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STOCHASTIC RESONANCE AND ITS PROVENANCE

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Stochastic resonance, in which the signal and/or signal/noise ratio in a nonlinear system can be enhanced by the addition of random fluctuations (noise) of appropriate intensity, is discussed. By revealing the relationship of stochastic resonance to earlier research, and especially to work by Debye in the 1920s, the phenomenon is set in a broad physical context. It is shown that the traditional techniques of statistical physics, for example linear response theory, are applicable to stochastic resonance and their implications for its range of occurrence are discussed.

1. Introduction

Stochastic resonance (SR) is a phenomenon of some topical interest in which a weak periodic signal in a nonlinear system can be optimally amplified by the addition of random fluctuations (noise) of appropriate intensity; even more surprisingly, the signal/noise ratio can often be enhanced as well. In its modern form [1,2] the concept of SR emerged from the ideas of Benzi et al. [3,4] and Nicolis [5]. They were seeking to account for the earth's periodic ~ 10^5-year ice-age cycle in terms of the small variations, also of period ~ 10^5 years, in the earth's orbital eccentricity: SR provided a possible mechanism by which a weak periodic effect of this kind could be sufficiently amplified by environmental noise to exert a strong influence on the climate, as observed.

Since then SR, and other phenomena closely related to SR, have been found to have a remarkably wide range of occurrence. They have been observed or are to be anticipated in, for example: a Schmitt trigger [6]; a bistable ring laser [7]; a variety of electronic circuits [8] -[13]; a passive optically bistable [14]; a laser with saturable absorber [15]; a magnetoelastic ribbon [16]; a hybrid ESR device [17]; a magnetoresistive oscillator [18]; single-domain uniaxially anisotropic magnetic particles [19]; a bistable superconducting quantum interference device (SQUID) [20]; a quantum two-level system with ohmic dissipation [21]; a system with a cyclic variable [22]; and a tunnel diode [23]. Synchronization of otherwise random switchings between the states has been observed for a Brownian particle in a bistable optical trap [24] and also for a two-state defect that modulates conductance of a mesoscopic wire [25]. The SR-related phenomenon of noise-enhanced heterodyning has been demonstrated in an electronic model [26] and observed experimentally in an all-optical bistable system [27]. Although most of the work has related to bistable systems that can be characterised by coexisting attractors corresponding to the minima of a static bistable potential, it is now understood that SR is not by any means confined to such system. It has also been investigated: in monostable systems [11,22]; in a system with one point attractor and one chaotic attractor

[10]; for coexisting periodic attractors, yielding a high frequency form of SR [12,13]; in the transient dynamics of an evolving system [28]; and in a level-crossing detector [29]. In relation to biology, it has been proposed [30,31] that SR may be relevant to transmission of information by sensory neurons; the effects of neuron coupling have been analysed [32]; and SR in a crayfish mechanoreceptor has been observed and investigated [33]. Many of these examples are treated in the proceedings [1,2] of two recent topical conferences on SR.

Perhaps in part because of the context in which SR was discovered [3]-[5], it was treated from the outset as a major challenge in its own right, so that the theoretical development of the subject (see, for example, [34]-[39]) was undertaken almost on an *ab initio* basis, without drawing significantly on related work in condensed matter physics that had occurred much earlier. Consequently, the simplest, most direct and straightforward approach (classical linear response theory (LRT): see below) was applied [9] to SR only a few years after its discovery and, even then, did not meet with immediate acceptance. One of the aims of the present paper is to demonstrate the close connection between SR and earlier work, especially that of Debye [40], thus enabling SR to take its proper place in the context of other phenomena in physics. A fuller discussion of some of these ideas will be presented in [41].

2. A physical picture of stochastic resonance

To set the scene, we first discuss the SR in terms of the adiabatic approximation introduced by McNamara, Wiesenfeld and Ray [7] and McNamara and Wiesenfeld [34] in order to account for SR phenomena observed in a ring laser. A particular advantage of this approach is that it provides a simple and intuitively appealing physical picture of the mechanism of SR. We consider a Langevin equation of the type

$$\dot{q} = -\frac{\partial U}{\partial q} + A\cos\Omega t + \xi(t), \quad \langle\xi(t)\rangle = 0, \quad \langle\xi(t)\xi(t')\rangle = 2D\delta(t-t') \tag{1}$$

where D is the characteristic noise intensity and the potential $U(q)$ is assumed to be bistable, with local minima at the two coordinate values $q_{1,2}$ when A=0. When $A\neq 0$, a delta-shaped spike occurs in the power-spectrum of $q(t)$ at frequency Ω (and its overtones), and it is the increase in the height of this spike with D that signals the occurence of SR. For small D, the fluctuations of q about $q_{1,2}$ are small compared to $q_2 - q_1$. Nevertheless, although the noise is weak on the average, there can occur, occasionally, outbursts large enough to cause switchings between the stable states. The probability W_{nm} of a switching from the nth to the mth state for a white-noise driven system was found by Kramers [42] to be of the activation type,

$$W_{nm} = \pi^{-1}[U''(q_n)|U''(q_s)|]^{1/2}\exp(-\Delta U_n/D), \quad \Delta U_n = U(q_s) - U(q_n). \tag{2}$$

Here, ΔU_n is the depth of the nth well of the potential $U(q)$ measured relative to the value $U(q_s)$ of $U(q)$ at its local maximum q_s between the minima of $U(q)$ at q_1 and q_2, $U'(q_s)$=0 and $U''(q_s)$<0. The balance equation for the average populations $w_{1,2}$ of the stable states can be written

$$\dot{w}_1(t) = W_{21}(t)w_2(t) - W_{12}(t)w_1(t), \quad w_1(t) + w_2(t) = 1. \tag{3}$$

The transition probabilities $W_{nm}(t)$ can readily be evaluated in the adiabatic approximation: i.e. assuming that the frequency Ω of the sinusoidal driving is small compared to the reciprocal relaxation time of the system. In this case the probability $W_{nm}(t)$ is determined by the *instantaneous* value of the potential well depth $\Delta\tilde{U}_n(A\cos\Omega t)$ where $\Delta\tilde{U}_n(A)$ is the depth of the nth potential well in the potential tilted by the external force, $U(q)$ -Aq. For small enough amplitude of the force only linear terms need be retained in the potential well depth. Using Eq. (2), one can then write $W_{nm}(t)$ in the form:

$$W_{nm}(t) = W_{nm}^{(0)}\exp(g_n\cos\Omega t), \qquad g_n = \tilde{g}_n A/D,\ \tilde{g}_n = -[\frac{\partial \Delta\tilde{U}_n(A)}{\partial A}]_{A=0} \tag{4}$$

where $W_{nm}^{(0)}$ is the value of the transition probability in the absence of the force.

It is evident from Eq.(4) that the parameter which determines the effect of the driving is proportional to the ratio of the amplitude of the force A to the noise intensity D. Thus, in agreement with what was said above, for small D we can have strong effects even for comparatively small A. Moreover, if A is so small that $|g_{1,2}| << 1$, the probabilities $W_{nm}(t)$ can be expanded in g_n and Eq.(3) can then be solved analytically. To first order in $|g_{1,2}|$ the dependence of the populations $w_{1,2}(t)$ on time is sinusoidal.

The solution is particularly simple in the important case of a symmetric potential $U(q)$, for which $\Delta U_1=\Delta U_2$, $g_1= -g_2$. An important byproduct of this solution is an interesting behaviour of the spectral density of fluctuations $Q(\omega)$ of the coordinate of the system $q(t)$:

$$Q(\omega) = \lim_{\tau\to\infty}(4\pi\tau)^{-1}|\int_{-\tau}^{\tau} dt\, q(t)\exp(i\omega t)|^2. \tag{5}$$

If the coordinate $q(t)$ is approximated by the sum of its values in the stable states $q_{1,2}$ weighted by the populations $w_{1,2}(t)$ (the two-state approximation), then, because of the sinusoidal dependence of the populations on time a δ-shaped spike occurs in $Q(\omega)$ at the frequency $\omega=\Omega$. The ratio of its intensity (area) to the value of the power spectrum $Q^{(0)}(\Omega)$ in the absence of the driving, i.e. the signal-to-noise ratio R, is given by the expression [7,34]:

$$R = \pi g^2 W^{(0)}/4, \qquad g \equiv g_1 = -g_2, \qquad W^{(0)} \equiv W_{12}^{(0)} + W_{21}^{(0)} = 2W_{12}^{(0)}. \tag{6}$$

It follows from (4) and (6) that the signal-to-noise ratio *increases exponentially* with increasing noise intensity for small enough D,

$$R \propto \exp(-\Delta U/D),\ \ \Delta U \equiv \Delta U_1 = \Delta U_2,\ \ \Delta U >> D$$

i.e. SR occurs. It was this amazing result that stimulated so much interest in the phenomenon of SR among physicists, biologists, and engineers.

3. Linear response theory and stochastic resonance

An alternative approach to SR which, as we shall see, makes it possible to place the phenomenon in its natural context within statistical physics and condensed matter physics, and to relate it to what had been done in these areas before, is based on linear response theory (LRT). According to LRT, if a system with a coordinate q is driven by a weak force $A\cos\Omega t$ (the addition to the Hamiltonian function of the system is of the form of $-Aq\cos\Omega t$), there arises a small periodic term in the ensemble-averaged value of the coordinate, $\delta\langle q(t)\rangle$, oscillating at the same frequency Ω and with amplitude a proportional to that of the force [44]:

$$\delta\langle q(t)\rangle = a\cos(\Omega t+\phi) \equiv \mathrm{Re}[\chi(\Omega)Ae^{-i\Omega t}],\ \ A\to 0, \tag{7}$$

$$a = A|\chi(\Omega)|,\ \ \phi = -\arctan[\mathrm{Im}\chi(\Omega)/\mathrm{Re}\chi(\Omega)].$$

The quantity $\chi(\Omega)$ here is the *susceptibility* of the system. Eq.(7) holds for dissipative and fluctuating systems that do not display persistent periodic oscillations in the absence of the force $A\cos\Omega t$. In the more general case of a system performing phase-locked oscillations with a period $2\pi/\omega_F$ (this case is of particular interest for systems driven by strong periodic fields with a frequency ω_F, e.g., by laser radiation) the linear response is described by the expression

$$\delta\langle q(t)\rangle = \mathrm{Re}\,\Sigma_{k=-\infty}^{\infty}\chi^{(k)}(\Omega)\, A\exp[i(k\omega_F - \Omega)t],\ A \to 0. \qquad (8)$$

In this case a weak force gives rise to vibrations not only at its own frequency, but also at the combination frequencies $|\Omega \pm k\omega_F|$, and $\chi^{(k)}(\Omega)$ are the corresponding susceptibilities.

The function $\chi(\Omega)$ (or the functions $\chi^{(k)}(\Omega)$) contains all information on the response of the system to a weak driving force. It gives both the *amplitude* of the signal, a, and its *phase lag* with respect to the force, ϕ (or partial amplitudes and phase lags for the vibrations at the combination frequencies). In fact, Eqs.(7), (8) still hold even if the force is of a more general nature than just an «additive» coordinate-independent force described by the extra term $-Aq\cos\Omega t$ in the Hamiltonian. In particular, the force can be coordinate-dependent (a multiplicative force), or it can be the intensity of the noise driving the system (e.g., the temperature, if the noise is of thermal origin) that is modulated periodically. In any case, if the amplitude of the modulation is weak enough, the response of the system is linear and is described by (7), (8). The onset of SR in response to the modulation of the noise intensity (temperature) was investigated in [45], and in [21] SR in response to the modulation of the temperature was considered for a mesoscopic wire with a two-state dissipating defect.

The periodic terms (7), (8) induced by the force give rise to δ-shaped spikes in the spectral density of fluctuations (SDF) $Q(\omega)$ (5) at the frequency of the force Ω (and at the combination frequencies $|\Omega \pm k\omega_F|$). The *intensity* (i.e., the area) of these spikes is equal to one quarter of the squared amplitude of the corresponding vibrations, i.e., to $\frac{1}{4}A^2|\chi(\Omega)|^2$, or to $\frac{1}{4}A^2|\chi^{(k)}(\Omega)|^2$. The signal–to-noise ratio R is thus expressed in terms of the susceptibility as

$$R = \tfrac{1}{4}A^2|\chi(\Omega)|^2/Q^{(0)}(\Omega),\quad A \to 0, \qquad (9)$$

and for periodically oscillating systems the signal-to-noise ratio $R^{(k)}$ at the combination frequency $|\Omega - k\omega_F|$

$$R^{(k)} = \tfrac{1}{4}A^2|\chi^{(k)}(\Omega)|^2/Q^{(0)}(|\Omega - k\omega_F|),\qquad A \to 0. \qquad (10)$$

Therefore, the evolution of the susceptibility and of $Q^{(0)}(\omega)$ with varying noise intensity D show immediately whether or not SR (understood as an increase, with the increasing D, of the signal or of the signal-to-noise ratio in a certain range of D) is to be expected at a given frequency.

Describing SR in terms of the susceptibility is particularly advantageous for systems that are in thermal equilibrium or in quasi-equilibrium. In this case the susceptibility can be expressed immediately in terms of the SDF $Q^{(0)}(\Omega)$ in the absence of periodic driving via the fluctuation-dissipation relations [44]:

$$\mathrm{Im}\,\chi(\omega) = \pi\omega/T\, Q^{(0)}(\omega),\qquad \mathrm{Re}\,\chi(\omega) = 2/T\,\mathrm{P}\int_0^{\infty} d\omega_1\, Q^{(0)}(\omega_1)\,\omega_1^2/(\omega_1^2 - \omega^2) \qquad (11)$$

where P implies the Cauchy principal part and T is the temperature in energy units. It follows from (10), (11) that the onset of SR can be predicted from purely *experimental* data on the evolution of the SDF of a system with temperature without assuming anything at all about the equations that describe its dynamics, i.e. for a system treated as a «black box».

The relevance of this approach to SR is seen from Fig.1 where some data from analog experiments for electronic systems simulating Brownian motion in a bistable [9]

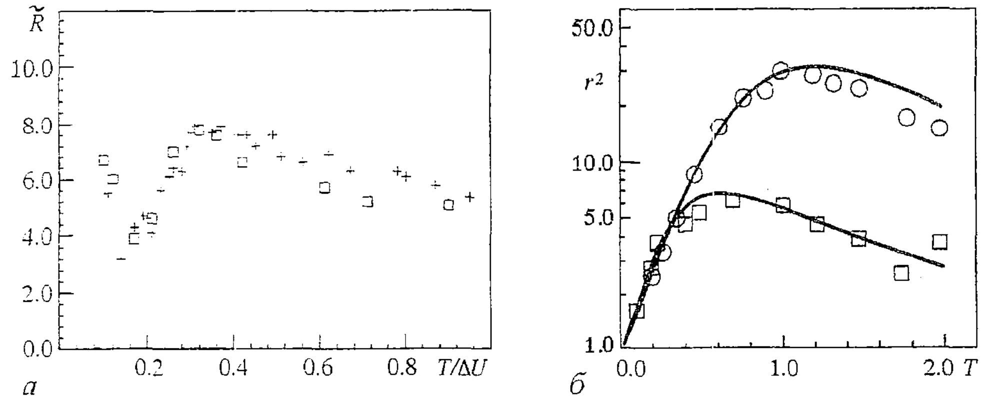

Fig. 1. Stochastic resonance for the Brownian motion $\ddot{q}+2\Gamma\dot{q}+U'(q)= (2\Gamma)^{1/2}\xi(t)+A\cos\Omega t$, for the white noise $\xi(t)$ (1) with the intensity $D\equiv T$ (after [9]): *a* - bistable potential $U(q)=-q^2/2+q^4/4$. The values of $\tilde{R}$=6.51·10^{-4}R are given for Ω=0.0695, A=0.1, Γ=0.125. ❑- Direct measurements, + - data calculated from the *measured* $Q^{(0)}(\omega)$ via fluctuation-dissipation relations (11); *b* - Monostable potential, $U(q)=Bq+q^2/2+q^4/4$. The stochastic amplification factors, $r^2=a^2(T)/a^2(0)$ equal to the squared ratio of the amplitude of the signal with and without noise, ❑ - B=0, O - B=2; A=0.02, Γ=0.011. The full curves represent theoretical predictions derived using LRT (after [11] (*b*))

and in a monostable [11] potential are compared with the LRT theoretical predictions. The simulated systems are quasi-thermal. The results of Fig. 1, *a* demonstrate that SR in the signal-to-noise ratio is described *quantitatively* by the fluctuation-dissipation relations (11), even in the range where an explicit analytic calculation of the susceptibility of the system was not possible. The data in Fig. 1, *b* show that, contrary to what had been commonly accepted, a noise-induced increase of the signal in a system does not require that it be a bistable one: the effect can arise in monostable systems as well. The particular mechanism explored [11] is based on the fact that the frequency of a nonlinear system depends on the amplitude (energy) of the vibrations. By varying the temperature of the system (the noise intensity) one varies the distribution of the system over the energy, and hence over the frequency. It is possible therefore to «tune» the system, and thus to increase the response at an appropriate frequency. The strong and rather interesting temperature dependence of the spectral density of the fluctuations $Q^{(0)}(\omega)$ of underdamped systems was reviewed in [46]. Recent results obtained for a special class of underdamped systems where the dependence of the eigenfrequency of the vibrations on the amplitude is nonmonotonic - the upper curve in Fig. 1,*b* refers to a system of this sort - are reported in [47].

4. Susceptibility and relaxation in solids with reorienting dipoles

To the best of our knowledge, analytical results for the susceptibility of a fluctuating symmetrical system with two coexisting stable states, which traditionally has been of primary interest in the context of SR, were first obtained by Debye [40]. Debye analyzed the dielectric response of polar molecules in a solid. He assumed that a molecule can switch between two *equivalent* positions within a unit cell, and that in these positions the dipole moment of the molecule is pointing in opposite directions. The expression for the transition probability W_{nm} he used was equivalent to Eq. (4), with $g_n= -\mathbf{E}\mathbf{d}_n/T$ where $\mathbf{E}$ is the amplitude of the electric field and $\mathbf{d}_n$ is the dipole moment in the *n*th position (n=1,2; $\mathbf{d}_1= -\mathbf{d}_2$); he linearized W_{nm} in $\mathbf{E}\mathbf{d}_n/T$ (however, he did not specify the form of the transition probabilities $W_{nm}^{(0)}$ in the absence of the external field).

The well-known expression for the susceptibility Debye derived was, in the present notation, of the form

$$\chi_D(\Omega) = \frac{d_1^2}{T}\frac{W^{(0)}}{W^{(0)} - i\Omega}, \qquad W^{(0)} \equiv W_{12}^{(0)} + W_{21}^{(0)} = 2\,W_{12}^{(0)}. \qquad (12)$$

This expression made it possible to explain the experimental data on the dispersion of the real part of the dielectric constant of ice. It is straightforward to see from the fluctuation-dissipation relations (11) that the signal-to-noise ratio R that follows from (9), (12) is precisely of the form (6) (cf. [9,48]) that describes SR in the linear regime.

In the context of condensed-matter physics, the quantity of special interest is usually the phase shift between the force and the signal, since it is the phase shift that determines the absorption of the energy from the force, in particular from the electromagnetic field in the case considered by Debye. In the symmetrical two-state model with thermally activated transitions between the states the phase shift ϕ as given by (7), (12) decreases monotonically with increasing temperature [40, 5, 34]:

$$(\phi)_{\text{two-state}} = -\arctan(\Omega/W^{(0)}). \qquad (13)$$

The phase shift is one of the characteristics used to describe the elastic properties of solids: in this case the force is stress, the signal is strain, and the phase lag is referred to as *internal friction* [49]. For finite frequency of the stress there arises a phase shift between the stress and the strain, even though the stress is linear in the strain (and thus reversible). In some metal alloys internal friction displays a strong nonmonotonic temperature dependence as shown in Fig. 2 taken from [50]. A simple mechanism of this dependence for body-centered cubic metals with interstitial impurity atoms was suggested by Snoek [51]. He assumed that an impurity occupies one of the equivalent interstitial points in an elementary cell thus forming an elastic dipole. The dipole can reorient as a result of thermal fluctuations. Uniaxial stress breaks the symmetry, like an electric field in the case of electric dipoles, and the response to the stress is given basically by Debye's theory, slightly modified to allow for a different number of equivalent stable states.

The strain measured experimentally arises as a combination of the strain related to the reorientation of the elastic dipoles and the strain due to the deformation of those cells that are free of impurities. This deformation is characterized by much faster relaxation than the reciprocal reorientation rate $1/W^{(0)}$ of the dipoles at room temperature. For low temperatures the reorientation rate $W^{(0)}$ is negligibly small, and the strain is equal to that for a crystal with immovable defects and is in phase with the stress (Hooke's law). Therefore the phase shift is equal to zero rather than to $-\pi/2$ as given by (13). Only for higher T does the reorientation of the elastic dipoles become «switched on» and the term described by (13) contributes to the phase shift. As a result $|\phi|$ sharply increases with temperature and displays a clearly resolved peak. The position of the peak (see the next subsection) may be used to determine the activation energy for reorientation of the elastic dipoles [49].

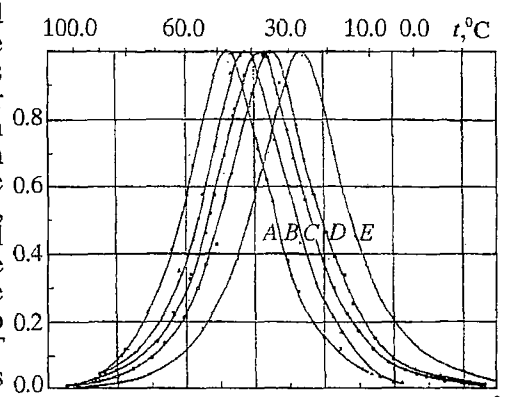

Fig. 2. Peaks of internal friction (normalized phase lag $-\phi$) vs temperature due to Snoek relaxation in an Fe-C alloy; the curves A to E correspond to the frequencies 2.1; 1.17; 0.86; 0.63; 0.27 Hz (after Wert and Zener [50])

5. Stochastic resonance in continuous dynamical systems

In many cases, the bistable dynamical systems where SR is investigated are *continuous* rather than being two-state ones. For such systems, the dependence of the phase lag on the noise intensity (temperature) is similar to that observed for internal friction in anelastic solids. This is clearly seen from a comparison of Fig.2 and Fig.3. In the latter case, the data [52] are from an analog simulation of overdamped Brownian motion (1) in a simple symmetric bistable potential

$$U(q) = -1/2\, q^2 + 1/4\, q^4. \tag{14}$$

The explicit LRT expressions for the phase shift and for the signal-to-noise ratio R of a continuous system (1) for low noise intensities and for low frequency Ω are of the form:

$$\phi = -\arctan[(\Omega/\Omega_r)(\Omega_r^2 W^{(0)} + \Omega^2 D)/(\Omega_r W^{(0)2} + \Omega^2 D)],$$

$$R = \frac{\pi A^2}{4D^2}\,(\Omega_r^2 W^{(0)2} + \Omega^2 D^2)/(\Omega_r^2 W^{(0)} + \Omega^2 D), \qquad \Omega, D \ll \Omega_r,\; W^{(0)} \ll D \tag{15}$$

where $\Omega_r \equiv t_{rel}^{-1} = U''(q_{1,2})$ is the reciprocal relaxation time for the intrawell motion (corrections to (15) of the order of Ω/Ω_r, $D/\Delta U$ have been dropped).

Measurements of the phase lag in an electronic model of (1), (14) are plotted as a function of noise intensity in Fig 3. The variation of $|\phi|$ with D exhibits a well-defined maximum and clearly has much in common with the internal friction data of Fig.2, although the peaks in the latter case are noticeably narrower, which shows that the system (1), (14) does not provide a completely adequate quantitative model for anelastic relaxation. We notice that the values of Ω in Fig.3 is very much higher than the effective value of Ω for the anelastic relaxation experiments (where the absolute value of the driving frequency is ~1 Hz as compared to ~10^{13} Hz for atomic vibrations in solids so that Ω ~10^{-13}). For $\Omega=10^{-13}$, the function of $-\phi(1/D)$ as given by (15) takes the form plotted in Fig. 4, showing a narrower and more symmetrical maximum than that in Fig. 3, albeit still slightly broader than those in Fig. 2. The positions of the maxima of $|\phi|$ as a function of D depend on Ω in both cases; for the model (1), (14) this position D_{max} is given by the equation

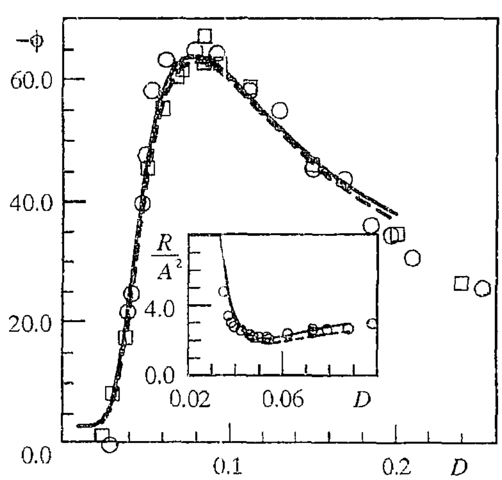

Fig. 3. Phase lag - ϕ between the coordinate $\langle q(t)\rangle$ of an overdamped Brownian particle oscillating in a potential (14) and the force of frequency $\Omega=0.1$ as measured in the electronic experiment; the force amplitude: o - $A=0.04$, □ - 0.2. The solid line represents the theoretical prediction based on LRT [52] (nonlinear corrections do not change this curve strongly for the actual value of A). The inset shows the normalized signal-to-noise ratio in the region of the minimum in R

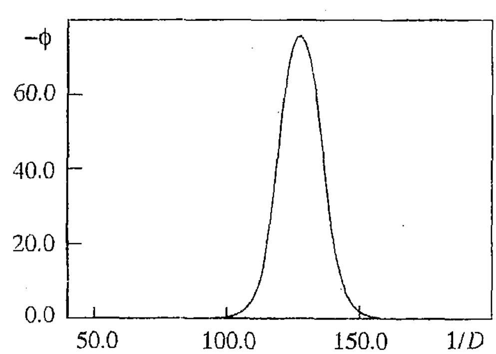

Fig. 4. Phase lag - ϕ for the system (1), (14), calculated [52] from (7), (11) for $\Omega=10^{-13}$ and plotted as a function of $1/D$ for more convenient comparison with the data of Fig.2

$$W^{(0)}(D_{max}) = \Omega(D_{max}/\Omega_r)^{1/2}.$$

The response of a continuous system differs markedly from that of a two-state one, not only in its non-monotonic rather than monotonic variation of ϕ with D, but also in the variation of its signal/noise ratio with D: for small D the function R *decreases* rather than increases with the increasing D. Such behaviour, seen in Fig.1, *a*, has a simple explanation. For small D the interwell transitions are frozen out: the susceptibility is then determined by the intrawell motion of the system, and is independent of noise, whereas the power spectrum is formed by the fluctuations about the minima of the potential and increases proportional to the noise intensity, so that $R \propto 1/D$, as seen from (15) for small $W^{(0)}$. The analysis of the position of the local maximum of R vs noise intensity was performed by Fox and Lu [53].

In general, of course, the motion of a bistable continuous system will not be described by the simple model (1) of overdamped Brownian motion in a symmetrical double-well potential. Neither will the noise be white, nor will the system be moving in a static potential. For example, the stable states of interest may be the states of stable periodic vibrations in a strong external force (periodic attractors), as is of interest in the context of optical bistability [54]. Analytic results for the fluctuations and for the response to a weak external force can be obtained [55] provided the noise intensity is small so that the probabilities of fluctuational transitions between the states are very much smaller than the reciprocal intrawell relaxation time, $W_{nm}^{(0)} << t_{rel}^{-1}$, and the fluctuations occur mostly within narrow vicinities of the stable states. The results hold for systems driven by an arbitrary Gaussian noise, in which case the dependence of the transition probabilities on the characteristic noise intensity D is of the activation type, and in the absence of the additional weak force

$$W_{nm}^{(0)} = \text{const} \exp(-R_n/D). \tag{16}$$

The activation energy of the escape from the state n, R_n, is given by the solution of a variational problem [56]. For certain types of non-white Gaussian noise R_n was found in Refs. [56, 57].

For small enough D there are two main contributions to the susceptibilities $\chi^{(k)}(\Omega)$. One comes from the motion close to the stable states where the system spends most of the time. This contribution is given by the sum of the partial susceptibilities $\chi_n^{(k)}(\Omega)$ (n=1,2) weighted by the populations of the stable states w_n. The other contribution, $\chi_{tr}^{(k)}(\Omega)$, is important in the case where the frequency Ω of the weak force is small or is close to the frequency ω_F of the strong external force. In this case the weak force modulates the probabilities of the transitions between the states and thus the populations of the states:

$$\chi^{(k)}(\Omega) = \Sigma_{n=1,2}\, w_n\, \chi_n^{(k)}(\Omega) + \chi_{tr}^{(k)}(\Omega),\quad w_1 = 1 - w_2 = W_{21}^{(0)}/W_{12}^{(0)}. \tag{17}$$

The partial susceptibilities $\chi_n^{(k)}(\Omega)$ can be easily found from the equations of motion linearized about the stable states in the absence of noise (noise determines the values of the populations w_n via the transition probabilities). They display dispersion on the frequency scale t_{rel}^{-1}, whereas in the range of interest for SR, $\Omega << t_{rel}^{-1}$ or $|\Omega - \omega_F| << t_{rel}^{-1}$, they are nearly frequency-independent.

The characteristic frequency scale which determines the dispersion of $\chi_{tr}^{(k)}(\Omega)$ is given by the relaxation rate of the populations, i.e., by $W^{(0)} = W_{21}^{(0)} + W_{12}^{(0)}$. A simple way to obtain $\chi_{tr}^{(k)}(\Omega)$ for $\Omega << \nu_c$ or $|\Omega - \omega_F| << \nu_c$ ($\nu_c = \min(t_{rel}^{-1}, t_{cor}^{-1})$, where t_{cor} is the correlation time of the noise) is based [55] on the fact that the major effect of the additional weak force $A\cos\Omega t$ on the populations of the states comes from the modulation of the activation energies of the transitions between the states R_n. For small Ω one can find this modulation just by evaluating R_n for a system biased by a *constant* force A, i.e., by finding $R_n \equiv \tilde{R}_n(A)$, and then by replacing A by $A\cos\Omega t$. In this case the escape probability can be written in the form similar to (4):

$$W_{nm}(t) = W_{nm}^{(0)}\exp(g_n\cos\Omega t), \quad g_n = \tilde{g}_n A/D, \quad \tilde{g}_n = -[\frac{\partial \tilde{R}_n(A)}{\partial A}]_{A=0}, \ \Omega \ll \nu_c. \qquad (18)$$

In the case of periodic attractors corresponding to forced vibrations in a strong periodic force $F\cos(\omega_F t + \phi_F)$, with the frequency $\omega_F \gg \nu_c$, the additional weak force $A\cos\Omega t$ with Ω very close to ω_F can be considered as a modulation of the amplitude of the strong force,

$$F\cos(\omega_F t + \phi_F) + A\cos\Omega t = \mathrm{Re}\tilde{F}(t)\exp(i(\omega_F t + \phi_F)),$$

$$\tilde{F}(t) = F + A\exp[i(\Omega - \omega_F)t - i\phi_F].$$

The activation energies $R_n \equiv R_n(F)$ are independent of the phase ϕ_F, and when the weak force $A\cos\Omega t$ is applied they take on time-dependent values corresponding to the instantaneous value of the amplitude $|\tilde{F}(t)|$, so that

$$W_{nm}(t) = W_{nm}^{(0)}\exp[g_n\cos((\Omega - \omega_F)t - \phi_F)], \qquad g_n = \tilde{g}_n A/D, \qquad (19)$$

$$\tilde{g}_n = -\frac{1}{F}\frac{\partial R_n(F)}{\partial F}, \qquad |\Omega - \omega_F| \ll \nu_c, \omega_F.$$

Eqs. (18), (19) can be inserted into Eq. (3) for the populations. For small amplitudes A, when $|g_n| \ll 1$, one can expand the transition probabilities in g_n. Terms linear in g_n are sinusoidal in time, and so also are the corresponding terms in the populations $w_{1,2}(t)$. If the value of the coordinate in the nth periodic attractor

$$q_n(t) = \Sigma_k q_n^{(k)}\exp(ik\omega_F t)$$

then the expression for the susceptibility $\chi_{tr}^{(k)}(\Omega)$ for $|\Omega - \omega_F| \ll \nu_c$ is of the form

$$\chi_{tr}^{(k)}(\Omega) = -\frac{W_{12}^{(0)}W_{21}^{(0)}}{W^{(0)}}\frac{\tilde{g}_1 - \tilde{g}_2}{D}\frac{q_1^{(k-1)} - q_2^{(k-1)}}{W^{(0)} - i(\Omega - \omega_F)}e^{i\phi_F}. \qquad (20)$$

The equation for the susceptibility with respect to a low-frequency force is very similar:

$$\chi_{tr}(\Omega) = -\frac{W_{12}^{(0)}W_{21}^{(0)}}{W^{(0)}}\frac{\tilde{g}_1 - \tilde{g}_2}{D}\frac{q_1^{(0)} - q_2^{(0)}}{W^{(0)} - i\Omega}, \qquad \Omega \ll \nu_c. \qquad (20a)$$

It can easily be seen that in the symmetrical case, $W_{12}^{(0)} = W_{21}^{(0)}$, $\tilde{g}_1 = -\tilde{g}_2$, $q_1 \equiv q_1^{(0)} = -q_2 \equiv -q_2^{(0)}$, this expression goes over into Debye's result (12).

Note that, for a simple model of *overdamped* Brownian motion in the bistable potential (1), the expressions for the susceptibility (17), (20a) (and also the explicit form of the partial susceptibility $\chi_n(\Omega)$) can be obtained at low noise intensities directly from an analysis of the eigenvalues and eigenfunctions [58] of the Fokker - Planck equation, both in the case of a symmetric [59] (a) and an asymmetric [59] (b) potential. A detailed numerical analysis of the Fokker - Planck equation for the system (1), (14) is [60] in full agreement with the analytic results derived above, and in particular with those for the phase shift shown in Fig. 3. Note also some earlier numerical work on the Fokker - Planck equation for periodically driven bistable systems [33].

From (18) - (20) (cf. [55]) it is clear: (i) that the susceptibility due to the transitions between the states *increases* exponentially sharply with noise intensity D in the range of very small D; (ii) that this susceptibility is greatest within a frequency range that is extremely narrow compared with the characteristic inverse relaxation time t_{rel}^{-1}; (iii) that the susceptibility is proportional to the reciprocal noise intensity, which is why it can become large, and (iv) that it becomes large only within the narrow range of the system parameters for which $R_1 \approx R_2$, and thus the transition probabilities, $W_{12}^{(0)}$ and $W_{21}^{(0)}$, and the populations, w_1 and w_2, are of the same order of magnitude (the range of the kinetic phase transition). All of these features have been observed in experiments and are immediately related to the onset of SR in bistable systems. In particular, the feature (iv) shows that SR in bistable systems is a kinetic phase transition effect. A demonstration of SR in a system with periodic attractors, obtained from an analog electronic experiment [13] is shown in Fig. 5. The experimental data (points) exhibit an increase of signal-to-noise ratio both at the frequency of the force Ω and at the combination frequency $2\omega_F - \Omega$; they agree well with the LRT theoretical predictions (curves).

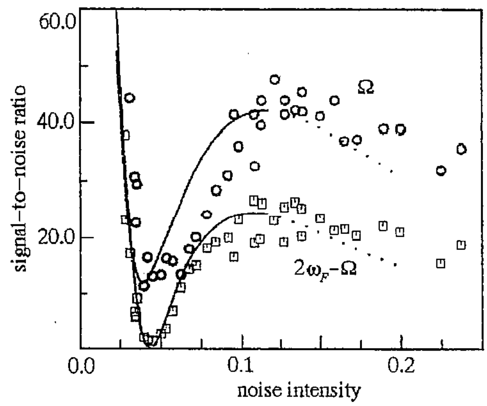

Fig. 5. The response of a noise-driven underdamped nonlinear oscillator with coexisting periodic attractors to an extra force $A\cos\Omega t$ (after [13]). The equation of motion of the oscillator for $A=0$ is of the form $\ddot{q}+2\Gamma\dot{q}+\omega_0^2 q+q^3=F\cos\omega_F t$. The quantities P and P provide signal-to-noise ratios at the frequencies Ω and $2\omega_F-\Omega$. The data refer to the kinetic phase transition range, $(\omega_F-\omega_0)/\Gamma=0.236$, $3F^2/32\omega_F^3(\omega_F-\omega_0)^3=0.0814$

Finally in this section, we would point out also that an important corollary of LRT is that, for small-amplitude signals, the signal-to-noise ratio at the output of a system driven by a stationary Gaussian noise does not exceed that at the input, even if the system displays SR. Indeed, the Fourier components of the noise are statistically independent and the total power of the noise $\Xi(\Omega)d\Omega$ in a small spectral interval $d\Omega$ about the frequency of the signal Ω is small. The signal-to-noise ratio at the input is given by $1/4A^2/\Xi(\Omega)$, whereas that at the output is $1/4|\chi(\Omega)|^2A^2/[|\chi(\Omega)|^2\Xi(\Omega)+Q'^{(0)}(\Omega)]$. The quantity $Q'^{(0)}(\Omega)$ gives the value of the spectral density of fluctuations in the system at frequency Ω as it would be if there was no signal and the spectral components of the noise at frequency Ω were suppressed, i.e., the power spectrum of the input noise had a hole at frequency Ω. By construction $Q'^{(0)}(\Omega)\geq 0$, which proves the statement. (In linear systems, on the other hand, which do not mix frequencies, $Q'^{(0)}(\Omega)=0$ and the signal-to-noise ratio at the output must be the same as at the input).

6. Conclusion

The main advantages of the LRT approach to SR are its simplicity, generality and predictive power. It follows on naturally from earlier work in physics, especially that of Debye [40]. Of course, LRT is by definition restricted to the small signal limit. Beyond this range of linear response, where LRT is inapplicable, other (perhaps numerical) methods become necessary (though analytic results can be obtained [9,43, 61] for the system (1), (14) in the low noise limit, still using classical techniques of theoretical physics), and the early theories of SR may then be expected to come into their own. Inevitably, they lack the generality of LRT and, in most cases, are directly applicable only to the particular type of system for which they were introduced. The LRT approach meets *a fortiori* the acid test of any theory in science, namely, the possession of predictive power. For example, it enables the presence or absence of SR in any given thermal equilibrium system to be predicted simply from the evolution of its SDF with increasing noise intensity in the absence of the periodic driving force: the discovery of SR in monostable systems [11] was arrived at in precisely this way.

With the benefit of hindsight, therefore, it can be seen that there is nothing particularly mysterious about SR, and its position in the mainstream evolution of physics has become clear. The next stage in the development of the topic seems likely to be in terms of applications, perhaps related to SQUIDs [20], or to information transfer in biological systems [30]-[33] or to communications, for example through noise-protected optical heterodyning [27].

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СТОХАСТИЧЕСКИЙ РЕЗОНАНС И ЕГО ПРОИСХОЖДЕНИЕ

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П.В.Е. Мак-Клинток, Н.Д. Стейн, Н.Г. Стокс

Обсуждается явление стохастического резонанса, при котором можно увеличить соотношение сигнал-шум (или сигнал-сигнал) в нелинейной системе за счет добавления случайных флуктуаций (шума) соответствующей интенсивности. Выявляя связь стохастического резонанса, с более ранними исследованиями и особенно с работой Debye в 20-е годы, авторы дают явлению широкую физическую трактовку. Показывается, что к стохастическому резонансу применимы традиционные методы статистической физики, например, теория линейного отклика, и обсуждаются их сложности в области возникновения стохастического резонанса.

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