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TOOLS FOR ANALYZING OBSERVED CHAOTIC DATA*

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1 Introduction

Most problems in physical and biological science which involve evolution in time can be cast as sets of differential equations where variables evolve in continuous time or discrete time maps where time is sampled at given intervals. If we allow infinite numbers of differential equations then partial differential systems and time delay systems and many integral equations also fit into this general category. Certainly the majority of engineering systems are typically cast into this framework, and it is the analysis of the evolution in time (continuous or discrete) which is the province of dynamical systems. When those dynamical systems involve the dependent variable $\mathbf{x}(t)$ in a nonlinear fashion in the evolution equations, and this is typically the case, phenomena occur which are substantially different and richer than those arise when the equations are constrained to be linear in $\mathbf{x}(t)$. In this review we expose some of the ideas which have been uncovered about the solutions to such nonlinear systems over the past several decades and discuss in detail how these new ideas allow the analysis of complex looking time series which might be dismissed as «noise» without the understanding achieved. Our viewpoint is to describe tools for the analysis of real data with an eye toward learning enough from that data to provide means to make models for prediction and control of the nonlinear systems one observes. We restrict the discussion to ordinary differential equations and maps, though with appropriate care on the kind of space one works in more general questions are encompassed as well.

The solution $\mathbf{x}(t)$ to sets of ordinary differential equations

$$d\mathbf{x}(t)/dt = \mathbf{F}(\mathbf{x}(t)) \qquad (1)$$

or discrete time iterated maps

$$\mathbf{x}(t+1) = \mathbf{F}(\mathbf{x}(t)), \qquad (2)$$

where $\mathbf{x}(t)$ is a vector in d-dimensional state space, and $\mathbf{F}(\mathbf{x})$ is a smooth nonlinear function of $\mathbf{x}$, has revealed numerous remarkable surprises over the past two decades. In the case where the physical system is dissipative, state space volumes shrink to zero in time, and time asymptotic motion occurs on a set of points to which all orbits $\mathbf{x}(t)$ from a large

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set of initial conditions **x**(0) are attracted. The set of initial conditions **x**(0) is called the *basin of attraction* associated with the particular time asymptotic set, and the set itself is called an *attractor*. A most remarkable feature of these attractors is that while they have zero volume, they may have a dimension $0< d_A <d$ which is fractional. Motion on such a *strange attractor* is typically nonperiodic in time.

The study of this nonperiodic motion, called *chaos*, has been the subject of abstract mathematical investigations as well as laboratory and numerical experimentation in physical, biological, chemical and other environments. The knowledge we have of such behavior has moved from the realm of analysis of given sets of differential equations and given maps to the stage where one can take experimental observations and determine characteristic features of the source of the signal. This «inverse» problem is critical for the application of ideas of nonlinear dynamics to real world issues. This article is devoted to the description of tools which are available for the systematic and quantitative analysis of measured data from laboratory experiments and field observations. Using these tools one can go from the observed data to model equations on the attractor. These are usable for prediction, control, and design of engineering systems in fields ranging from communications to fluidized bed flows. In this article we report on «methods that work» and allow one to analyze motions which appear complex when viewed one way and see them in other ways where they appear amenable to quantitative analysis and practical use.

Chaotic motion of physical systems - mechanical, fluid, electromagnetic, hydrologic, optical, ... - is known to occur as a pervasive feature of their motion [1,2]. To a traditional linear analyst of these signals the output of these systems when undergoing chaotic motion appears complex in the time domain and broadband in the Fourier domain. Broadband not only by virtue of occupying a wide band of frequencies but also by dint of being continuous in its spectral band. In other words, and this is the critical feature of chaotic spectra, it is continuous, broadband, so the motion in time domain is *non-periodic*. Nonetheless, the motion is totally deterministic and often that of a system which is composed of a small number of degrees of freedom. Chaotic signals represent a middle ground between (1) our traditional view of «noise» which is spectrally continuous broadband but not predictable and is composed of a very large number of degrees of freedom - in principle, an infinite number - and (2) our traditional view of acceptable signals which are regular, completely predictable, and spectrally composed of a number of sharp signals. The time domain version of such regular signals looks complex because of the presence of many sinusoids in it, but in Fourier domain its simplicity is revealed.

Chaotic signals are also simple when properly viewed, and the main thrust of this review is to discuss how to establish and then use the space in which this simplicity is revealed. The simplicity is essentially geometric, and using these geometric properties one can perform the usual tasks associated with signal processing:

- **signal separation** - given observations contaminated by a signal which is not of direct interest, how do we separate the signal of interest from the combination presented in the observation? If one of the signals is «noise», this is often called noise reduction, but it is quite important to recognize that separating one signal from another is the main problem. This opens up the use of many of the techniques we will discuss for communications applications. The task of separating signals when one is chaotic is both simpler and more difficult than in the conventional case where one signal is spectrally broad («noise») and the other is spectrally narrow («signal»). One uses the structure of chaos in state space to differentiate it from another signal, and the geometric and dynamical features of this structure allow one to separate the signals.
- **establishing the proper state space for the signal** - given an observation which is a single scalar quantity, how can we reestablish the essential features of the multivariate space which is required for chaotic motion? The analysis of this question will occupy the center of this paper, and in the absence of contamination of the signal is often enough to pin down many of the needed properties needed in model making and system identification.
- **extracting invariant characteristics of the system from the observed**

signal - sometimes called system identification. In linear systems we characterize the system producing a signal by the collection of narrow lines associated with resonant behavior of the dynamics. If the system is driven harder, the energy under such lines will change, but their frequency, as long as the system remains linear, is unchanged. Similarly if the system is started at a different time, the phase of the signal will be altered, but the characteristic lines in the spectrum will not change. In nonlinear, chaotic system, these lines are not present, so we turn to other characteristics such as fractal dimensions and Lyapunov exponents which are unchanged under changes in initial conditions or changes of coordinate system. These quantities allow one to identify the system which originates the signal. It is not known what constitutes a complete set of such invariants. Nonetheless, their use in characterizing the source of the chaos is clear.

• **model building in the state space - for prediction and control.** This is the main goal in engineering practice. One studies systems not to catalogue their fractal dimensions, regardless of how expert one may become at that exercise, but to establish a set of evolution equations which govern the motion and then use these equations for predicting future behavior of the system or for providing a framework within which to devise controls to make the system perform «better» according to some criterion established by the user. We will show how to use the neighborhood structure in the system phase space (or state space) to make local or global models of the dynamics revealed by the observations.

There is an important sense of model building which becomes evident when one views the geometric structure defined by orbits of a chaotic system: since one sees motion of the system only on the set of points in state space to which all orbits are attracted (a strange attractor in the case of chaotic motions) after system transients have died out, the natural model one can build for evolution of the system will evolve it along the attractor or perhaps within the whole basin of attraction. The attractor is typically located on a small subspace of the original system state space, so one should not expect to be able to determine the original differential equations (partial or ordinary) which govern the system dynamics in its larger state space -infinite dimensional in the case of partial differential equations. Instead one should expect to be able to determine only an effective set of equations which allows the analysis of motion on the attractor subspace alone. The effective equations may not have any global analytic expression in the phase space, but may be composed of a large collection of local evolution rules having no expression beyond a kind of lookup table on a computer hard disk. This redefines the traditional view of finding some version of Newton's equations as the goal of the signal analysis. Since one can use the effective model to perform most of what can do with the differential equations, substituting the effective equations for analytic expressions does not diminish the utility of the analysis for practical tasks.

Another important use of the analysis tools is to determine the appropriateness of a proposed set of differential equations for the description of the observations. One cannot compare the detailed orbits (time series) of any of the variables in the solution of the differential equations with the observed chaotic orbits because chaotic motion is associated with unstable orbits throughout the system state space so any roundoff error or difference in initial conditions is exponentially amplified in chaotic motion. Any two orbits of the same system are uncorrelated in any sense - linear or nonlinear - after a characteristic time. Comparison of the output from the model with data must be carried out on other terms: comparison of the attractor properties such as fractal dimensions or Lyapunov exponents or other invariant aspects of the dynamics. Even though the system may be completely deterministic, the essential instabilities which underlie the chaotic behavior means that the terms of comparison are statistical.

It is the purpose of this brief review article to explain many of these statements in enough detail and with sufficient elaboration so that the reader can fully understand how to go about the kind of time series analysis we describe. We will stick to practical algorithms and give examples of their application to quite varied data sets. The algorithms described here all run on standard serial workstations and, depending on the length of the data sets and the dimension of the state space, run in minutes to a few hours. None of the computation is prohibitive and much of it can be parallelized with accompanying speed-ups in execution.

Many topics will be passed by in this short article. We do not touch at all on the applications of the methods to medical or biological problems, though the tools are general enough to be useful there without any alteration. We focus on establishing the proper state space for the signal and extracting invariant characteristics of the system from the observed signal - two of the items from our list above. Signal separation, though not difficult, requires some ideas beyond what is useful to present in this kind of introduction. We will only touch on the ideas in model building since that is both the easiest and the hardest part of the whole process: it is easy to construct models which work; it is difficult to construct models which encompass aspects of the physics one is trying to capture. The development of efficient and usable models is a wide open subject which knows no hard and fast rules. It is unlikely there will be a «handbook» of models to use for this or that data. Much will depend on the interests and goals of the user, and that makes the subject rich indeed. We will outline what is known and works, but we expect that to grow rapidly with the application of the methods to a wide variety of practical problems. Some of the topics not contained here are touched on in [2,3] and additional detail will be found in [4].

The analysis of chaotic time series is by no means a closed or finished subject, so what we write on many of these topics will be superseded over the next few years as one applies the general viewpoint to numerous real world and practical problems. Out of this we anticipate will emerge a set of practices moving within the general sort of guidelines we are able to present and even more by what works and what doesn't work in practical situations. At this stage we are able to content ourselves with a consistent set of successful applications of the framework without having either the chutzpah or necessity to dictate how *all* problems should be approached or solved when dealing with nonlinear chaotic systems.

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