of perturbations along the chain.

Let us consider first the effects of external synchronization in the homogeneous chain (6) with fixed length $m=100$ without noise sources ($D=0$). The homogeneous chain consists of identical elements. So, we suppose that $\omega_j=\omega_0=1$, $j=1,2,\ldots m$. The value $\Delta_{ex}=\omega_{ex}-\omega_0$ determines the mismatch of the excitation frequency from the unperturbed frequencies of oscillators. We also introduce the phase differences $\theta_j(t)=\phi_{j+1}(t)-\phi_j(t)$, $\theta_{ex}(t)=\omega_{ex}t-\phi_1(t)$. A perfect phase locking of oscillators on the main tone is given by the following conditions:

$$\dot{\theta}_j \equiv 0, \quad \dot{\theta}_{ex} \equiv 0, \quad \dot{\rho}_j \equiv 0, \quad j = 1,2,\ldots m. \tag{7}$$

In this case the phase differences are constant, and the oscillations are harmonic with the period equal to the period of the external force. According to a more general definition of the phase locking [9] only the phase differences limitation (3) are needed.

The synchronization of the j-th oscillator on the external frequency was detected numerically by the condition

$$|p_j - 1| \leq 10^{-4}, \tag{8}$$

where $p_j=\tilde{\omega}_j/\omega_{ex}$ is the relative mean frequency (winding number) of the j-th oscillator. We consider the validity of condition (8) for all oscillators of the chain as a numerical criterium of a global synchronization of a chain (i.e. synchronization of all chain elements). In the case of global synchronization the condition (7) must also be valid for all oscillators. If only a part of oscillators is synchronized, the oscillations are not perfectly periodic (as in the case of frequency clusters exist). In this situation the synchronization in the sense of (8) does not correspond to (7).

We will study a region of the global synchronization as the parameters Δ_{ex}, C, g_1, g_2 are varied. Disregarding perturbations of the partial amplitudes ρ_j it is possible to estimate the region of global synchronization. Supposing in (6) that $\rho_j=1$, $j=1,2,\ldots m$ we obtain from (7) the approximate synchronization conditions:

$$C \leq |\Delta_{ex}|\,(g_1+g_2)\,/\,g_2; \quad |\Delta_{ex}| \leq g_2. \tag{9}$$

The equality $C=|\Delta_{ex}|(g_1+g_2)/g_2$ at $|\Delta_{ex}|\leq g_2$ determines the external synchronization boundary of the first oscillator of the chain, assuming that all elements of the chain are mutually synchronized. The equality $|\Delta_{ex}|=g_2$ at $C\leq|\Delta_{ex}|(g_1+g_2)/g_2$ corresponds to the boundary of mutual synchronization of all chain elements in condition that the first oscillator is synchronized with the external force. The estimation of the global synchronization region in accordance with (9) does not reflect the dependence of synchronization effect on a chain length m. Notwithstanding, in some cases these estimations agree rather well with results of numerical simulations. The conditions (9) result in the impossibility of global synchronization of the chain with purely diffusive interaction of oscillators (g_2=0). On the contrary, numerical simulation shows that the finite length chain can be synchronized. But the synchronization region is very small even for a short chain (m=10-20).

The calculation results of the global synchronization boundaries are given in Fig. 5, *a*. The global synchronization region S is obtained for a chain with m=100 elements on the parameter plane (Δ_{ex},C). The signs ($\circ$) and ($*$) mean the numerically obtained points of a boundary for the cases of one-direction and combined coupling, respectively. Dashed lines mark the boundaries described by (9). The estimation (9) for the case of Fig. 5, *a* is in a well agreement with the numerical data. When a diffusive component of coupling prevails the one-direction coupling, then a distribution of partial amplitudes ρ_j influences essentially on the dynamics of phases ϕ_j. In these circumstances relations (9) do not give a satisfactory estimation for synchronization boundaries. As it follows from simulation data, there exists a certain maximal value of the diffusive coupling parameter value g_1^{max} when global synchronization is yet possible in a chain of any length. Fig. 5, *b* shows the numerically calculated values of a half-width band Δ_s of global synchronization as functions of a chain length at fixed external amplitude and different coupling component relations. The signs ($\circ$), ($*$), ($\times$) mark the results for the cases of one-direction coupling, combined coupling and purely diffusive coupling, respectively. It is well seen that the value Δ_s tends to the certain constant level as the m increases. For $m\geq20$ a value of Δ_s does not practically depend on the next increasing of a chain length. So, it is reasonable to suppose that in the presence of one-direction interaction of oscillators directed from an external excited element, a global synchronization can be observed for the chain of an infinite length. It takes place in the band of external frequencies of nonzero width. This width essentially depends on the relationship between the coupling parameters g_1 and g_2. As the diffusive coupling parameter g_1 increases, the synchronization band becomes more narrow. For the pure diffusive interaction the band width quickly goes to zero with m increasing.

With crossing different parts of the region S boundaries two scenarios of the transition to global synchronization may be realized. In Fig. 5 two routes (Q and R) on the parameter plane corresponding to these scenarios are marked with arrows. The rout R corresponds to the gradual simultaneous approaching of the mean frequencies $\tilde{\omega}_j$ of all partial oscillators to the external frequency ω_{ex}. Approaching to the boundary of the region S from outside along direction Q we can observe external synchronization of several first oscillators in the chain. If $g_1\neq0$, the synchronization of the first oscillators

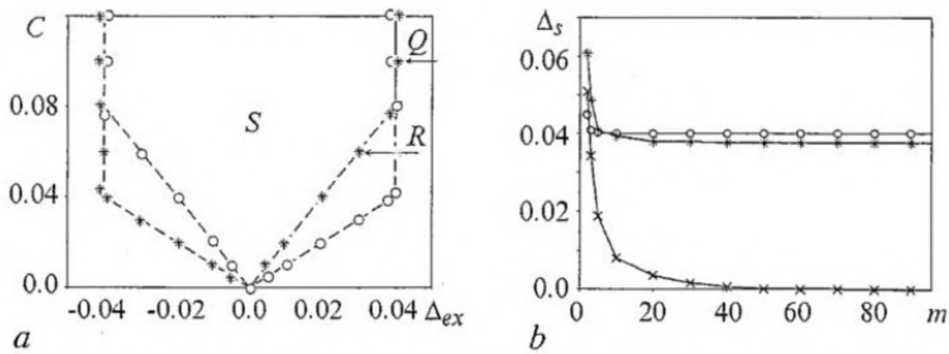

Fig. 5. *a*) Region S of global synchronization of the homogeneous chain of self-sustained oscillators (6) on the parameter plane (Δ_{ex},C) for g_1=0, g_2=0.04 ($\circ$) and g_1=g_2=0.04 ($*$). The dashed lines denote the boundaries of region S, defined by the conditions (9). The directions Q and R marked by arrows correspond to different scenarios of global synchronization; *b*) dependence of the half-width of the global synchronization band on the chain length m for g_1=0, g_2=0.04 ($\circ$); g_1=0.08, g_2=0.04 ($*$); g_1=0.08, g_2=0 ($\times$). The amplitude of excitation is assumed to be equal C=0.1

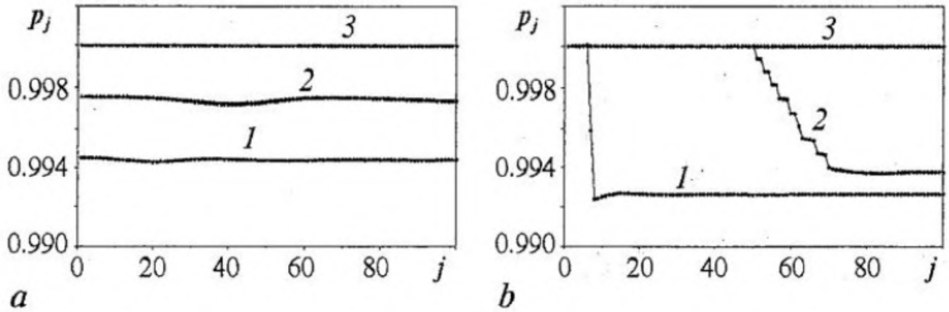

Fig. 6. Distribution of relative mean frequencies $p_j=\tilde{\omega}_j/\omega_{ex}$ along the chain when crossing different parts of the region S boundary on the parameter plane (Δ_{ex},C) for $g_1=g_2=0.04$: *a*) direction R (Fig. 5, *a*), Δ_{ex}=0.031; 0.0315; 0.0305; *b*) direction Q (Fig.5, *a*), Δ_{ex}=0.04; 0.0395; 0.039

must be understood in a sense of (3). The frequency synchronization cluster is formed (j=1, 2,...k,m). For all oscillators belonging to the cluster the relative frequencies p_j satisfy the condition (8). Oscillatory regimes for all elements of the chain (including the synchronized ones) are quasi-periodic. With a distance till the synchronization boundary decreases a length of cluster k increases tending to m. The evolution of the distribution of the relative frequencies p_j along the chain with approaching to the region S boundary is given in Fig. 6, *a*, *b*.

Consider the noise influence on the effect of global chain synchronization with the external frequency. In the presence of Gaussian δ-correlated noise sources the perfect synchronization is impossible. In this case we can speak about effective synchronization of self-oscillator only, i.e. about a phase locking during finite time intervals. However, if the noise intensity is small enough, the times of locking may be very long for some nonzero region of mismatch Δ_{ex} values, and a mean frequency of self-sustained oscillations is equal to the external force frequency with high accuracy. Therefore, the usage of (8) as a criterion of the j-th oscillator synchronization is justified from an experimental point of view both without noise and in the presence of noise.

We will study the robustness of global synchronization effect in the presence of independent sources of Gaussian δ-correlated noise in each oscillator of a chain. Consider the chain (6) of fixed length m=100 consists of identical oscillators with combined type of interaction. In Fig. 7, *a* some curves illustrate the dependence of relative frequency of oscillator with number j=100 on mismatch Δ_{ex} for different noise levels. It is well seen

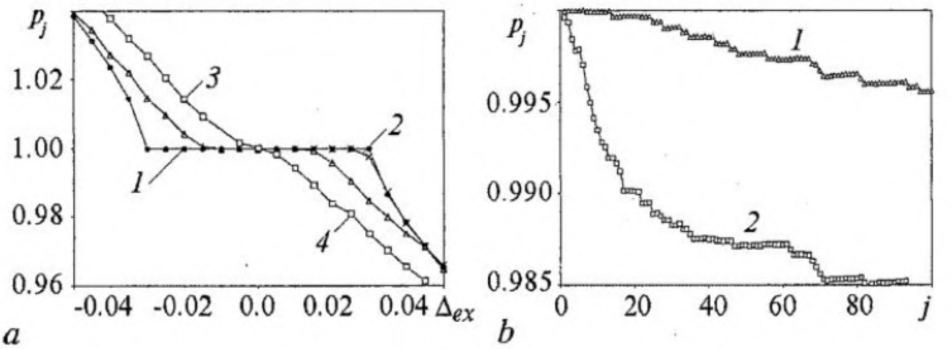

Fig. 7. Effect of noise on global synchronization of the chain with $g_1=g_2=0.04$, C=0.06. *a*) Dependences of the relative frequency of the self-sustained oscillator with j=100 on mismatch Δ_{ex}, obtained for different noise levels: D=0.0 (curve *1*), D=0.001 (curve *2*), D=0.005 (curve *3*), D=0.01 (curve *4*); *b*) variation of the relative frequency p_j along the chain for the given value of mismatch Δ_{ex}=0.02 and for two different noise intensities D=0.005 (curve *1*) and D=0.01 (curve *2*)

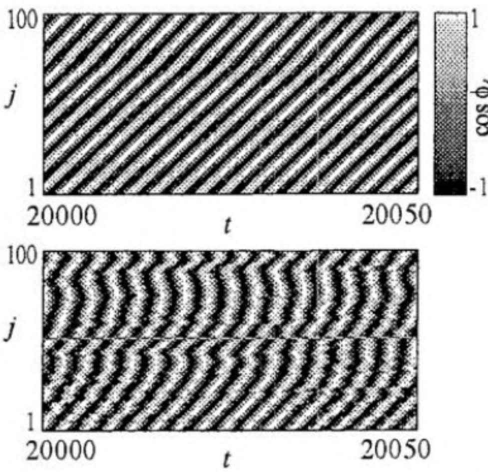

Fig. 8. Spatio-time diagrams of the phase behavior in a chain (6) at $g_1=g_2=0.04$, $C=0.06$, $\Delta_{ex}=0.02$ without noise (*a*) and with a noise of intensity $D=0.01$ the tone of gray colour correspond to a value of $\cos\phi_j$ from -1 (black) till +1 (white)

that the noise influence on the chain synchronization is analogous to the case of one noisy oscillator synchronization. The region of effective synchronization diminishes when the noise intensity D increases. For a small enough noise $D=0.001$ (×) the width of a global synchronization region consists nearly 80% (∘). For the noise $D=0.005$ (△) it is nearly 30%. For the case of $D=0.01$ (□) there is no any synchronization region because the condition (8) is valid for all oscillators of the chain only if $\Delta_{ex}=0$. However, a few first oscillators are still remain synchronized. Fig. 7, *b* illu-strates how values of p_j change along the chain for the given mismatch $\Delta_{ex}=0.02$ and two different noise levels: $D=0.005$, and $D=0.01$. In a case of $D=0.005$ the $k=13$ first oscillators can be considered as synchronized ones. At $D=0.01$ the condition (8) is not satisfied even for the first oscillator of the chain. In Fig. 8 the spatio-temporal diagrams of the chain are given. They show the behavior of phases of all oscillators of the chain in a region of global synchronization. Without noise excitation a well- distinguishable structure of diagonal strips is observed (Fig. 8, *a*). It corresponds to the regime of a phase wave propagation along a chain. The noise excitation destroys this structure (Fig. 8, *b*). It is clear, that in the presence of any noise (even very small) synchronization with the external frequency $\omega_{ex}\neq\omega_0$ can be achieved only for a finite number (thought it may be very large) of the first oscillators of the chain.

5. Conclusions

In this paper we have numerically studied the synchronization phenomena in a chain of coupled Van der Pol oscillators with noise. The numerical results obtained allow us to make a number of important conclusions.

The frequency cluster structure observed in a chain of non-identical elements with diffusive coupling appears to be sufficiently stable against uncorrelated Gaussian fluctuations added to each element. The cluster structure can be considerably destroyed in the presence of noise of large intensities.

Cluster synchronization in the chain in the presence of fluctuations should be understood as effective synchronization and characterized by the effective diffusion coefficient D_{eff}.

The amplitude dynamics may play an essential role in creating the cluster structure. Cluster synchronization can also be observed in a chain modeled by the phase equations only. But this effect is realized in a considerably narrow range of coupling parameter values. Besides, the cluster structure appears to be more sensitive to noise perturbations.

In a case of harmonic external excitation of the first chain element the character of interaction of oscillators plays the principal role. The presence of one-direction component of coupling results in a possibility of a realizing global synchronization with an external frequency for a chain of any length. There exists a region of external frequency variations in which the synchronization is observed. The width of this region tends to a constant level when the chain length m increases. In the case of pure diffusive coupling external synchronization is possible only for the chain of a finite length.

Two scenarios of the chain transition to the regime of global external synchronization are observed. One of them corresponds to simultaneous frequency synchronization for all oscillators, and the other one - to the formation of group of externally synchronized oscillators. The number of elements of this group (cluster) increases with approaching to the synchronization boundary.

And finally, the action of independent sources of Gaussian δ-correlated noise restricts a number of chain elements which can be considered as synchronized in the sense of effective synchronization. The synchronization of a chain of infinite length becomes impossible in the presence of noise.

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КЛАСТЕРНАЯ И ГЛОБАЛЬНАЯ СИНХРОНИЗАЦИЯ В ЦЕПОЧКЕ КВАЗИГАРМОНИЧЕСКИХ АВТОГЕНЕРАТОРОВ С ШУМОМ

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В работе исследуется влияние шума на эффекты синхронизации в цепочке осцилляторов Ван дер Поля. Анализируется режим частотных кластеров в неодно-

родной цепочке с шумом. Понятие эффективной синхронизации обобщается на случай пространственно распределенной системы. Исследуется также влияние амплитудных соотношений на поведение фаз осцилляторов. Рассматривается возможность вынужденной синхронизации однородной цепочки. Выясняется роль двух компонент связи (диффузионной и однонаправленной) и источников шума по отношению к эффекту глобальной синхронизации.

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