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When Integers Embrace the Beauty of Complex Numbers

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1. Introduction

Exploring the nature of reality requires addressing the challenge of precision. What happens if we abandon the illusion of the infinitesimal and consider a fundamentally discrete reality? This article explores that possibility, where computational mathematics provides clues about the underlying structure of the world. Within the discrete domain, a defined and elegant geometry emerges, revealing ordered patterns that defy conventional interpretations based on noise and numerical error.

Traditionally, physics and mathematics have relied on the assumption that reality is continuous, allowing for infinitely small divisions in space, time, and other magnitudes. The continuous conception of physical reality has yielded many results. The discipline of mathematical analysis, originating in the 17th century with Fermat, Newton, and Leibniz [19, §6.1], has enabled the attainment of valuable predictive models. However, not all scholars accept the continuity of all magnitudes. Finitistic mathematicians argue that the continuum is a mathematical fiction with no representation in the real world [11, Chapter 3]. Contemporary physics has begun to explore the possibility that nature may be fundamentally discrete, with space and time having a minimal granular structure. Quantum theory points to the existence of a minimum distance in the universe, below which no measurement is possible [20–22]; string theory posits a lower limit marked by the Planck length [10]. The real number set is an infinitely dense structure that cannot be represented in any finite computational system [9, 13]. If instead reality is discrete and finite, this challenges the validity of continuous models and suggests that our knowledge may be shaped as much by the tools we use as by the phenomena themselves.

Precision plays a crucial role in understanding the physical world. Measurement errors in nonlinear systems can lead to significant issues, as inaccuracies grow exponentially with truncation and rounding [6, 26]. If reality is indeed discrete and finite, determining its granularity could enable an exact representation of its properties, eliminating certain types of errors intrinsic to continuous models. Furthermore, the predictability of dynamical systems depends on the precision with which their initial conditions are known; even infinitesimal inaccuracies can lead to divergent outcomes over time [4]. In quantum mechanics, due to Heisenberg's uncertainty principle [8], this limitation in measurement is ontological rather than merely epistemological. The precision of measurement may therefore be constrained not only by technology, but also by the discrete nature of reality itself.

Mathematics serves both as a tool and as a candidate for the fundamental language of reality. The assumption that continuous mathematics is the best framework for describing the universe has guided modern physics, from Newtonian mechanics to quantum field theory. However, if reality is fundamentally discrete, a new mathematical approach may be necessary: one that replaces continuous functions with recursive, finite arithmetic models more faithfully aligned with the universe's actual structure [25]. In such a discrete framework, emergent behaviours can arise from simple rules, revealing symmetries and regularities that would otherwise remain hidden [15]. The presence of these emergent structures suggests that continuity may be an illusion, a large-scale effect emerging from fundamentally discrete interactions.

One of the most fascinating consequences of a discrete approach is the emergence of patterns and structures that are not apparent in continuous models. To illustrate these ideas, this article presents two case studies: one on the logistic map and another on the Mandelbrot set. These examples show how fixed-point arithmetic reveals hidden patterns and symmetries that remain obscured under traditional floating-point simulations. The findings suggest that discrete mathematics may provide a more faithful and structured representation of complex systems, potentially leading to a paradigm shift in how we model physical reality.

The remainder of this paper is organized as follows. Section 2 introduces the fixed-point data type used in our computations. The first part of the study includes Sections 3 and 4, which present the case studies on the logistic map and the Mandelbrot set, highlighting the emergent patterns observed through discrete arithmetic. The second part, comprising Sections 6 and **??**, discusses the broader implications of these findings, including the possible need for a new mathematical framework based on discrete principles. Finally, Section 7 offers concluding remarks on the potential impact of this perspective for future research in physics and computational modelling.

1.1. Objectives of the Study This article aims to explore the consequences of abandoning the traditional assumption of continuity in favor of a fundamentally discrete view of physical reality. The main objectives of this work are:

- To argue that discrete, finite mathematical structures can reveal emergent patterns that are often overlooked by continuous models.
- To present two case studies —the logistic map and the Mandelbrot set— demonstrating that discrete arithmetic introduces structured behavior, not merely numerical error.
- To propose a new modelling framework based on integer-valued computations that avoids rounding and bias, potentially redefining how precision is handled in physical systems.
- To introduce and analyze the concepts of *symmetric points* and *emergent scalar symmetry* as manifestations of moiré-type interference in discrete dynamical systems.

1.2. Contributions and Novelty This paper makes the following original contributions:

- It introduces the concept of moiré interference patterns as emergent structures resulting from the interaction between discrete measurement grids and the underlying arithmetic of recursive systems.
- It constructs a fixed-point data type and applies it to well-known chaotic systems, showing that discrete arithmetic can reveal ordered behavior that remains hidden in continuous formulations.
- This highlights the need to develop a mathematics based on new paradigms. Two examples have demonstrated how working with recursive and discrete models reveals novel behaviors, which are overshadowed by continuous and linear mathematics.

Compared to previous works [1–3], which focused on the visual and empirical properties of these patterns, this article expands the analysis by formalizing the computational structures, symmetry classes, and theoretical implications within a broader philosophical and physical context.

2. Fixed-point data type

When using a computer, whether working with floating-point arithmetic (e.g., IEEE754 [12]) or with the fixed-point data type presented here, results are never exact. By 'exactness' we refer to the symbolic precision of mathematical expressions. Saying π is not the same as saying $3.141592\ldots$: no matter how many digits we compute, we cannot fully represent its value, if such exactness even exists.

This limitation is well known and widely studied. So much so that listing references, even from the last year alone, would be impractical.

The aim of this article is not to analyze rounding or bias errors, but to highlight a deeper issue: what if these errors arise not from computational limits, but from the mismatch between continuity-based models and a fundamentally discrete reality? If reality is recursive and discrete, then determining its granularity becomes crucial. In such a context, accuracy would no longer concern numerical approximations of $\mathbb{R}$, but the finite set of actual values that constitute the fabric of physical reality. Without real-number approximations, bias and rounding errors disappear. What then emerges? The case studies below aim to offer partial answers.

All simulations presented here were implemented in C# using a custom fixed-point data type, which is briefly introduced in this section.

Each value is defined by:

- `raw` — a 64-bit signed integer (`Int64`),
- `precision` — a 16-bit unsigned integer (`UInt16`) specifying how many least-significant bits are interpreted as fractional.

Each fixed-point value is interpreted as (p: `precision`) $x = \texttt{raw}/2^p$.

This allows representing rational numbers with denominators of the form 2^p, ranging from -2^{63-p} to $+2^{63-p} - 1$. Between any two integers, the representation includes 2^p evenly spaced values. For example, with `raw = 0x4`, we obtain 4.0, 2.0, 1.0, 0.5, ... down to 0.03125 depending on `precision` in $\{0, \ldots, 7\}$. Conversely, the value 0.5 can be encoded by `raw = 0x01, 0x02, ...` up to `0x80` depending on `precision` from 1 to 8.

Increasing precision by 1 doubles the number of representable values between integers. For instance, between $k/2^p$ and $(k+1)/2^p$, the midpoint $(2k+1)/2^{p+1}$ is introduced.

Although rationals such as 1/3 are not exactly representable, high-precision approximations are possible: with `precision = 15`, $1/3 \approx 0.33325...$ encoded as `0x2AAA`; with `precision = 60`, the representation is `0x55...55`, encoding 0.333...330442....

We omit implementation details here. Overflow and rounding are implementation dependent, though the phenomena discussed in this article appear regardless of the chosen representation, whether fixed-point or IEEE754 [2, §3]. Implementation specifics can be found in [3, §3.2].

Importantly, in all simulations presented here —with `precision` up to 32 bits— overflow never occurred. The fixed-point arithmetic remains closed, deterministic, and free of floating-point artifacts.

Additional details on the arithmetic logic, including bit-level implementation and recursive behavior, are provided in Appendix 7.1.

3. Case study on the logistic map

As a first case study, we analyze the behavior of the logistic map and its bifurcation diagram under discrete arithmetic. The expected occurs: when periods and bifurcations are calculated using a fixed-point data type, the results differ from those predicted by analytical methods. And the unexpected also occurs: the hypothetical "errors" introduced by finite and discrete encodings exhibit a surprising degree of structure and harmony. Perhaps it is precisely this structure —information that emerges when reality is treated as both recurrent and discrete— that deserves attention.

Definition 1. *The logistic map is a discrete-time dynamical system. It models iterative processes like population dynamics with limited resources.*

$$F_\lambda(x) = \lambda\, x\, (1.0 - x) \tag{1}$$

In this equation, x represents a normalized population size (typically $x \in [0, 1]$), and λ is a positive parameter ($\lambda \in [0, 4]$) controlling the growth rate.

Iterating the map —$x_{i+1} = \lambda\, x_i\, (1.0 - x_i)$— demonstrates various long-term behaviors depending on the value of λ: from stable fixed points and periodic cycles to highly sensitive dependence on initial conditions, characteristic of deterministic chaos. As such, it serves as a foundational model in chaos theory and for exploring phenomena like bifurcation cascades and the emergence of complexity in population dynamics and other fields.

3.1. Background Concepts The definitions included in this subsection are well-known in the study of dynamical systems. They are provided here to support readers less familiar with the context.

Definition 2. *(cfr. [5, Definition 3.1.])*
*Given a discrete process, defined by the successive iteration of a function $f(x)$ initiated at the value x_0, the sequence of points $x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots$ is called the **orbit** of x_0.*

Example 1. *For $\lambda = 2.1$ and $x_0 = 0.25$, the iterative sequence evolves as follows: $x_1 = 0.393750$, $x_2 = 0.501293$, $x_3 = 0.524996$, $x_4 = 0.523688$, $x_5 = 0.523822$, $x_6 = 0.523808$, $x_7 = 0.523810$, $x_8 = 0.523810$, $x_9 = 0.523810$, and so on.*

*This sequence exemplifies the convergence of the iterative process towards a **fixed point**, observed here at approximately $x \approx 0.523810$.*

Definition 3. *(cfr. [5, Definition 3.2.])*
*A point x is a **fixed point** if $f(x) = x$; it is a **periodic point** of period n if $f^n(x) = x$.*

A fixed point represents an equilibrium state where, if the population (or any modeled variable) reaches that value, it will remain constant. A periodic point indicates that the population does not stabilize at a single value but oscillates cyclically among a set of n distinct values.

Example 2. *For $\lambda = 2.0$, $x_0 = 0.5$ is a fixed point: $x_1 = 2.0 \times 0.25 = 0.5$.*

The logistic map $F_\lambda(x) = \lambda\, x\, (1.0 - x)$ has two fixed points: $p_1 = 0$ and $p_2 = (\lambda - 1)/\lambda$.

The stability of a fixed (or periodic) point determines whether an orbit starting nearby tends to approach or move away from that point. To classify this stability, we use the magnitude of the function's derivative at the fixed point (or the n-th iterate of the function for periodic points).

Definition 4. *(cfr. [5, Definitions 4.1., 4.5., and 4.7.])*
*A periodic point x of period n is called an **attracting point** (or **attractor**) if $|(f^n)'(x)| < 1$, and a **repelling point** (or **repellor**) if $|(f^n)'(x)| > 1$.*

- An **attractor** is a stable equilibrium point: if the population deviates slightly from it, the function's iterations will guide it back towards this point. The orbit converges to the attractor.
- A **repeller** is an unstable equilibrium point: if the population deviates slightly from it, the iterations will push it further away from that point. The orbit diverges from the repeller.

The derivative of the logistic map $F_\lambda(x)$ is $F'_\lambda(x) = \lambda(1 - 2x)$.

Example 3. *At the fixed point $p_1 = 0$, the derivative is $F'_\lambda(p_1) = \lambda$. Therefore, $|F'_\lambda(p_1)| = \lambda$*

- *If $\lambda < 1$: $p_1 = 0$ is an attractor. This implies that, with a very low growth rate, the population will tend to extinction.*
- *If $\lambda > 1$: $p_1 = 0$ is a repeller. In this case, if the population is minimally greater than zero, it will not go extinct but will start to grow.*

Example 4. *For the fixed point $p_2 = (\lambda - 1)/\lambda$, the derivative is $F'_\lambda(p_2) = 2 - \lambda$. Thus, $|F'_\lambda(p_2)| = |2 - \lambda|$*

- *If $1 < \lambda < 3$: p_2 is an attractor (since $|2 - \lambda| < 1$). Within this range of λ, the population tends to stabilize at a positive, non-zero value, as observed in Example 1. This is supported by [5, Proposition 5.3] and [17].*
- *If $\lambda > 3$: p_2 is a repeller (since $|2 - \lambda| > 1$). When λ exceeds 3, this equilibrium point loses its stability, triggering more complex dynamic behaviors including higher-period cycles and, eventually, chaos.*

A particularly interesting scenario occurs when the absolute value of the derivative is exactly 1, i.e., $|f'(x)| = 1$. In these situations, the first derivative criterion alone is inconclusive, and the fixed point is considered marginally stable or neutral. These are the critical values of the parameter λ where the dynamic system undergoes bifurcations: qualitative changes in its behavior as λ varies.

- When $\lambda = 1$: The two fixed points, $p_1 = 0$ and $p_2 = (\lambda - 1)/\lambda$, merge at $x = 0$. At this point, $F'_\lambda(0) = 1$. While $x = 0$ remains an attractor, its attraction is "weak" or "neutral" and the system undergoes a transcritical bifurcation. This means that as λ crosses 1, the fixed point $x = 0$ changes from being an attractor (for $\lambda < 1$) to a repeller (for $\lambda > 1$), yielding its role as an attractor to p_2.

- When $\lambda = 3$: The fixed point $p_2 = 2/3$ has $|F'_\lambda(p_2)| = 1$. At this point, the stability of the fixed point p_2 is lost, and the system experiences a period-doubling bifurcation (or "flip bifurcation"). The population no longer converges to a single fixed point but begins to oscillate between two distinct values, forming a period-2 cycle. This is the first step in a cascade of period-doublings that, as λ increases, will lead to the chaotic behavior of the logistic map.

Thus, the function F_λ exhibits rich dynamics:

- p_1 is an attractor when $\lambda < 1$.
- p_2 is an attractor when $1 < \lambda < 3$.
- For $\lambda \geqslant 3$, the fixed points lose their stability, leading to higher-order periodic cycles and, finally, to chaos.

Definition 5. *(cfr. [5, §2.10.], [23])*
Sarkovskii Ordering

$$
\begin{aligned}
&3 \rhd 5 \rhd 7 \rhd \cdots \rhd \\
&2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd \cdots \rhd \\
&2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \rhd \cdots \rhd \\
&2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd 2^3 \cdot 7 \rhd \cdots \rhd \\
&\cdots\cdots \\
&2^3 \rhd 2^2 \rhd 2^1 \rhd 2^0
\end{aligned}
$$

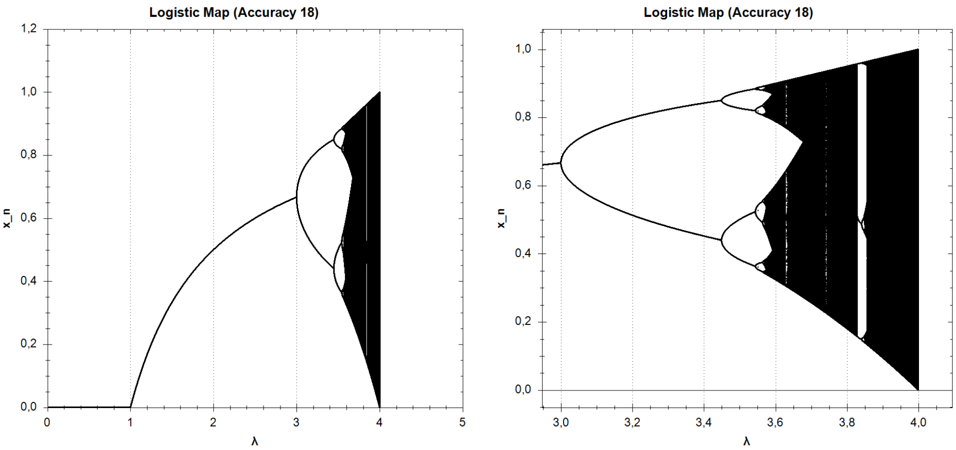
Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, Sarkovskii's Theorem states that if f has a periodic point of prime period k, and $k \rhd l$ in the above order, then f also has a periodic point of period l [5, Theorem 10.2].

From Sarkovskii's ordering and theorem, we can draw two significant conclusions (cf. [5, §1.10]):

1. If a system has a periodic point whose period is not a power of two, then it necessarily has infinitely many periodic points. Conversely, if all periodic points are finite in number, their periods must be powers of two.
2. If a function has a point of period 3 (the highest element in the ordering), then it also has points of every other period. This is the essence of the famous result "period three implies chaos" by Tien-Yien Li and James Yorke [16].

The evolution of periods as λ increases can be visualized using a **bifurcation diagram**, as shown in Figure 1, where the horizontal axis represents λ values, and the vertical axis shows the final values of orbits under F_λ. The logistic map's bifurcation diagram graphically plots the attracting period (fixed point or periodic orbit) where any given orbit eventually settles for each value of the parameter λ. It thus illustrates the universal transition from stable states to period-doubling and deterministic chaos, independent of the initial x_0.

Equation 1 has been used to illustrate many foundational concepts in chaos theory. Robert May [17] was among the first to observe that, starting from $\lambda = 3$, the period of orbits begins to double with increasing λ, following the initial entries of Sarkovskii's ordering specifically, the powers of 2. Both May and Yorke noted this period-doubling behavior and how it occurs



(a) (b)

Рис. 1. Bifurcation diagram of the logistic map computed with `precision = 18`. (a) Global view; (b) zoomed-in view for λ between 3 and 4.

over progressively smaller intervals of λ. Mitchell Feigenbaum [7, Appendix A4] calculated the constant that bears his name, defined as the limiting ratio of successive bifurcation intervals:

$$\delta = \lim_{x \to \infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \approx 4.66920160910299067 \tag{2}$$

(value taken from [14, 27]).

The **Feigenbaum constant** (δ) is a universal mathematical constant central to chaos theory. It quantifies the rate at which successive **period-doubling bifurcations** occur as a non-linear dynamical system transitions from stable to chaotic behavior. In Eq. 2, λ_n denotes the parameter value at which the n-th period-doubling bifurcation occurs. This constant highlights a remarkable **universality**: it is found in any one-dimensional, unimodal map exhibiting this route to chaos, signifying a profound quantitative regularity in the emergence of complex dynamics across diverse systems.

3.2. Period variation under discrete encoding The behavior of orbits under the logistic map depends both on the initial condition x_0 and on the level of precision used in the arithmetic. With fixed-point arithmetic, these factors can significantly affect the final orbit, its period, and even the apparent dynamics of the system.

As a first example, consider `precision = 3`, which allows eight values between consecutive integers. For a fixed value of λ, different initial values x_0 can lead to different attractors. For instance, with $\lambda = 4$ (`0x20`), initial values such as $1/8$, $3/8$, $5/8$, and $7/8$ all converge to a period-2 cycle; while others, such as $1/4$ or $3/4$, reach fixed points.

Precision itself also plays a decisive role. For example, with $x_0 = 1/8$ and $\lambda = 3 + 3/8$, we observe:

- `precision = 3`: the orbit enters a period-2 cycle.
- `precision = 4`: the orbit converges to a fixed point.
- `precision = 5 - 6`: longer-period cycles appear.

As λ increases, the diversity of observed periods becomes more pronounced. The bifurcation structure, classically tied to the continuous limit, now reveals strong sensitivity to encoding.

3.3. Emergence and distribution of period 3 A particularly significant moment in the logistic map is the first appearance of an orbit of period 3, which —by Sarkovskii's Theorem—

implies the presence of all other periods. In fixed-point arithmetic, the first appearances of period-3 orbits vary depending on both precision and the value of x_0.

For example, with `precision` = 4 and $x_0 = 1/4$, period-3 cycles are detected at $\lambda = 3{+}3/4$, $3 + 13/16$, $3 + 7/8$, etc. As precision increases, the number of λ values yielding period-3 orbits also increases rapidly (Table 1).

This dependency is not random. It reflects the interplay between arithmetic precision and the discretization of both x_0 and λ. The presence or absence of chaos in this context cannot be treated as a sharp boundary (as in the continuous case), but rather as a property distributed across the arithmetic space.

Таблица 1Observed period 3 orbits for different values of *lambda* and fixed point precision. Working with fixed point and low precisions, the behavior of the logistic map differs depending on the initial value x_0. The logistic map plot has been calculated for all possible initial values x_0 at each precision: the table shows at which values of x_0 (encoded in hexadecimal) the period three has appeared, and how many times. There are values of x_0 for which period three does not appear at all. Columns (1): precision; (2): # λ with period 3; (3): λ values.

(1)	**(2)**	**(3)**	
4	5	$3 + 3/4$: `0x3C` (twice)	$3 + 13/16$: `0x3D` (3 times)
		$3 + 7/8$: `0x3E` (3 times)	$3 + 15/16$: `0x3F` (5 times)
		4 : `0x40` (6 times)	–
5	0	–	
6	6	$3 + 53/64$: `0xF5` (3 times)	$3 + 27/32$: `0xF6` (6 times)
		$3 + 55/64$: `0xF7` (8 times)	$3 + 7/8$: `0xF8` (22 times)
		$3 + 57/64$: `0xF9` (7 times)	$3 + 29/32$: `0xFA` (8 times)
7	6	$3 + 55/64$: `0x1EE`	$3 + 111/128$: `0x1EF`
		$3 + 7/8$: `0x1F0`	$3 + 113/128$: `0x1F1`
		$3 + 57/64$: `0x1F2`	$3 + 115/128$: `0x1F3`
8	5	(not listed in detail)	
9	9	(not listed in detail)	
10	12	(not listed in detail)	
11	21	(not listed in detail)	
16	576	(not listed in detail)	

3.4. Highlights and implications The logistic map's classical bifurcation diagram predicts the first period-doubling at $\lambda = 3$. However, under finite fixed-point encoding, this bifurcation occurs at lower values, and the system exhibits a period-2 orbit long before $\lambda = 3$.

Figure 2 shows how this behavior unfolds with `precision` = 14. A total of 16384 rational values are encoded between any two integers. The first bifurcation (period doubling) occurs at $\lambda = 1 + 10267/16384 = 1.626647\ldots$ (`0x681B`). A zoom near $\lambda = 3.0$ shows that period-2 cycles persist for a wide range of λ values, forming a stable, ordered structure.

These diagrams diverge from classical predictions not due to computational errors, but because we are not working with real numbers: the system operates over a finite, uniformly distributed subset of rationals. Within this discrete space, dynamical behavior is not chaotic in the traditional sense, but highly structured.

The emergence of period-3 orbits at lower than expected values of λ —and the sensitivity to both x_0 and precision— suggest a new paradigm for interpreting complexity. The classical bifurcation framework may need reinterpretation when arithmetic itself becomes part of the system's ontology.

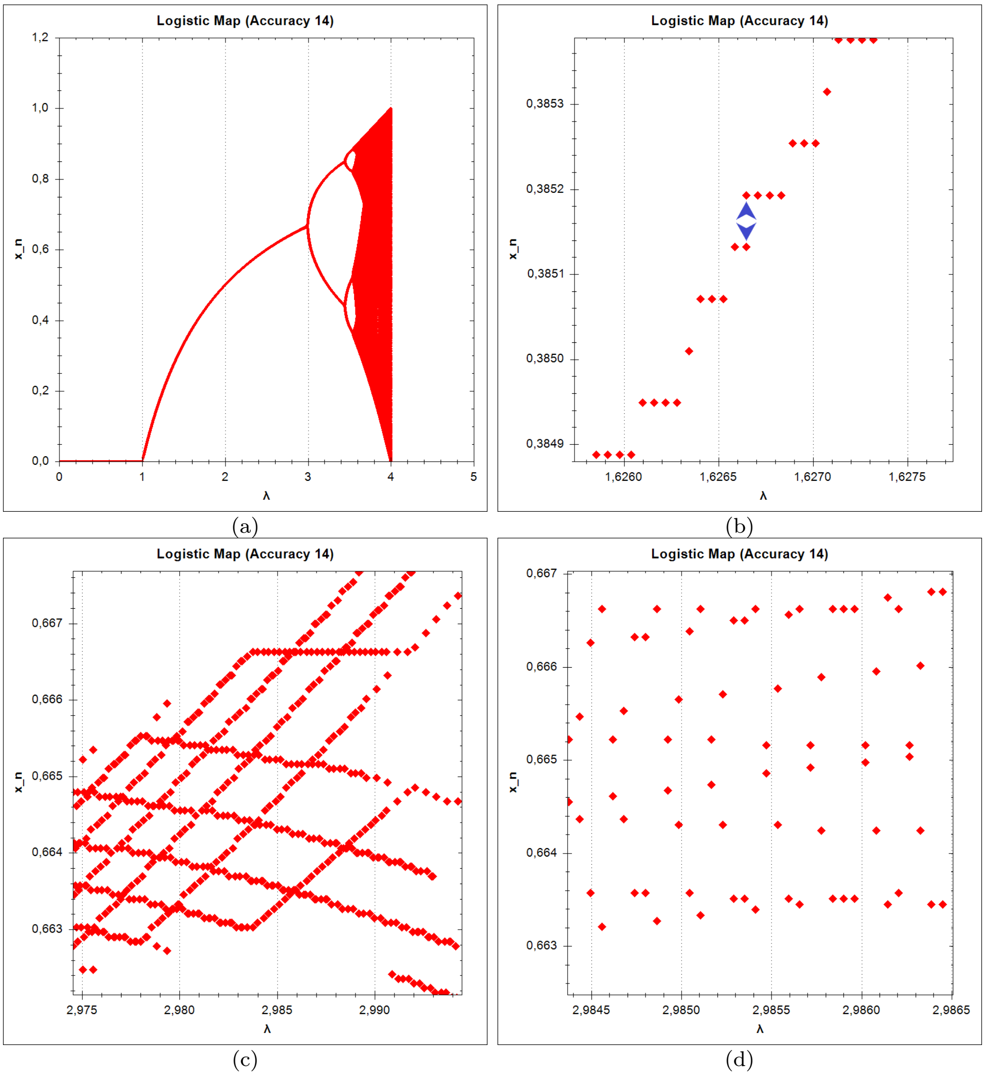


Рис. 2. Views of the bifurcation diagram of the logistic map, computed with `precision = 14`.

Future work may include studying the evolution of Feigenbaum constants under varying precisions, the behavior of Lyapunov exponents in discrete spaces [18, 24], and discrete analogs of Sharkovskii's ordering.

4. Case Study on the Mandelbrot Set

The Mandelbrot set (henceforth *M-set*) is a paradigmatic object in the study of complex dynamics. Each point $c \in \mathbb{C}$ defines an orbit through the iterative process $z_{n+1} = z_n^2 + c$, with $z_0 = 0$. A point c belongs to the M-set if this orbit remains bounded under iteration.

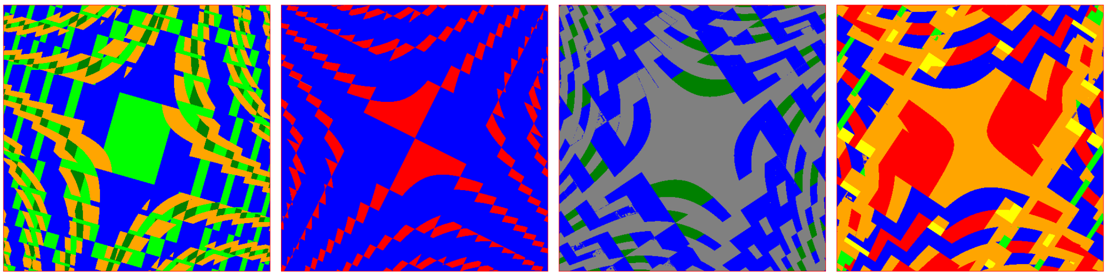
In this study, we examine the discrete *period map* of the M-set, computed using fixed-point arithmetic at various levels of precision. Unlike the continuous case, which produces uniform colorings across hyperbolic components (each corresponding to a single orbit period), our discrete simulations reveal highly structured and reproducible interference patterns. These are not numerical artifacts, but emergent features arising from the interaction between the sampling grid and the arithmetic structure of the iteration.

This behavior is analyzed in detail in [3], where it is shown that discrete sampling interacts with the modular structure of the recurrence, giving rise to *moiré interference patterns*. These patterns are organized around specific locations in the complex plane —referred to as *symmetric points*— which act as centers of discrete periodic organization.

Around each symmetric point, a finite set of distinct geometric arrangements —called *configurations*— emerge. These are governed by a characteristic parameter C, known as the *class divisor*. For a fixed C, the configurations are grouped into *classes* defined by the residue `IncR` mod C, where `IncR` is the increment used to sample the complex plane. Transitions between configurations follow modular arithmetic laws.

Rather than reproducing the full classification introduced in [3], this article focuses on its implications: the emergence of order, invariance, and symmetry from a purely discrete computational system; a phenomenon that is obscured in continuous formulations.

Figure 3 shows four representative configurations near symmetric points, for different values of C and `IncR`. Each image was computed using fixed-point arithmetic at consistent resolution. The regularity of each configuration is determined entirely by the congruence class `IncR` mod C, illustrating the central role of modularity.



(a) (b) (c) (d)

Рис. 3. Representative configurations around symmetric points in the Mandelbrot Set, computed with fixed-point arithmetic. **(a)** Configuration 4.4 ($C = 52$, `IncR` $= 0xD4$); **(b)** 6.6 ($C = 6$, `IncR` $= 0xF0$); **(c)** 13.13 ($C = 13$, `IncR` $= 0xB6$); **(d)** 2.2 ($C = 6$, `IncR` $= 0xD0$). Each configuration is uniquely determined by the congruence class `IncR` mod C, and exhibits modular symmetries and regularities that reflect the arithmetic structure of the sampling.

4.1. Scalar Symmetry under Emergence A remarkable property of these configurations is their behavior under resampling. When `IncR` is a multiple of the class divisor C, and the sampling grid is downsampled by a factor F, the same configuration reappears. This phenomenon —termed *emergent scalar symmetry*— shows that a structure can manifest at multiple scales with identical geometry, governed by modular arithmetic alone.

Figure 4 illustrates this effect for configuration 20.20. The leftmost panel shows the original image ($F = 1$), and the subsequent panels show the same configuration re-emerging under grid downsampling with $F = 2$, 5, and 10. Despite the reduction in resolution, the structure is preserved exactly, confirming the symmetry induced by discrete modular rules.

5. Implementation and Finite-Precision Effects

The numerical simulations of the logistic map were performed using a custom-developed class that implements 64-bit fixed-point arithmetic. This approach was chosen to explicitly control the effects of finite precision. A key feature of our implementation is the ability to set a variable parameter that controls the number of bits allocated to the fractional part of the number, allowing us to precisely define the simulation's precision within a range of 5 to 32 bits.

The core iterative equation for the logistic map was implemented as follows:

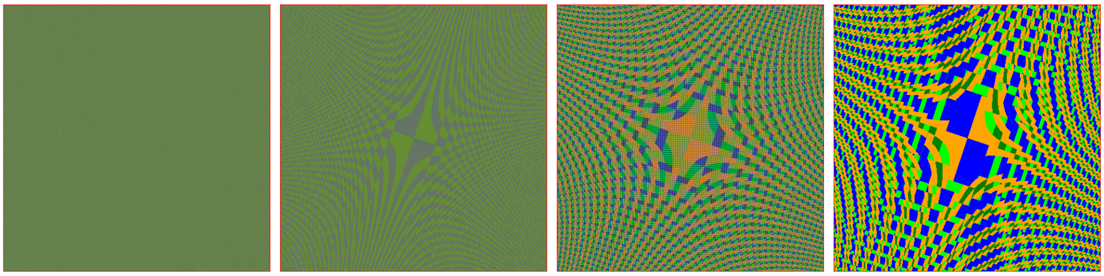


Рис. 4. Example of **emergent scalar symmetry** in configuration 20.20. From left to right: original image ($F = 1$), and emergent images obtained by selecting one row and column out of every $F = 2$, 5, and 10, respectively. The structural invariance across scales arises from the arithmetic condition $\texttt{IncR} \equiv 0 \mod C$.

```
% Pseudocode of the logistic map iteration
for i from 0 to N:
  x[i + 1] = lambda * x[i] * (1 - x[i])
```

While mathematically equivalent in a continuous domain, different formulations of the logistic map equation are not identical under finite precision. An alternative implementation, for example, could be:

```
% Pseudocode of the alternative implementation
for i from 0 to N:
  x[i + 1] = lambda * x[i] - lambda * x[i] * x[i]
```

The first variant, $x_{next} = \lambda x_{current}(1 - x_{current})$, involves one subtraction and two multiplications. The second variant, $x_{next} = \lambda x_{current} - \lambda x^2_{current}$, performs two multiplications and one subtraction. The accumulation of rounding errors after each operation can lead to subtle variations in the resulting trajectory, which could manifest as different visual patterns or attractors in the simulations.

The analysis of such finite word length effects is a well-known problem in digital signal processing and control systems. The delta operator, for instance, has been proposed as a substitute for the conventional shift operator to improve filter performance and reduce sensitivity to finite precision [?, ?]. Our findings are consistent with this body of work, as they demonstrate that the specific implementation and the underlying arithmetic structure directly influence the emergent dynamics of the system. Therefore, the choice of a specific arithmetic and algorithm formulation is not a secondary detail but an integral part of the system's behavior.

6. What if precision is the essence?

The results presented in Sections 3 and 4 suggest that the use of discrete-variable mathematics and finite arithmetic can reveal an emergent order that is not the result of randomness but of the intrinsic structure of integer-based representations. These patterns highlight properties concealed within traditional continuous models.

6.1. Emergence in Discrete Systems In this study, emergence is a descriptive framework for the structured behaviors observed in our simulations, where discrete recursion gives rise to patterns not evident in continuous models. These emergent properties are not mere artifacts but reveal a new layer of structure.

6.2. The Emergence of Moiré Patterns: A Window into a Structured Reality The moiré patterns we observe are often dismissed as a simple interference artifact from the superposition of two periodic processes—the internal dynamics of the system and the external grid imposed by computation. However, in our simulations, these patterns are not a trivial subproduct. They are a direct visual manifestation of the **intrinsic arithmetic structure** of the computation itself.

We propose that these patterns arise from the interaction between an underlying "reality grid" of the finite arithmetic domain and our "observational grid" imposed by the simulation. While the configuration of the underlying grid remains hidden, the resulting moiré pattern is an emergent structure generated by this interaction. This emergent geometry reveals properties and symmetries that are absent in both continuous and trivial discrete models, providing a structured view of the system's dynamics that would otherwise remain inaccessible.

This interpretation aligns with the visual configurations observed in the discrete Mandelbrot period map (Section 4) and the logistic map, where moiré-like patterns and modular symmetry emerge as reproducible features of the arithmetic structure. Our findings suggest that the finite nature of numbers and the operations of truncation are not just sources of error, but constructive elements of the system, offering a new perspective on the organization of chaotic systems.

7. Conclusions

This article has suggested that fixed-point arithmetic, when applied to discrete recursive systems, can reveal an emergent order that is concealed within continuous-variable models. Through the simulation of the logistic map and the Mandelbrot set, we have observed that geometric regularities and modular symmetries are not merely artifacts of rounding errors but appear to be intrinsic features of the underlying arithmetic structure.

Our findings support the notion that finite-precision effects are not exclusively a source of error or a limitation of digital simulation. Instead, they may serve as a powerful tool for discovering new layers of structure in nonlinear systems. The emergent moiré-type patterns, in particular, provide a compelling visual example of how the interaction between the system's dynamics and a discrete computational framework can reveal a structured reality.

Furthermore, we have shown that the specific implementation of the iterative algorithm, though mathematically equivalent in the continuous domain, can significantly influence the resulting dynamics. This highlights the importance of analyzing not only the theoretical models but also their concrete computational realizations.

In conclusion, our work supports a paradigm shift from a continuous to a discrete perspective in the analysis of nonlinear dynamics. By embracing the finite nature of numbers, we can uncover a new set of properties and behaviors that would otherwise remain hidden, opening new avenues for research in fields ranging from computational physics to structured system design.

7.1. Future Research Directions and Applications Based on the findings presented in this article, several promising avenues for future research emerge. First, further investigation is needed to explore the implications of finite-precision arithmetic in a broader range of nonlinear dynamical systems, including those with higher dimensionality. This could lead to the discovery of new emergent behaviors and symmetries.

Second, the structured moiré patterns observed in our visualizations present a unique opportunity for inverse problem research. It would be valuable to develop a mathematical framework to deduce the properties of a hidden arithmetic substrate from the analysis of the emergent interference patterns.

Finally, we plan to conduct a systematic comparison of different algorithmic implementations, including those based on both fixed-point and floating-point arithmetic. The goal is to quantify the impact of each on the long-term dynamics of a system and to establish a set of best practices for computational modeling of chaotic phenomena.

Appendix A. Implementation Details

A.1 Fixed-Point Data Type All simulations in this paper were performed using a fixed-point representation implemented in C#. The fixed-point type consists of two components:

- `raw`: a signed 64-bit integer (`Int64`) that encodes the numerical value.
- `precision`: an unsigned 16-bit integer (`UInt16`) that determines how many of the lower bits in `raw` are interpreted as the fractional part.

Each value is interpreted as:

$$x = \frac{\texttt{raw}}{2^{\texttt{precision}}}$$

Only rational numbers whose denominator is a power of two can be represented exactly. Increasing the value of `precision` refines the granularity between two consecutive integers.

A.2 Arithmetic Operations The arithmetic follows exact integer operations:

- Addition and subtraction are performed directly on the `raw` values, assuming the same `precision`.
- Multiplication involves scaling correction: the product of two `raw` values is right-shifted by `precision` bits.
- Division reverses the process, with a left shift before performing integer division.

No rounding, truncation, or floating-point computations are used. Overflow behavior is deterministic within 64-bit limits and no bias correction is applied.

A.3 Orbit Computation Procedure For both the logistic map and the Mandelbrot set:

- Initial values (x_0, λ, or c) are encoded as fixed-point values at a selected `precision`.
- Iterative formulas are applied using only integer arithmetic:

$$x_{i+1} = \lambda \cdot x_i \cdot (1 - x_i) \qquad \text{or} \qquad z_{n+1} = z_n^2 + c$$

- Period detection is done by storing previous orbit points and comparing them at each iteration.

A.4 Grid Exploration and Increment Parameter To explore the period space:

- Values of λ (for the logistic map) or c (for the Mandelbrot set) are swept across a range using a uniform grid.
- The grid increment is determined by a fixed `raw` increment and the current `precision`.
- The behavior of symmetric points and their configurations depends on the class divisor C and on the value of `raw` mod C.
- For a given symmetric point, the resulting image configuration is independent of `precision` and determined only by the raw encoding of the increment.

Figures were generated using a range of `precision` values from 14 to 32, and increment steps carefully selected to isolate distinct configurations.

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