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Eliminating transverse vibrations on an elastic beam using absorbers

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Abstract. The *purpose* of this study is to investigate the optimization of parameters of an elastic beam equipped with dynamic vibration absorbers under transverse oscillations. Special attention is given to improving the efficiency of vibration suppression by selecting optimal system parameters, as well as analyzing the influence of mass ratio and installation positions of the absorbers. *Methods*. The problem is formulated within the framework of classical beam theory. The solution of transverse vibration equations is obtained using the Bubnov–Galerkin method, which allows reducing the governing partial differential equations to a system of ordinary differential equations. Additionally, the vertical tangents method is applied to determine optimal tuning conditions of dynamic vibration absorbers based on amplitude–frequency characteristics. For the case of multiple absorbers, the method of expansion in eigenfunctions (natural modes) is used to construct approximate analytical solutions. *Results*. Analytical expressions describing the damping efficiency of transverse vibrations are obtained. It is shown how the optimal parameters of dynamic absorbers depend on the mass ratio and their spatial configuration along the beam. The study demonstrates that the use of two parallel-installed dynamic absorbers significantly improves vibration suppression over a wider frequency range. The amplitude–frequency characteristics of the system are analyzed, and optimal tuning parameters are identified for various boundary conditions. *Conclusion*. The proposed approach provides an effective framework for optimizing vibration control systems in elastic beams. The combined use of the Bubnov–Galerkin method and the vertical tangents method ensures high accuracy and computational efficiency. The obtained results can be applied in engineering design of structures requiring enhanced vibration suppression.

Keywords: elastic beam, Laplace operator, bending moment, dynamic vibration absorber, transverse vibrations, amplitude-frequency response.

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Подавление поперечных колебаний упругой балки с использованием динамических поглотителей

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Аннотация. *Целью* данной работы является исследование задачи оптимизации параметров упругой балки с динамическими гасителями колебаний при поперечных вибрациях. Особое внимание уделяется повышению эффективности подавления колебаний за счёт оптимального подбора параметров системы, а также анализу влияния массового отношения и расположения гасителей вдоль балки. *Методы.* Постановка задачи выполнена в рамках классической теории упругих балок. Для решения уравнений поперечных колебаний используется метод Бубнова–Галёркина, позволяющий свести исходную краевую задачу к системе обыкновенных дифференциальных уравнений. Дополнительно применяется метод вертикальных касательных для определения оптимальных параметров настройки динамических гасителей на основе анализа амплитудно-частотных характеристик. В случае системы с несколькими гасителями решение строится с использованием метода разложения по формам собственных колебаний. *Результаты.* Получены аналитические зависимости, характеризующие эффективность демпфирования поперечных колебаний балки. Установлено влияние массового отношения и положения динамических гасителей на их оптимальные параметры. Показано, что использование двух параллельно установленных динамических гасителей позволяет существенно расширить диапазон эффективного подавления колебаний. Проведён анализ амплитудно-частотных характеристик системы и определены оптимальные параметры настройки при различных граничных условиях. *Заключение.* Предложенный подход является эффективным инструментом для оптимизации систем виброзащиты упругих балок. Совместное применение метода Бубнова–Галёркина и метода вертикальных касательных обеспечивает высокую точность и вычислительную эффективность. Полученные результаты могут быть использованы при проектировании инженерных конструкций с повышенными требованиями к снижению вибраций.

Ключевые слова: упругая балка, оператор Лапласа, изгибающий момент, динамический гаситель колебаний, поперечные колебания, амплитудно-частотная характеристика.

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Introduction

Modern development of engineering and technology requires the design of elastic beams with economical and low material consumption. In this context, problems associated with transverse vibrations of elastic beams often arise. Insufficient investigation of methods for solving transverse vibration problems leads to the necessity of introducing additional design modifications during the stages of design, construction, and commissioning. This, in turn, results in increased development time or changes in the main performance characteristics of the product, thereby reducing the consumer properties of elastic beams [1–4].

Numerous scientific studies are devoted to the problem of vibration suppression in systems with distributed parameters using dynamic vibration absorbers. It has been shown in [5, 6] that when a dynamic vibration absorber is attached to a beam, an additional natural frequency of the system appears.

This frequency is close to the partial natural frequency of the absorber and, depending on the system parameters, may be lower than, higher than, or equal to the absorber's partial frequency.

Experimental investigations [7] present a comparative analysis of beam vibrations with two dynamic vibration absorbers symmetrically positioned relative to the beam ends. In this case, the governing differential equations of motion are nonlinear and require the application of appropriate analytical and numerical solution methods.

In works [8], nonlinear vibration problems of a beam with a dynamic vibration absorber are analyzed, taking into account elastic–damping properties of the hysteresis type under harmonic excitation. Solutions of the governing equations are obtained in the form of transfer functions.

The dynamics of nonlinear oscillatory systems [9], as well as their stability [10], have been extensively studied. Based on the above, it follows that the investigation of beam vibrations and vibration suppression remains a relevant problem in modern mechanics. This article focuses on the optimization of system parameters during steady-state vibrations of a beam equipped with two dynamic vibration absorbers.

A vibration control device is presented in [11], consisting of compression and tension springs operating jointly to resist vertical and horizontal loads caused by permanent, temporary, and seismic actions. In addition to absorbing vertical and horizontal loads, the device is capable of restoring the span structure to its initial position after seismic excitation. Furthermore, it eliminates resonance effects without increasing construction costs for the span, supports, or foundations and does not complicate installation conditions.

1. Proposed methodology, experiments and results

In the present study, the objective is to suppress transverse vibrations of an elastic beam using dynamic vibration absorbers (DVAs). An algorithm describing the sequence of operations required to achieve the desired dynamic properties is developed based on a systematic set of procedures. Fig. 1 illustrates the algorithm of operations and the main stages of the proposed technical process.

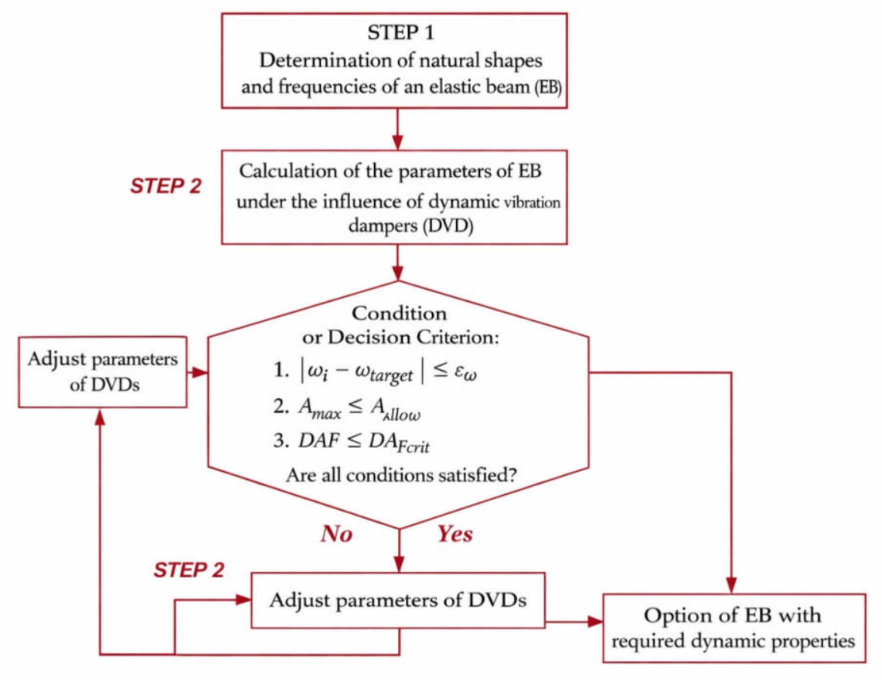

Fig. 1. Algorithm of operations and stages of the technical process

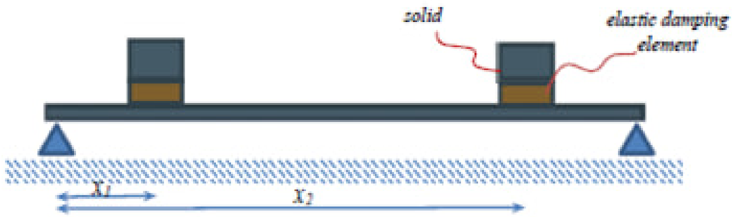

Fig. 2. Design diagram of an elastic beam (EB) with two dynamic vibration absorbers (DVAs)

The solution to the problem of transverse vibrations of a beam with two parallel-installed dynamic vibration absorbers is considered using the method of series expansion in vibration modes. This approach is particularly convenient for optimizing the parameters of dynamic vibration absorbers for various types of beam vibrations under different boundary conditions, especially when repeated calculations of the amplitude–frequency characteristics (AFC) of the system are required.

The results of the studies cited above confirm that, for sufficiently large vibration decrement of the material of the elastic–damping element of the dynamic vibration absorber (DVA), the nonlinearity of the internal resistance characteristics of the beam material has a negligible effect on the beam vibrations and on the determination of the optimal parameters of the DVA.

Consider a beam of length l, width b, and height h, rigidly fixed to a vibrating base. The motion of the base is prescribed along the Oz axis. Dynamic vibration absorbers are installed at points of the beam with coordinates x_1 and x_2 [12–14] (Fig. 2).

2. Analysis of experiments and research results

We write the governing differential equations of motion of an elastic beam with two dynamic vibration absorbers (DVAs) under kinematic excitation in the following form:

$$\begin{gathered}
\frac{\partial^2 M}{\partial x^2} + \rho F \frac{\partial^2 w}{\partial t^2} - c_1 R_1 \delta(x - x_1)\zeta_1 - c_2 R_2 \delta(x - x_2)\zeta_2 = -\rho F \frac{\partial^2 w_0}{\partial t^2}, \\
m_1 \frac{\partial^2 w(x_1, t)}{\partial t^2} + m_1 \frac{\partial^2 \zeta_1}{\partial t^2} + c_1 R_1 \zeta_1 = -m_1 \frac{\partial^2 w_0}{\partial t^2}, \\
m_2 \frac{\partial^2 w(x_2, t)}{\partial t^2} + m_2 \frac{\partial^2 \zeta_2}{\partial t^2} + c_2 R_2 \zeta_2 = -m_2 \frac{\partial^2 w_0}{\partial t^2}.
\end{gathered} \tag{1}$$

Here, M denotes the bending moment; the terms involving the Dirac delta functions represent the interaction forces between the beam and the dynamic vibration absorbers. In particular, the absorber force acts as a concentrated transverse force applied at the installation points $x = x_1$ and $x = x_2$. ρ is the material density; F is the cross-sectional area of the beam; $w(x, t)$ is the transverse deflection of the beam; $w_0(t)$ is the prescribed displacement of the vibrating base; $w(x_1, t)$ and $w(x_2, t)$ are the beam displacements at the DVA installation points; c_1 and c_2 are the stiffness coefficients of the elastic–damping elements of the DVAs; m_1 and m_2 are the absorber masses; ζ_1 and ζ_2 denote the relative displacements of the DVAs with respect to the beam; $\delta(x - x_1)$ and $\delta(x - x_2)$ are the Dirac delta functions indicating the locations of the dynamic vibration absorbers; x_1 and x_2 are the coordinates of the DVA installation points.

$$R_1 = 1 + (-\nu_1 + j\nu_2)\left[D_0 + g_1(\zeta_{1rel})\right]. \tag{2}$$

$$R_2 = 1 + (-\theta_1 + j\theta_2)\left[E_0 + g_2(\zeta_{2rel})\right]. \tag{3}$$

Here, $j^2 = -1$, $\nu_1, \nu_2, \theta_1, \theta_2$ are coefficients determined by the dissipative properties of the materials, and $g_1(\zeta_{1\text{rel}})$, $g_2(\zeta_{2\text{rel}})$ denote the vibration decrements [15].

$$g_1(\zeta_{1rel}) = \sum_{K_1=1}^{r_1} D_{K_1} \zeta_{1rel}^{K_1}. \tag{4}$$

$$g_2(\zeta_{2rel}) = \sum_{K_2=1}^{r_2} E_{K_2} \zeta_{2rel}^{K_2}. \tag{5}$$

The parameters $D_0, D_1, \ldots, D_{r_1}, E_0, E_1, \ldots, E_{r_2}$ are material-dependent parameters of the elastic elements of dynamic absorbers and are determined experimentally [8]. The relationship between the normal stress σ_n and the relative strain ξ_{rel} is given as follows [9].

$$\sigma_n = E\Big(1 + (-\eta_1 + j\eta_2)\left[C_0 + f\left(\xi_{rel}\right)\right]\Big)\xi_{rel}. \tag{6}$$

Here, E is the modulus of elasticity of the rod material; η_1, η_2 are coefficients determined by the dissipative properties of the materials; and $f(\xi_{\text{rel}})$ denotes the vibration decrement of the rod, which can be expressed as follows:

$$f(\xi_{rel}) = \sum_{j_1=1}^{r_3} C_{j_1} \xi_{rel}^{j_1}. \tag{7}$$

Here, $C_0, C_1, \ldots, C_{r_3}$ are experimentally determined coefficients of the hysteresis loop that characterize the nonlinear dissipative properties of the rod material. In this study, the coefficients C_0, C_1, C_2, C_3 are adopted from the model proposed by Pisarenko and Boginich [8].

For the relative strain, the following expression is written:

$$\xi_{rel} = \frac{\partial^2 w}{\partial x^2} z, \tag{8}$$

here, the z-axis is directed along the cross section of the rod.

We calculate the bending moment acting on the cross section of the rod:

$$M = 2b\int_0^{h/2} \sigma_n z\, dz = EJ\frac{\partial^2 w}{\partial x^2}\left[1 + C_0(-\eta_1 + j\eta_2) + \frac{24}{h^3}(-\eta_1 + j\eta_2)\int_0^{h/2} f(\xi_{\text{rel}}) z^2\, dz\right], \tag{9}$$

here, $J = \dfrac{bh^3}{12}$ is the second moment of area; b is the width of the rod, and h is the height (thickness) of the rod.

Substituting the obtained expression for the bending moment into the first equation of system (1), we obtain the following system of differential equations:

As a result, the system of differential equations takes the form

$$\begin{aligned}
&EJ\left[1 + C_0(-\eta_1 + j\eta_2)\right]\frac{\partial^4 w}{\partial x^4} + \frac{24}{h^3}EJ(-\eta_1 + j\eta_2)\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 w}{\partial x^2}\int_0^{h/2} f(\xi_{rel}) z^2\, dz\right] + \\
&\qquad + \rho F\frac{\partial^2 w}{\partial t^2} - c_1 R_1 \delta(x - x_1)\zeta_1 - c_2 R_2 \delta(x - x_2)\zeta_2 = -\rho F\frac{\partial^2 w_0}{\partial t^2}, \\
&m_1\frac{\partial^2 w(x_1, t)}{\partial t^2} + m_1\frac{\partial^2 \zeta_1}{\partial t^2} + c_1 R_1 \zeta_1 = -m_1\frac{\partial^2 w_0}{\partial t^2}, \\
&m_2\frac{\partial^2 w(x_2, t)}{\partial t^2} + m_2\frac{\partial^2 \zeta_2}{\partial t^2} + c_2 R_2 \zeta_2 = -m_2\frac{\partial^2 w_0}{\partial t^2}.
\end{aligned} \tag{10}$$

To solve system (10), the displacement of the rod is represented in the following form:

$$w(x,t)=\sum_{i=1}^{\infty}u_i(x)\,q_i(t). \tag{11}$$

Here, $q_i(t)$ is a function of time, while $u_i(x)$ represents the mode shape (eigenfunction) of the rod and satisfies the following equation:

$$EJ\frac{d^4u_i(x)}{dx^4}-\rho F p_i^2 u_i(x)=0. \tag{12}$$

Here, p_i denotes the natural frequency of the rod.

Substituting solution (11) into system (10) and taking relation (12) into account, after the corresponding transformations we obtain

$$\begin{gathered}\sum_{i=1}^{\infty}\Bigg(\Big(\ddot{q}_i+\big[1+C_0(-\eta_1+j\eta_2)\big]p_i^2q_i\Big)u_i+\\
+\frac{3EJ}{\rho F}(-\eta_1+j\eta_2)q_i\sum_{j_1=1}^{r_3}C_{j_1}\,q_{ia}^{j1}\,\frac{j_1^{h_{j_1}}}{2^{j_1}(j_1+3)}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2u_i}{\partial x^2}\int_0^{h/2}z^2f(\xi_{rel})\,dz\right)\Bigg)-\\
-\frac{c_1}{\rho F}\zeta_1R_1\delta(x-x_1)-\frac{c_2}{\rho F}\zeta_2R_2\delta(x-x_2)=-W_0,\\
\sum_{i=1}^{\infty}u_{i1}\,\ddot{q}_i+\ddot{\zeta}_1+n_1^2R_1\zeta_1=-W_0,\\
\sum_{i=1}^{\infty}u_{i2}\,\ddot{q}_i+\ddot{\zeta}_2+n_2^2R_2\zeta_2=-W_0,\end{gathered} \tag{13}$$

here,

$$W_0=\frac{\partial^2w_0}{\partial t^2}$$

is the base acceleration.

In the particular case, the dynamic absorbers are assumed to be linear, i.e., $\nu_1=\nu_2=\theta_1=\theta_2=0$. Using the Bubnov–Galerkin method for the first equation of system (13) and employing the orthogonality condition of the functions $u_i(x)$, we obtain the following system of differential equations:

After performing the corresponding transformations, the system of equations is reduced to the following form:

$$\begin{gathered}\ddot{q}_i+\big[1+(-\eta_1+j\eta_2)N_i\big]p_i^2q_i-\mu_1\mu_{0i}n_1^2u_{i1}\zeta_1-\mu_2\mu_{0i}n_2^2u_{i2}\zeta_2=-d_iW_0.\\
u_{i1}\ddot{q}_i+\ddot{\zeta}_1+n_1^2\zeta_1=-W_0;\\
u_{i2}\ddot{q}_i+\ddot{\zeta}_2+n_2^2\zeta_2=-W_0,\end{gathered} \tag{14}$$

where p_i is the natural frequency of the beam; the dimensionless mass ratios are defined as

$$\mu_1=\frac{m_1}{m_s},\qquad \mu_2=\frac{m_2}{m_s},$$

$$\mu_{0i}=\frac{l}{d_{2i}},\qquad d_i=\frac{d_{1i}}{d_{2i}},$$

with

$$d_{1i}=\int_0^l u_i(x)\,dx,\qquad d_{2i}=\int_0^l u_i^2(x)\,dx,$$

where

$$m_s = \rho F l,$$

$$N_i = C_0 + \frac{3EJ\,\mu_{0i}}{m_s p_i^2} \sum_{j_1=1}^{r_3} C_{j_1}\, q_{ia}^{j_1} \frac{h^{j_1}}{2^{j_1}(j_1+3)}\, G_{ij_1},$$

$$G_{ij_1} = \int_0^l u_i(x)\, \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 u_i}{\partial x^2} \left| \frac{\partial^2 u_i}{\partial x^2} \right|^{j_1} \right) dx,$$

is the mass of the beam; m_1 and m_2 are the masses of the dynamic vibration absorbers; $u_i(x)$ are the natural vibration modes of the beam; W_0 denotes the base acceleration. The modal shape functions at the installation points of the absorbers are given by

$$u_{i1} = u_i(x_1), \qquad u_{i2} = u_i(x_2),$$

where x_1 and x_2 are the coordinates of the DVA installation points.

The natural frequencies of the dynamic vibration absorbers are defined as

$$n_1 = \sqrt{\frac{c_1}{m_1}}, \qquad n_2 = \sqrt{\frac{c_2}{m_2}},$$

where c_1 and c_2 are the stiffness coefficients of the elastic elements of the absorbers.

The base displacement is denoted by $w_0(t)$, and the corresponding base acceleration is defined as

$$W_0 = \frac{\partial^2 w_0}{\partial t^2}.$$

In the case of harmonic excitation, the base acceleration is assumed in the form

$$W_0 = w_b \cos(\omega t),$$

where w_b is the amplitude of the base acceleration and ω is the excitation frequency.

We seek steady-state solutions of the system in the following form:

$$\begin{aligned} q_{mi}(t) &= a_{mi}(t)\cos\big(\omega t + \alpha_{mi}(t)\big), \\ \zeta_1(t) &= b_1(t)\cos\big(\omega t + \beta_1(t)\big), \\ \zeta_2(t) &= b_2(t)\cos\big(\omega t + \beta_2(t)\big). \end{aligned} \tag{15}$$

Substituting these expressions into the governing differential equations of motion and assuming that the coefficients vary slowly with time, we obtain the following system of averaged (normal) equations for the system under consideration:

$$\begin{aligned} \dot a_{mi} &= \frac{1}{2\omega}\Big[d_i w_b \sin\alpha_{mi} - a_{mi} p_i^2 \eta_2 N_i + l_{1,i} n_1^2 b_1 \sin\phi_1 + l_{2,i} n_2^2 b_2 \sin\phi_2\Big], \\ \dot\alpha_{mi} &= \frac{1}{2a_{mi}\omega}\Big[d_i w_b \cos\alpha_{mi} + a_{mi} p_i^2 (1 - \eta_1 N_i) - a_{mi}\omega^2 - l_{1,i} n_1^2 b_1 \cos\phi_1 - l_{2,i} n_2^2 b_2 \cos\phi_2\Big], \\ \dot b_1 &= \frac{1}{2\omega}\Big[(1 - d_i u_{i1}) w_b \sin\beta_1 - l_{2,i} n_2^2 u_{i1} b_2 \sin\phi_3 - u_{i1} p_i^2 a_{mi} \sin\phi_1\Big], \\ \dot\beta_1 &= \frac{1}{2b_1\omega}\Big[(1 - d_i u_{i1}) w_b \cos\beta_1 + b_1 n_1^2 T_{i,6} - b_1\omega^2 + l_{2,i} n_2^2 u_{i1} b_2 \cos\phi_3 - u_{i1} p_i^2 a_{mi} \cos\phi_1\Big], \\ \dot b_2 &= \frac{1}{2\omega}\Big[(1 - d_i u_{i2}) w_b \sin\beta_2 - l_{1,i} n_1^2 u_{i2} b_1 \sin\phi_3 - u_{i2} p_i^2 a_{mi} \sin\phi_2\Big], \\ \dot\beta_2 &= \frac{1}{2b_2\omega}\Big[(1 - d_i u_{i2}) w_b \cos\beta_2 + b_2 n_2^2 T_{i,7} - b_2\omega^2 + l_{1,i} n_1^2 u_{i2} b_1 \cos\phi_3 - u_{i2} p_i^2 a_{mi} \cos\phi_2\Big]. \end{aligned} \tag{16}$$

where

$$\phi_1 = \beta_1 - \alpha_{mi}, \qquad \phi_2 = \beta_2 - \alpha_{mi}, \qquad \phi_3 = \beta_2 - \beta_1,$$
$$l_{1,i} = \mu_1\mu_{0i}u_{i1}, \qquad l_{2,i} = \mu_2\mu_{0i}u_{i2}.$$

From the system of equations (16), putting zeros instead of the derivatives on the left-hand side, we obtain the stationary solutions in the following form:

$$|q_{mi}| = |a_{mi}| = \sqrt{\frac{(B_1(\omega))^2 + (B_2(\omega))^2}{(B_3(\omega))^2 + (B_4(\omega))^2}}\, w_b,$$
$$|\zeta_1| = |b_1| = \sqrt{\frac{(B_5(\omega))^2 + (B_6(\omega))^2}{(B_3(\omega))^2 + (B_4(\omega))^2}}\, w_b, \tag{17}$$
$$|\zeta_2| = |b_2| = \sqrt{\frac{(B_7(\omega))^2 + (B_8(\omega))^2}{(B_3(\omega))^2 + (B_4(\omega))^2}}\, w_b.$$

The coefficients A_k $(k = 1, \ldots, 18)$ are determined as follows:

$$B_1(\omega) = d_i\omega^4 - A_1\omega^2 + A_2, \qquad B_2(\omega) = -A_3\omega^2 + A_4,$$
$$B_3(\omega) = -\omega^6 + A_5\omega^4 - A_6\omega^2 + A_7,$$
$$B_4(\omega) = A_8\omega^4 - A_9\omega^2 + A_{10},$$
$$B_5(\omega) = (1 - d_iu_{i1})\omega^4 - A_{11}\omega^2 + A_{12},$$
$$B_6(\omega) = -A_{13}\omega^2 + A_{14},$$
$$B_7(\omega) = (1 - d_iu_{i2})\omega^4 - A_{15}\omega^2 + A_{16},$$
$$B_8(\omega) = -A_{17}\omega^2 + A_{18},$$

$$A_1 = n_1^2T_{i,1} + n_2^2T_{i,2}, \qquad A_2 = n_1^2n_2^2T_{i,3}, \qquad A_3 = 0, \qquad A_4 = 0,$$
$$A_5 = p_i^2(1 - \eta_1N_i) + n_1^2T_{i,6} + n_2^2T_{i,7},$$
$$A_6 = (n_1^2 + n_2^2)p_i^2(1 - \eta_1N_i) + n_1^2n_2^2T_{i,8},$$
$$A_7 = n_1^2n_2^2p_i^2(1 - \eta_1N_i), \qquad A_8 = p_i^2\eta_2N_i,$$
$$A_9 = (n_1^2 + n_2^2)p_i^2\eta_2N_i, \qquad A_{10} = n_1^2n_2^2p_i^2\eta_2N_i,$$
$$A_{11} = p_i^2(1 - \eta_1N_i) + T_{i,4}n_2^2, \qquad A_{12} = n_2^2p_i^2(1 - \eta_1N_i),$$
$$A_{13} = p_i^2\eta_2N_i, \qquad A_{14} = n_2^2p_i^2\eta_2N_i,$$
$$A_{15} = p_i^2(1 - \eta_1N_i) + n_1^2T_{i,5}, \qquad A_{16} = n_1^2p_i^2(1 - \eta_1N_i),$$
$$A_{17} = p_i^2\eta_2N_i, \qquad A_{18} = n_1^2p_i^2\eta_2N_i.$$

The auxiliary coefficients $T_{i,j}$ are given by

$$T_{i,1} = d_i + \mu_{0i}\mu_1u_{i1}, \qquad T_{i,2} = d_i + \mu_{0i}\mu_2u_{i2},$$
$$T_{i,3} = d_i + \mu_{0i}(\mu_1u_{i1} + \mu_2u_{i2}),$$
$$T_{i,4} = 1 + \mu_{0i}\mu_2u_{i2}(u_{i2} - u_{i1}) - d_iu_{i1},$$
$$T_{i,5} = 1 + \mu_{0i}\mu_1u_{i1}(u_{i1} - u_{i2}) - d_iu_{i2},$$
$$T_{i,6} = 1 + \mu_{0i}\mu_1u_{i1}^2, \qquad T_{i,7} = 1 + \mu_{0i}\mu_2u_{i2}^2,$$
$$T_{i,8} = 1 + \mu_{0i}(\mu_1u_{i1}^2 + \mu_2u_{i2}^2).$$

3. Results of numerical studies

Numerical analysis is performed to determine the first eigenmode in two separate cases. First, a numerical analysis is carried out by varying the mass ratios μ_1 and μ_2, ($\mu = \mu_1 = \mu_2$) which represent the ratios of the masses of the dynamic vibration absorbers to the mass of the beam. Second, based on the obtained relationships, the amplitude–frequency characteristics (AFC) of the system are constructed, and approximate optimal locations for installing the dynamic vibration absorbers are determined.

In the study of the motion of a rod equipped with dynamic vibration dampers, the choice of the damper mass plays a crucial role. In this work, the ratios of the dynamic damper masses to the mass of the rod are considered (Fig. 3). For each case, the corresponding amplitude–frequency characteristic is presented. In the first case, a mass ratio of 0.05 is selected (red curve); in the second case, a ratio of 0.04 is chosen (blue curve); and in the third case, a ratio of 0.03 is considered (yellow curve).

The obtained graphs show that decreasing the damper mass, together with increasing the excitation frequency, leads to a noticeable shift of the stable and unstable regions of the system. At the same time, an increase in the mass of the dynamic vibration damper reduces the resonant frequency of the system and decreases the vibration amplitudes. As the damper mass grows, the resonance peak becomes lower; at a certain damper mass the vibration amplitude reaches a minimum, which is taken as the principal criterion for selecting the optimal damper mass. At this optimal mass, resonance is suppressed most effectively and the overall vibration level of the system is minimized.

When the installation points of the dynamic vibration dampers are shifted symmetrically along the length of the rod without changing their masses, the dominant stability region varies depending on the excitation frequency (Fig. 4). In the first case, the dampers are installed at the points $l/3$ and $2l/3$ (red curve); in the second case, at the points $l/4$ and $3l/4$ (blue curve); in the third case, at the points $l/5$ and $4l/5$ (yellow curve); and in the fourth case, at the points $l/6$ and $5l/6$ (black curve).

An analysis of the obtained graphs shows that, as the installation points of the dynamic vibration dampers are moved symmetrically away from the center of the rod, the stability regions tend to approach each other.

Fig. 5 presents the amplitude–frequency characteristics of the system for a fixed mass ratio $\mu = 0.04$. The calculations are performed for different installation locations of the dynamic vibration dampers, which are arranged symmetrically with respect to the center of the rod. The damper positions correspond to the coordinate pairs $l/3$ and $2l/3$ (red curve), $l/4$ and $3l/4$ (blue curve), $l/5$ and $4l/5$ (yellow curve), and $l/6$ and $5l/6$ (black curve).

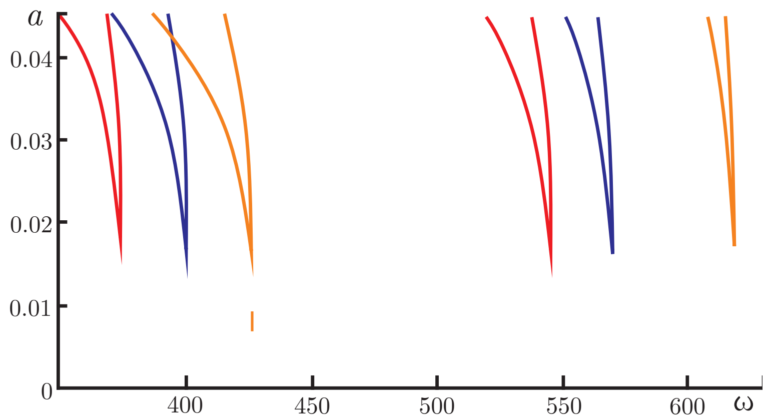

Fig. 3. Amplitude–frequency characteristics of a rod with a dynamic vibration damper (DVD) for different values of the mass ratio $\mu = m/m_s = 0.05,\ 0.04,\ 0.03$, corresponding to the first eigenmode $u_i(x) = \sin\left(\frac{i\pi x}{l}\right)$ $(i = 1)$, with $p_1 = 470.517\ \mathrm{s}^{-1}$ and different installation coordinates of the DVD (color online)

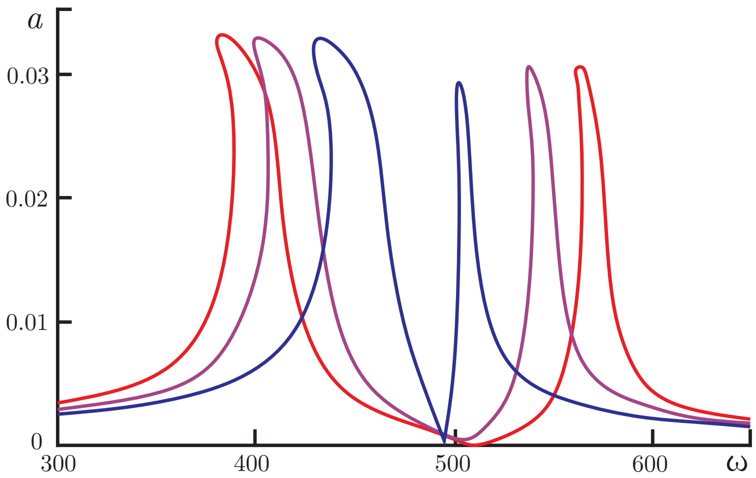

Fig. 4. Amplitude–frequency characteristics (AFCs) for a mass ratio of $\mu = 0.04$ at different installation locations of the dynamic vibration dampers: $l/3$ and $2l/3$ (red curve), $l/5$ and $4l/5$ (black curve), and $l/12$ and $11l/12$ (blue curve) (color online)

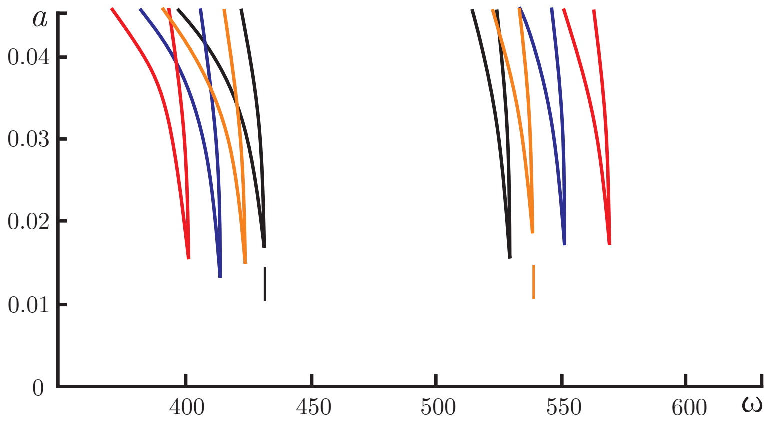

Fig. 5. Amplitude–frequency characteristics of the system for $\mu = 0.04$ (color online)

It can be observed that a symmetric displacement of the damper installation points away from the rod center significantly affects the amplitude–frequency response of the system and leads to noticeable changes in the location of the stability regions.

A horizontal displacement of the system leads to a significant shift in the resonant frequencies while preserving the overall vibration pattern. This makes it possible to control the resonance regions without altering the dynamic stability of the system. Consequently, varying the installation positions of the dynamic vibration dampers along the beam provides an effective and convenient approach for optimizing resonant frequencies and controlling vibrations.

Conclusion

The problem of optimizing transverse vibrations of an elastic beam with two parallel-installed dynamic vibration dampers equipped with elastic elements under harmonic base excita-tion has been investigated. A systematic approach for determining the optimal installation parameters of dynamic vibration dampers for an elastic beam has been developed, taking into account the amplitude–frequency characteristics associated with transverse vibrations.

Numerical analyses of the system vibrations were carried. Numerical investigations demons-trate that the dynamic response of the beam can be effectively controlled by adjusting two key parameters: the mass ratios of the dynamic vibration dampers and their installation positions along the beam. Proper selection of these parameters leads to a significant reduction in vibration amplitudes and a controlled shift of resonant frequencies toward a desired operating range, thereby improving the overall dynamic stability of the system. As a result, the system attains dynamic stability, enhanced protection against resonance, and improved energy efficiency.

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