Review of actual problems of nonlinear dynamics

Official English translation

«Exotic» models of high-intensity wave physics: Linearizable equations, exactly solvable problems and non-analytic nonlinearities

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Topic and aim. A brief review of publications and discussion of some mathematical models are presented, which, in the author's opinion, are well-known only to a few specialists. These models are not well studied, despite their universality and practical significance. Since the results were published at different times and in different journals, it is useful to summarize them in one article. The goal is to form a general idea of the subject for the readers and to interest them with mathematical, physical or applied details described in the cited references. **Investigated models**. Higher-order dissipative models are discussed. Precisely linearizable equations containing nonanalytic nonlinearities – quadratically-cubic (QC) and modular (M) – are considered. Equations like Burgers, KdV, KZ, Ostrovsky–Vakhnenko, inhomogeneous and nonlinear integro-differential equations are analyzed. **Results**. The appearance of dissipative oscillations near the shock front is explained. The formation in the QC-medium of compression and rarefaction shocks, which are stable only for certain parameters of the «jump», as well as the formation of periodic trapezoidal sawtooth waves and self-similar N-pulse signals are described. Collisions of single pulses in the M-medium are discussed, revealing new corpuscular properties (mutual absorption and annihilation). Collisions are similar to interactions of clusters of chemically reacting substances, for example, fuel and oxidizer. The features of the behavior of «modular» solitons are described. The phenomenon of nonlinear wave resonance in media with QC-, Q- and M-nonlinearities is studied. Precisely linearizable inhomogeneous equations with external sources are used. The shift of maximum of resonance curves relative to the linear position, which is determined by the equality of velocities of freely propagating and forced waves, is indicated. Simplified models for diffracting beams obtained by projecting 3D equations onto the beam axis are analyzed. Strongly nonlinear waves in systems with holonomic constraints are discussed. Integro-differential equations with relaxation type kernel, and the possibility of reducing them to differential and differential-difference equations are considered. **Discussion.** The material is outlined on a popular level. Apparently, these studies can be continued if the readers find them interesting enough.

Key words: dissipative models, shock fronts, linearizing equations, QC-, Q- and M-nonlinearities.

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Introduction

Dmitry Ivanovich Trubetskov is well known not only as an outstanding scientist, but also as a brilliant popularizer of science, on whose books several generations of specialists grew. I was lucky to be a scientific editor of the tutorial [1] written by D.I. Trubetskov together with M.I. Rabinovich by the time when the «nonlinear» literature was in great demand. In the following years I got a great pleasure of the books written by Dmitry Ivanovich and his lectures at the Nonlinear Waves Schools, which were organized in Nizhny Novgorod by A.V. Gaponov-Grekhov.

It's a great responsibility – to write an article for the special issue of the journal, devoted to D.I.Trubetskov. It's hard to maintain the proper scientific and educational level fit to the jubilee celebrator. Especially, it's hard to do it in such limited time according to the Journal requirements. On the other hand, it's impossible to refuse to participate in the jubilee, even if the material is rather raw. I hope that the readers and Dmitry Ivanovich shall not judge the result of my efforts strictly.

The exactly solvable problems attracted my attention during many years. Perhaps, it was because I didn't have the skill of informational technologies. I was «dissuaded» to practice computer calculations by academicians R.V. Khokhlov (my scientific supervisor) and N.S. Bakhvalov (the famous expert in numerical methods). In 1974 they organized the group of physicists and mathematicians [2] focused of computer solution of nonlinear wave problems. After many years of collaboration with N.S. Bakhvalov and his colleagues I understood that the development and application of numerical methods is the special occupation requiring special skills. It's better for everyone to go about his own business, and my business is analytical theory oriented upon experiment and applications.

1. Burgers-type equations with higher derivatives

In this section we shall speak about evolutional equations of the following type

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} = \pm\Gamma\frac{\partial^n V}{\partial \theta^n}, \tag{1}$$

here Γ is the dissipation parameter. In most of the wave problems the dimensionless variables $z,\ \theta$ have the sense of «slow» coordinate along the direction of wave propagation and of the «delayed» time in the reference system which moves with the speed of the natural wave in the medium. For $n = 1$ (1) transforms into Hopf-type equation, for $n = 2$ – Burgers equation, for $n = 3$ – Korteweg–de Vries equation. All three belong to the most well-known equations, and it's clear, why. Besides their interesting mathematical properties, they provide correct modeling of the processes of different physical nature. The second-order equation (Burgers equation, invented by Bateman) is generally unique: it admits exact linearization with the help of simple conversion of variable (Cole-Hopf transformation, devised by Florin).

There is a question: what is happening when $n > 3$? Are these equations practically useful or they must be considered just formally continuing the series $n = 1, 2, 3, ...$? As an example of interesting but complex model, we can name the one-dimensional Kuramoto–Sivashinsky equation with second-order, third-order and fourth-order derivatives. Nevertheless, the equation (1), which contains fourth-order derivative only, didn't attract any attention. But it is surely interesting for practical application. The following equation [3]

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} = -\Gamma\frac{\partial^4 V}{\partial \theta^4} \tag{2}$$

corresponds to dispersion relation

$$k = \frac{\omega}{c} + i\chi\omega^4,$$

i.e. describes nonlinear waves in dissipative medium, where dissipation is proportional to the fourth degree of frequency with some coefficient χ. The dissipation of acoustic waves proportional to $\chi\omega^4$ is observed in media with low-dimensional heterogeneities (rocks [4], skull bones [5]). For example, in liquids containing gaz bubbles with radius a, volume concentration n and natural frequency ω_*, the coefficients $\chi = 4\pi n a^2/\omega_*^2$. The same dissipations take place because of Rayleigh scattering in the media with low-scale fluctuations of refractive index μ. The coefficient $\chi = 8\left\langle \mu^2 \right\rangle a^3/c^4$, where a is the correlation radius.

It's not difficult to find the partial solution (2) as a stationary wave [3]

$$\theta = \left(\frac{40}{9}\Gamma\right)^{1/3} \int\limits_0^V \frac{dt}{(1-t^2)^{2/3}}. \tag{3}$$

The exact solution (3) in quadratures describes the shock wave of compression (see Fig. 1) with finite width of front, defined by competition between nonlinearity and dissipation, proportional to $\beta\omega^4$. The curves *1–5* correspond to the values of dissipation parameter Γ equal to: $2.2 \cdot 10^{-4}$, $6.1 \cdot 10^{-3}$, $2.8 \cdot 10^{-2}$, $7.7 \cdot 10^{-2}$, $2.2 \cdot 10^{-1}$, respectively. The front expands with the increasing of dissipation.

The value $V = 1$ is achieved in finite time $\theta = \theta_* \approx 3.4 \cdot \Gamma^{1/3}$. In the point $\theta = \theta_*$ the first and second derivative are zero. Nevertheless, the third derivative is positive and the increasing of V over the inflection point $\theta > \theta_*$ continues. Thus, the stationary wave (3) forms for the perturbation, which indefinitely grows at infinity. This is the difference between this equations and Burgers equation, for which the stationary wave $V = \tanh(\theta/2\Gamma) \to 1$ for $\theta \to \infty$. The cause of this difference is that the more strong dependence of dissipation on frequency (ω^4 instead of ω^2) can be compensated by increased inflow of energy to the wave front, which is supplied by growing of V when $\theta \to \infty$. But there also is the limited stationary solution. The numerical solution of (2) shows that the asymptotic increasing of $V \to 1$ when $\theta \to \infty$ is not monotone but is accompanied by fading oscillations [3].

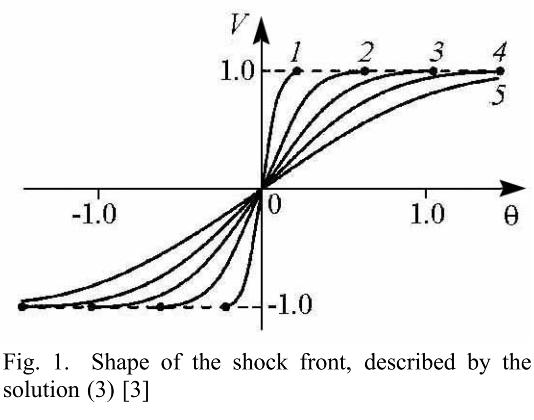

Fig. 1. Shape of the shock front, described by the solution (3) [3]

Later the properties of symmetry of the model (2) were investigated by the methods of Lie group theory [6]. In [7] the equation (2) and the equation (1) of the sixth order have been studied in detail. Computer analysis confirmed the fact of appearance of the fading oscillations near

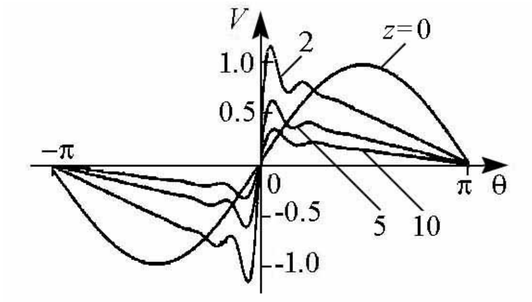

Fig. 2. One period of a harmonic (at $z = 0$) signal and dissipative smoothing of an oscillating shock front [7]

the shock front. The oscillations intensify with the increasing of the order of equation. Such oscillations appear at propagation of arbitrary form signal, in consequence of appearance of an abrupt shock front as the result of signal evolution. In Fig. 2 [7] one can see the oscillating front for the initial harmonic signal in model (1) with $n = 6$. The different profiles correspond to different values of z and $\Gamma = 2.45 \cdot 10^{-6}$.

It is known that front oscillations can appear in other cases, for example, in Korteweg–de Vries–Burgers model [8]. But the mechanisms in these cases are different. In dispersive medium oscillations appear because of the «scattering» (disphasing) of harmonies, which form the front. In our case oscillations are the consequence of «Fourier phenomenon». The dissipation proportional to ω^{2n} removes the high harmonics from signal spectrum, consequently the forming of «monotonous» front is impossible. The sequence ω^{2n} with $n \to \infty$ converges non-uniformly: in the diapason $0 \le \omega < 1$ – to zero, for $\omega > 1$ – to infinity. Thus, only the low harmonics, belonging to the area $0 \le \omega < 1$, «survive» in the wave: they form the oscillating signal.

2. Exactly linearized equations with modular, quadratic and quadratic-cubic nonlinearities

Nonlinear equations in partial derivatives of the second order, allowing linearization, are interesting not only for mathematical physics, but also for understanding of nonlinear phenomena. The equations adequate to real systems are the most useful. Let us consider the following equation:

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial \theta}\left(\alpha\left|V\right| + \frac{\beta}{2}V^2 + \frac{\gamma}{2}V\left|V\right|\right) + \Gamma\frac{\partial^2 V}{\partial \theta^2}. \tag{4}$$

Coefficients α, β, γ correspond to the terms with the nonlinearities, which we shall call modular (M), quadratic (Q) and quadratically-cubic (QC) nonlinearities. It's easy to see that the replacement

$$V = \frac{2\Gamma}{\beta \pm \gamma}\frac{\partial}{\partial \theta}\ln U \tag{5}$$

brings (4) to linear equation for auxiliary function U:

$$\frac{\partial U}{\partial z} = \pm\alpha\frac{\partial U}{\partial \theta} + \Gamma\frac{\partial^2 U}{\partial \theta^2}. \tag{6}$$

In the expressions (5) and (6) the upper signs are used for $V > 0$, and the lower ones for $V < 0$. Linearization allows to find many solutions describing real phenomena.

In the case we have only the Q-nonlinearity $\alpha = \gamma = 0$, $\beta \neq 0$), we obtain the common Burgers equation from (4).

Not long ago the study of quadratically-cubic equation (equation (4), in which α=β=0, $\gamma \neq 0$ [9–13]) (4) began. It also can be linearized and has important physical sense.

The QC-equation describes: shock waves of compression and rarefaction, which are stable only for certain values of jump parameter (see Fig. 3); transformation of harmonic wave into a «saw» with trapezoidal prongs (see Fig 4); effects of self-action, nonlinear attenuation, etc.

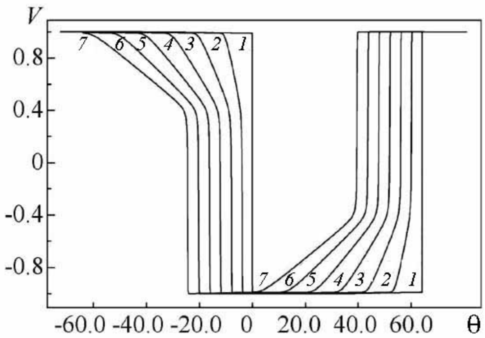

Fig. 3. The process of forming stable compression shock fronts (right-hand curves *1–7*) and rarefaction shock fronts (left curves *1–7*). Number $\Gamma = 0.03$. Curves *1–7* correspond to distances $z = 0, 10, 20, 30, 40, 50, 60$ [11]

In Fig. 5 [13] one can see the self-similar solution of QC-equation (4) as N-wave. According to one of the symmetries of equation (4) it can be brought to an ordinary differential equation by the following substitution

$$V = \sqrt{\frac{2\Gamma}{z}}\, y\left(x = \frac{\theta}{\sqrt{2\Gamma z}}\right) \qquad (7)$$

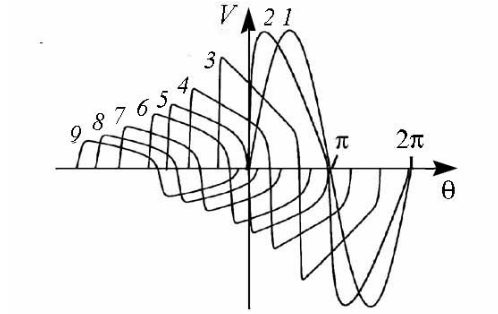

Fig. 4. The process of forming a periodic trapezoidal sawtooth wave in a QC-medium. Number $\Gamma = 0.01$. Curves *1–9* correspond to distances $z = 0$, 1, 4, 8, 12, 16, 24, 32, 40 [12]

Positive and negative branches of the solution joint in the point x_*. We can mark that this solution generalizes the self-similar solution of Q-equation [14, 15] and transforms into the latter in the case of unipolar impulse. Let us point at some applications of QC-model.

Example 1. It is known that for high-level sounds a hole in a plate exhibits nonlinear response. It is shown experimentally [16] that the relation between acoustic pressure and speed is the following $p' = \varsigma\, u\, |u|$. The similar term appears in Cauchy–Lagrange integral in the case of oscillating dynamics of medium.

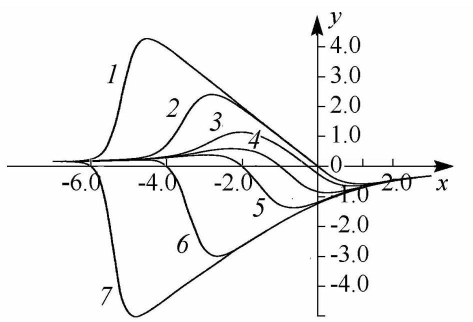

Fig. 5. A self-similar solution of the QC-equation (4) in the form of an N-wave. Curves *1–7* correspond to points:$(-x_*) = 10^{-5}, 10^{-2}, 0.25, 1, 2, 4, 6$ [13]

Tables of coefficients ς for different streamlined obstacles can be found in the reference book [17]. Adding this nonlinear term to the state equation of fluid equations system, we can easily derive the QC-equation (4) using standard procedures. Metamaterial with nonlinearity $p' = \varsigma\, u\, |u|$ can be made by placing streamlined elements into a fluid. The effect of this type of nonlinearity appears in the neck of Helmholtz resonators with fiberfill, which are used for loud sounds absorption [18].

Example 2. As the media parameters are usually defined from the data of accurate spectrum measurements, let us discuss harmonic series expansion of the solution of QC-equation (4) for monochromatic initial signal. First of all the lowest (third) harmonic is

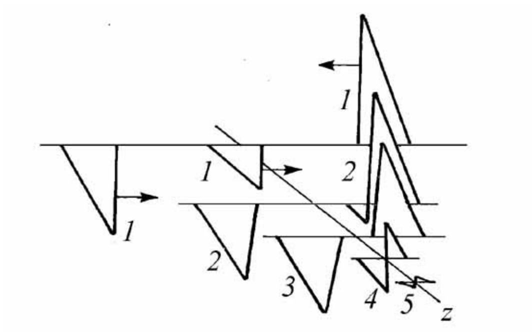

Fig. 6. The collision of three single signals with zero total momentum. Curves *1–5* represent successive moments of processes of mutual absorption and annihilation of signals. Number $\Gamma = 0.003$ [24]

generated. At small distances it is proportional to amplitude square p'^2_0 of the wave of fundamental frequency, and increases in proportion to z^1. At the other hand, in common Q-medium these relations are usually the follows: p'^3_0 and z^2, but often there are some deviations. For example, in aluminum alloy polycrystalline the deviation from p'^3_0, z^2 is conduced by nonlinear friction on intercrystallite boundaries [19]. The behaviour of QC-model is also described for shear waves (for example in soft biological tissues), in which the symmetry doesn't allow quadratic effects. In any case the dependence of the third harmonic from the distance (proportional to z^m for $1 < m < 2$) and the amplitude (proportional to p'^k_0 for $2 < k < 3$) evidence that there are QC-elements in the medium volume. Other applications and experiment are described in the review [10].

The study of the waves in the media with «module» nonlinearity ($\beta = \gamma = 0$, $\alpha \neq 0$) have been also began not long ago [20–22]. M-media, which can be found in mechanics, have different elasticities for tensile and compression deformations. For example –reinforced polymers and concretes [23, Ch. 1, p. 10]. In this case the M-equation is even simpler than the two linearized equations of Q- and QC-type, mentioned above. This equation is linear for the function which keeps the sign, i.e. for $V > 0$ or $V < 0$. Nonlinear effects appear only for alternating solutions.

An example is the process of collision of two [9] or three [24] pulses of different polarity (see Fig. 6). At first (the curve *1*) one positive and two negative pulses are spaces and begin to converge without interaction. As the result of collision of the positive and the nearest negative pulses, the coupled state with common shock front forms (curve *2*). In that moment the nonlinear attenuation «switches on» (compare the curves *2* and *3*), and goes on up to negative pulse vanishing. The formed positive pulse delays in phase and has smaller amplitude in comparison with the original one. Further it propagates without changing its form and collides with the second negative pulse. The mutual absorption and annihilation of the pulses takes place (compare the curves *4* и *5*).

Thus, the interaction of solitary waves in M-media demonstrates properties, which differ from the ones observed for elastic collisions of solitons and non-elastic shock waves. There is an analogy with interaction of clots of chemically reacting substances, for example, fuel and oxidizer. As result of such reaction one (smaller) component vanishes and the mass of the other (larger) decreases.

Solutions of Korteweg–de Vries [25] M-equation demonstrate interesting properties.

$$\frac{\partial V}{\partial z} - \frac{\partial}{\partial \theta} |V| = D \frac{\partial^3 V}{\partial \theta^3}. \tag{8}$$

A modular soliton can't propagate in undisturbed media, i.e., can't have a profile shown in Fig. 7, for which $V(\theta) \to 0$, $|\theta| \to \infty$. The cause of it is the fact that solitons form when there is a competition between nonlinear «steepening» and dispersive «widening» of the wave, and in M-model there is no nonlinearity for perturbation with constant sign. It's the main difference from the common solitary solutions of KdV Q-nonlinear equation; it is connected with disappearance of one of the symmetries of M-equation (8).

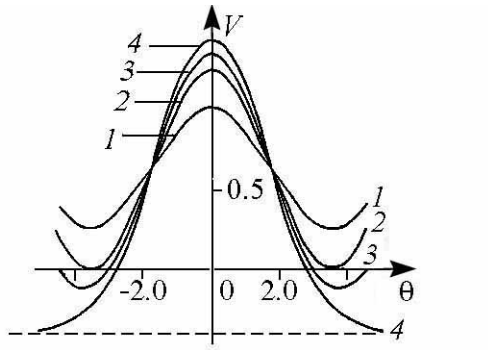

Fig. 7. The process of transformation of a periodic wave (curve *1*) into a soliton (curve *4*), which occurs with increasing wave amplitude [25]

Effects of module nonlinearity were observed in the experiments [26, 27]. It is found that the dependence of the second harmonic amplitude from the first harmonic amplitude, in the solids with inhomogeneous structure, differs from the quadratic law $A_2 = KA_1^2$. It has the following form: $A_2 = KA_1^m$, where exponent index belongs to diapason $1 < m < 2$. It has sense to assume that besides the classic Q-nonlinearity the medium has the second type nonlinearity, which is responsible for deviation from $m = 2$. This could be the «molular» nonlinearity.

In the presence of Q- and M-nonlinearities simultaneously, the solution of the equation

$$\frac{\partial \sigma}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial \tau}\left(g\,|\sigma| + \delta\frac{\sigma^2}{c^2\rho}\right) = 0 \qquad (9)$$

by perturbation method for mechanical tension σ, has the following form

$$\sigma = -A_1 \sin(\omega\tau) - \frac{\omega}{c}x\left(\frac{4}{3\pi}gA_1 + \frac{\delta}{2c^2\rho}A_1^2\right)\sin(2\omega\tau). \qquad (10)$$

We can see that the modular nonlinearity gives linear dependence between the second harmonic amplitude and the first harmonic amplitude ($A_2 \sim A_1$), while the quadratic nonlinearity gives another law ($A_2 \sim A_1^2$). In common case when the both nonlinearities are essential, the exponent index in $A_2 = KA_1^m$ lies in the diapason $1 < m < 2$. Making several measurements, we can solve the inverse problem [28] and restore the moduli g, δ. It's not hard to make concentrated M-nonlinear elements artificially [29], and then include them into metamaterial matrix.

3. Inhomogeneous equations and wave resonance

Here we discuss the following equation

$$\frac{\partial V}{\partial z} - \frac{\partial}{\partial \theta}\left(\alpha\,|V| + \frac{\beta}{2}V^2 + \frac{\gamma}{2}V\,|V|\right) - \Gamma\frac{\partial^2 V}{\partial \theta^2} = F(\theta + \delta z). \qquad (11)$$

The presence of «external sources» i.e. function $F(\theta)$, in the right-hand-side of (11) means that this equation can discribe not only the waves freely propagating in dissipative medium with triple nonlinearity, but also «forced» waves, including the process of their excitation. As the homogeneous equation ($F(\theta) \equiv 0$), the equation (11) can be linearised. The model (11) is convenient to use for simplified description of nonlinear wave resonance in the systems of different physical nature. For the first time this equation (with α=γ=0) has been suggested as a model of generation of intensive hypersound by laser and a process of stimulated Mandelstam–Brillouin scattering taking the acoustic nonlinearity into account [30]. Further (11) has been applied to describe: excitation of underwater signals by thermic optoacoustic method; excitation of sea surface waves by traveling pressure wave; aerodynamical perturbations with transonic flow around of laser ray or solid profile and many other applied problems (the bibliography can be found in [31]). Later there appeared some mathematical studies devoted to the properties of inhomogeneous Burgers equation and approximate methods of its analysis (see for example [32–34]).

Let's remind that nonlinear wave resonance takes place when the «external force» velocity c (the exciting source is meant) is equal to the velocity of the natural wave c_0. The condition $c \to c_0$ is similar to the condition $\omega \to \omega_0$, corresponding to realization of simple vibration resonance.

If we neglect the dissipative and nonlinear terms, the solution (11) has the following form:

$$V = \frac{1}{\delta}\left[\tilde{F}\left(\theta + \delta z\right) - \tilde{F}\left(\theta\right)\right], \quad V|_{\delta\to 0} = z\,F\left(\theta\right), \tag{12}$$

where $\tilde{F}$ is the primitive of F. Solution (12) satisfies the condition $V\left(z = 0, \theta\right) = 0$, i.e. the increasing of the wave begins from zero level. When $\delta \to 0$, $c \to c_0$, the uncertainty appears in (12). It means that in the case of exact linear resonance the wave increases unlimitedly proportional to the distance z, wherein the wave form repeats the function $F\left(\theta\right)$.

When there is the velocity detuning ($\delta \neq 0$), the energy of the source is brought into the area of its localization (where $F \neq 0$), and then «flows» to the right or to the left, because the excited wave travels faster or slower than the sources. The result is the stationary form of the wave. For example, for the source like «Lorentz bell» we obtain:

$$F\left(\theta\right) = \frac{A}{1 + b^2\theta^2}, \quad \tilde{F} = \frac{A}{b}\arctan\left(b\theta\right), \quad \left|V_{\max}\left(\delta\right)\right|_{z\to\infty} = \frac{\pi A}{2\left|\delta\, b\right|}. \tag{13}$$

Dissipation usually eliminates the peculiarity of resonance curve $\left|V_{\max}\left(\delta\right)\right|$ (13) with zero detuning $\delta \to 0$, as in common oscillating system. For example, for harmonic excitation $F = A\sin\left(\theta\right)$ we have:

$$\left|V_{\max}\right| = \frac{A}{\omega_0\sqrt{\Gamma^2\omega_0^2 + \delta^2}}. \tag{14}$$

Resonance curve (14) has the finite maximum with $\delta \to 0$, which is equal to $\left|V_{\max}\right| = A/(\Gamma\omega_0^2)$. It's obvious that with $\delta = 0$ and weak dissipation ($\Gamma \to 0$) the wave may grow to the level demanding nonlinear limiting.

The stationary resonance profiles ($z \to \infty$) for zero detuning ($\delta = 0$) can be calculated analytically. For M-nonlinearity ($\alpha = 1$, $\beta = \gamma = 0$) and for the source $F = A\sin\left(\theta\right)$ within one period $-\pi \leq \theta \leq \pi$ we have

$$V = \frac{A}{2}\frac{\operatorname{sgn}\theta}{\left(1 + \Gamma^2\right)}\left[2\frac{1 - \exp\left(-\left|\theta\right|/\Gamma\right)}{1 - \exp\left(-\pi/\Gamma\right)} + \cos\theta + \Gamma\left|\sin\theta\right| - 1\right]. \tag{15}$$

For Q-nonlinearity ($\beta = 1$, $\alpha = \gamma = 0$) and the same conditions the solution is expressed through Mathieu function

$$V = 2\Gamma \frac{d}{d\theta} \ln ce_0 \left(\frac{\theta}{2}, \frac{A}{\Gamma^2} \right), \quad -\pi \leq \theta \leq \pi. \tag{16}$$

For QC-nonlinearity the result is more complex, because the odd QC-nonlinearity leads to the self-action effects and additional shift of resonance condition [35].

In Fig. 8 we can see the profiles of one period of solution (15) and (16) for M- and Q-media, and also for QC-media for $\Gamma \to 0$. In the first two cases (even nonlinearity) compression shock front appears. In the third case (odd nonlinearity) in each period there is a compression shock wave and a rarefaction shock wave. A sawtooth wave with trapezoid «teeth» forms. It's interesting that the ratio of larger and smaller «jumps» of different sign, which appear on the breaks over the level $V = 0$ is equal to $(\sqrt{2} - 1)$; the same takes place also in the wave propagating freely in QC-medium [10].

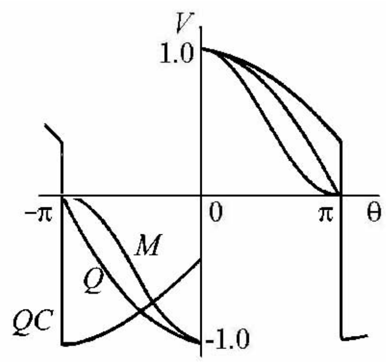

Fig. 8. The steady-state profiles of a one wave period in media with M-, Q- and QC-nonlinearities for excitation by harmonic sources

The process of establishing the stationary wave profile is conditioned by the flow of energy from the source, nonlinear-dissipative absorption and redistribution of energy over the spectrum and the time (within the wave period). The cooperative influence of these factors can be studied only by numerical modelling.

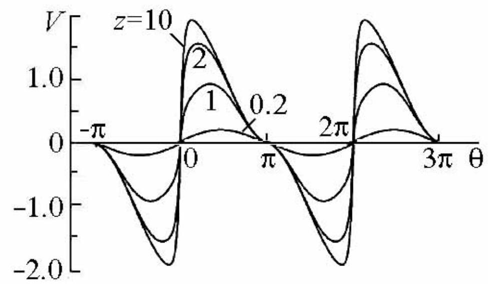

Fig. 9. Profiles of a periodic wave excited by sinusoidal running sources in an M-nonlinear dissipative medium at values of numbers $\Gamma = 0.1$, $A = 1$, $\delta = 0$

In Fig. 9 the process of generation of the periodic profile in dissipative medium with M-nonlinearity is shown. For small z the profile follows the sinusoid, and for $z = 10$ it is close to the stationary form (see Fig. 8).

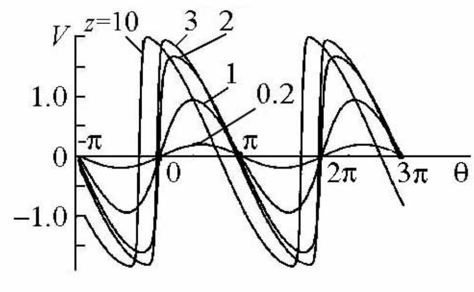

Fig. 10. Profiles excited in a Q-nonlinear medium with the same values of the parameters as in Fig. 9. Here, the speed detuning is taken into account, $\delta = 0.1$

For the curves in Fig. 9 the detuning δ is equal to zero. The same curves for Q-media, calculated with account for detuning, are shown in Fig. 10. The result of the detuning is that the new

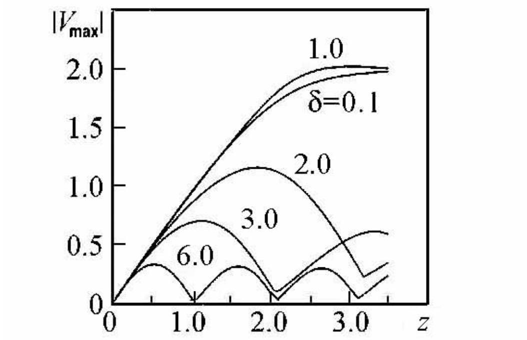

Fig. 11. Dependences on the distance of the maximum «amplitude» of the wave in the Q-medium at $\Gamma = 0.1$, $A = 1$. As the detuning increases, the spatial frequency of the beats increases, and their «swing» decreases

portions of energy begin to enter into the travelling wave in anti-phase and damp it. Thus the beat oscillations (the dependance of wave profile maximum from the distance) appear, which are shown in Fig. 11.

The wave profiles for QC-medium and zero detuning ($\delta = 0$) are shown in Fig. 12. The other parameters are the same as in Fig 9 and 10. The shift of the curves along the axis θ to the left means that the wave in the medium moves faster than c_0. This is the result of self-action in the odd QC-nonlinearity. It this case the space beatings appear even for $\delta = 0$ (see Fig. 11).

If Fig. 13 we demonstrate the resonance curves for absolute maximum of wave amplitude and of the intensity, averaged over the period. These curves are calculated for QC-medium, but similar curves are obtained for Q- and M-nonlinearities. The maximum shift $c \to c_0$ is common for these three cases. This phenomenon is well known in aerodynamics. During transonic flight the plane goes through maximal radiation resistance with small excess of sound speed. This effect is described by Khokhlov–Zabolotskaya (KZ) model or Lin–Reissner–Tsien model [2, 36].

The model of inhomogeneous equation (11) is widely used for strongly deformed wave profiles, excited in acoustic resonators. Herewith many types of resonance curves can be calculated analytically (see the Review [37] and [38, Ch. 11]). Such resonators can be found in aeroacoustic applications and are used for nondestructive testing of solids [39]. The equation (11) is also used to describe the Burgers turbulence (Burgulence), because it takes into account the third principle factor, the flow of energy to low frequency spectrum [40], besides high-frequency dissipation and nonlinear flow of energy over the spectrum.

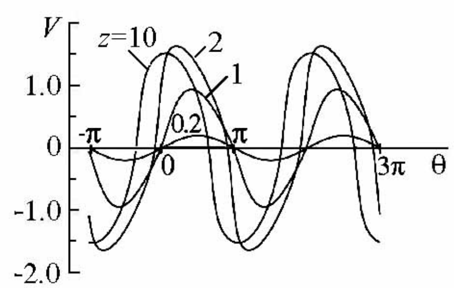

Fig. 12. Profiles excited in a QC-nonlinear medium at $\Gamma = 0.1$, $A = 1$. Here the velocity detuning $\delta = 0$

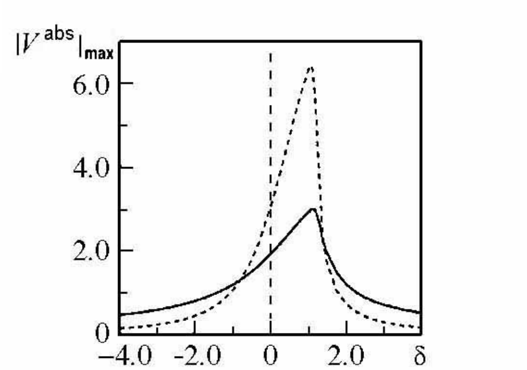

Fig. 13. Nonlinear resonance curves – the dependence on the detuning δ of the absolute maximum of the «amplitude» in the QC-medium (solid line). The dashed line is a resonance curve for averaged intensity

4. Simplified equations for wave beams

The Khokhlov–Zabolotskaya model mentioned above has stimulated the progress in understanding physics of nonlinear diffracting beams in Q-media. This model has the following form:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p}{\partial x} - \frac{\varepsilon}{c^3\rho}p\frac{\partial p}{\partial \tau}\right) = \frac{c}{2}\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right). \qquad (17)$$

Here p is the acoustic pressure; x is the axial coordinate; r is the radial polar coordinate; $\tau = t - x/c$ is the time in the reference system accompanying the wave; ε is the medium nonlinear parameter; ρ is the density. The history of obtaining (17) and the following results is set forth in the view [2].

There are no physically interesting solutions of Khokhlov–Zabolotskaya equations, but the large amount of information is obtained by numerical integration [42]. These data help to develop a series of approximate analytical methods.

The most interesting nonlinear problems concern obtaining maximally strong fields in focus. For «sharp» focusing we can consider the wave before focus as spherically converging, and the wave in focal area – as plane one [43]. In this case the waist area has the form close to cylindrical. For the beams with round cross section the length of this cylinder is equal to l_*, and cross section radius – a_*

$$l_* = \frac{2R^2}{l_d} \ll R, \quad a_* = \frac{aR}{l_d} \ll a. \qquad (18)$$

In this expression $l_d = \omega a^2/(2c)$ is the diffraction distance, R is the focal distance, a is the initial beam radius. Here and further we mean that the focusing is strong and diffraction is weak. Just in this case when $l_d \gg R$, we can form the most intensive fields in the focus.

Based on these assumptions, the authors of the work [44] suggested a model describing the wave within the waist

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p}{\partial x} - \frac{\varepsilon}{c^3\rho}p\frac{\partial p}{\partial \tau}\right) = -\frac{2c}{a_*^2}p. \qquad (19)$$

The equation (19) may be obtained from the equation (17) if we formally assume that the acoustic field close to the axis $(r = 0)$ has parabolic dependence on r

$$p(x, \tau, r) = \left(1 - \frac{r^2}{a_*^2}\right)p(x, \tau). \qquad (20)$$

Substituting (20) into (17) and confining by paraxial area $(r \to 0)$, we come to the model (19), which radically simplifies the analysis. It's obvious that the one-dimensional equation (19) is much simpler than two-dimensional (17) one for analytical study as well as for numerical analysis.

For brevity the equation (19) is further used in dimensionless form

$$\frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta}\right) = -\gamma^2 V. \qquad (21)$$

We should indicate that the equation (21) is just a common model in wave theory. It is called Ostrovsky–Vakhnenko equation [45]. It has been obtained also for oceanology problems (internal waves in rotating ocean [46]) and mathematical physics (soliton-like solutions [47]). The generality of equation (21) is obvious: it is connected with universal law of low-dimensional dispersion $k = \omega/c - \chi/\omega$ and corresponds, for example, to evolutional version of Klein–Gordon equation.

The stationary solution of the equation (21) $V = V(\theta + \beta z)$

$$\frac{dV}{d\theta} = \pm\gamma\sqrt{\frac{2}{3}}\frac{\sqrt{V^3 - (3/2)V^2 + (\beta^3/2)(1-C)}}{|\beta - V|} \tag{22}$$

is expressed through elliptical functions. It is shown in Fig. 14, marked as

$$V = \beta U, \quad T = \theta\gamma\sqrt{2/3}.$$

As it follows from (22), the maximal and minimal values of acoustic pressure are given by the formula [44]

$$p_{\max} = 2\,|p_{\min}| = \frac{\pi^2}{18}\frac{c^2\rho}{\varepsilon}\left(\frac{a}{R}\right)^2. \tag{23}$$

The limiting of intensity is achieved in stationary wave field, which has not only the special profile form but certain amplitude (23), which doesn't depend from initial amplitude and frequency. This «saturation phenomenon» was for the first time studied in the work [48]. For strongly focused wave in the water with focus angle 60° estimation (23) gives $p_{\max} \approx 100$ MPa and intensity $I \approx 50$ kW/cm^2. The limit of focal intensity 96 kW/cm^2 is found by numerical modelling and 33 kW/cm^2 is measured experimentally [49]. The intensities close to the limit are already achieved in medical devices [50].

Let us mark that in focal area we often observe asymmetric profiles (see Fig. 14), but with shocks. But these waves are not stationary and change their form because of energy losses in shock fronts. Just stationary waves bring the energy from source to focus with minimal losses.

The nonstationary waves near the focus are investigated with modified model (19)

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p}{\partial x} + \frac{d\ln f}{dx}p - \frac{\varepsilon}{c^3\rho}p\frac{\partial p}{\partial\tau}\right) = -\frac{2c}{a^2 f^2(x)}p, \quad f^2 = \left(1 - \frac{x}{R}\right)^2 + \left(\frac{x}{l_d}\right)^2. \tag{24}$$

The calculated results of distortion of the wave passing through the focal area, are shown in Fig. 15 [51]. The ratio between focal length and diffraction length R/l_d=0.2 and between focal length and fracture length $R/l_{sh} = 0.4$. The curves in Fig. 15, calculated from the model (24), agree with the results of direct calculation based on model [42].

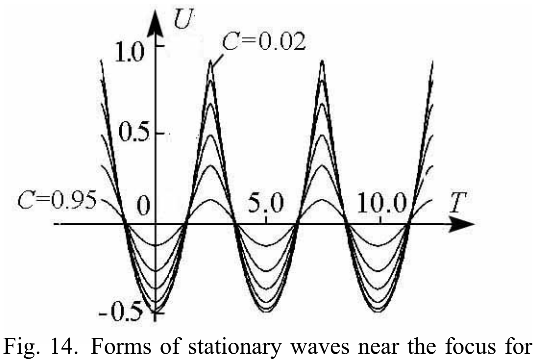

Fig. 14. Forms of stationary waves near the focus for different values $C = 0.95, 0.75, 0.5, 0.25, 0.1, 0.02$. With increasing amplitude ($C \to 0$) the profile is distorted and the positive peak becomes sharp [44]

Let us mark that the equation (21) with variable coefficients of type (24) has been investigated by physicists and mathematicians [52] for many years. Models like (21) were not long ago generalized for M-, cubic and QC-nonlinearities in the works [53] and [54].

Here we don't discuss the models of nonlinear diffraction in the inhomogeneous media (see [55]). Such models are even more «exotic». They are very complicated and almost always need numerical analysis or essential simplification [56]. The exception includes the systems written for nonlinear geometry acoustic approximation. These 3D models admit exact solutions and in this sense are unique, as the «record» 3D Landau–Slyozkin solution of submerged jet [57] (see the historical Review [58]). The latter problem can be also called «exotic». The result is well known but doesn't stimulate research streams, because it's impossible to add anything.

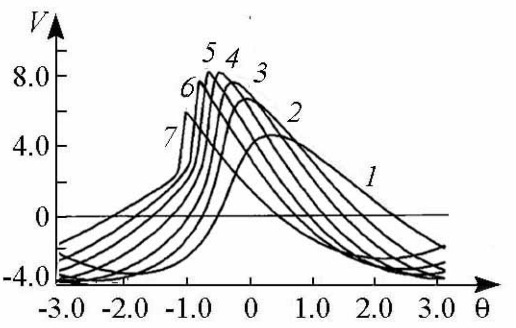

Fig. 15. Profiles of one period of a harmonic (for $x = 0$) wave in a neighborhood of the focus. Curves *1–7* are constructed for distances $x/R = 0.8, 0.9, 0.95, 1.0, 1.05, 1.1, 1.2$ [51]

5. Strongly nonlinear systems with holonomic constraints

It's useful to distinguish strongly nonlinear waves and weak waves with strongly pronounced nonlinearity [59]. Generally we have weak waves, and the defining equations which describe them, can be expanded in power series or functional series. An example is, adiabat «acoustic expansion» in density and pressure powers in the neighborhood of equilibrium state. The terms of these series correlate with quadratic, cubic nonlinearity and the nonlinearities of high degree. In optics the polarisation vector is expanded in powers of the ratio of the electric field to the atomic field. Nevertheless such expansions are not convenient is practical use at least in three cases: firstly, when the defining equations have singularities; secondly, when series diverge in strong fields; thirdly, when the expansion doesn't contain linear term, but the high nonlinearities dominate.

An example of the first type can be found in distributed systems with M- and QC-nonlinearities. There are no limit transitions to linear problems even for very weak signals. The systems with holonomic constraints, discussed below, can be attributed to the third type of strongly nonlinear systems.

As an example of such structure we can consider a crystal cell model shown in Fig. 16. It's important to emphasise that the movement is bounded: the lattice sites can move only along the axis x. Such systems with constraints may be artificially produced but also may have natural origin. For example, mica plates, which can easier be shifted parallel to the planes than orthogonally.

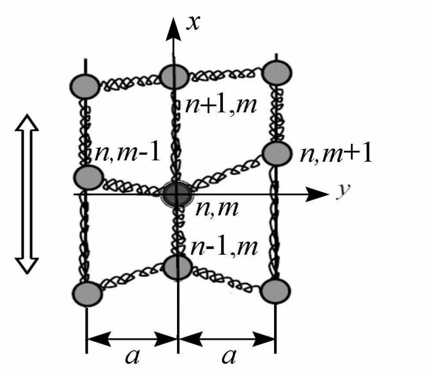

Fig. 16. Lattice of the same masses, connected with the «nearest neighbors» by linearly deformed unstretched springs. The motion of the masses is limited by rods parallel to the x-axis

Let us consider a lattice cell defined by a pair of numbers (n, m). A particle with mass M in this cell is coupled with four nearest neighbours by the same linearly deformable springs with hardness coefficient γ. Let us designate the displacement of the mass from equilibrium along the axis x as $\xi_{n,m}$. The Lagrange function of the structure (see Fig. 16) is obviously the following

$$\begin{aligned} L = \sum_{n,\,m} \Bigg\{ \frac{1}{2} M \dot{\xi}_{n,m}^2 - \frac{\gamma}{2} \left[\sqrt{a^2 + (\xi_{n,m} - \xi_{n,m-1})^2} - a \right]^2 - \\ - \frac{\gamma}{2} \left[\sqrt{a^2 + (\xi_{n,m} - \xi_{n,m+1})^2} - a \right]^2 - \\ - \frac{\gamma}{2} (\xi_{n,m} - \xi_{n+1,m})^2 - \frac{\gamma}{2} (\xi_{n,m} - \xi_{n-1,m})^2 \Bigg\}. \end{aligned} \tag{25}$$

If the displacements of the masses are small in comparison with cell period, i.e. $(\xi_{n,m} - \xi_{n,m-1})^2 \ll 2a^2$, then the Lagrange function is simplified

$$\begin{aligned} L = \sum_{n,m} \Bigg\{ \frac{1}{2} M \dot{\xi}_{n,m}^2 - \frac{\gamma}{8a^2} (\xi_{n,m} - \xi_{n,m-1})^4 - \frac{\gamma}{8a^2} (\xi_{n,m} - \xi_{n,m+1})^4 - \\ - \frac{\gamma}{2} (\xi_{n,m} - \xi_{n+1,m})^2 - \frac{\gamma}{2} (\xi_{n,m} - \xi_{n-1,m})^2 \Bigg\}. \end{aligned} \tag{26}$$

The chain of equations of motion for (26) is the following:

$$M \frac{d^2 \xi_{i,j}}{dt^2} = -\frac{\gamma}{2a^2} \left[(\xi_{i,j} - \xi_{i,j-1})^3 + (\xi_{i,j} - \xi_{i,j+1})^3 \right] + \gamma \left(2\xi_{i,j} - \xi_{i+1,j} - \xi_{i-1,j} \right). \tag{27}$$

It's easy to see that the relations (25)–(27) contain strong nonlinearity. For example, if the springs inside the vertical layers (see Fig. 16) don't deform, and the neighbours move identically but in antiphase ($\xi_{i,j-1} = \xi_{i,j+1} = -\xi_{i,j}$), then (27) changes into equations that doesn't contain linear term $\sim \xi_{i,j}$,

$$\frac{d^2 \xi_{i,j}}{dt^2} + \frac{8\gamma}{a^2 M} \xi_{i,j}^3 = 0. \tag{28}$$

Periodical solutions of the equation (28) are expressed through elliptic integrals. With the increasing of amplitude the oscillation period of $\xi_{i,j}$ grows, and when the amplitude decreases to zero the period tends to infinity and the oscillations vanish.

Discrete systems like (25)–(28) are discussed in papers [60, 61]. Besides, the work [60] describes an experiment with torsional oscillations of coupled disks with bounded movement. Nevertheless there are few works in this direction and it's desirable to go on.

Let us go to the limit of the continuum so that instead of differential-difference equations [27] we would have partial differential equation

$$\frac{\partial^2 \varsigma}{\partial t^2} - c^2 \frac{\partial^2 \varsigma}{\partial x^2} = \frac{c^2}{2} \frac{\partial^2 \varsigma^3}{\partial y^2} + \frac{c^2 a^2}{12} \frac{\partial^4 \varsigma}{\partial x^4}, \quad \varsigma = \frac{\partial \xi}{\partial y}, \quad c^2 = \frac{\gamma a^2}{M}. \tag{29}$$

The dimensionless quantity ς has the sense of deformation along the axis y, and c^2 is the velocity square of the wave propagating in x-direction.

The equation (29) takes into account the nonlinearity, dispersion and anizotropic properties of the structure shown in Fig 16. If the deformation doesn't depend on coordinate x, for pure transverse waves we derive

$$\frac{\partial^2 \varsigma}{\partial t^2} = \frac{1}{2} c^2 \frac{\partial^2 \varsigma^3}{\partial y^2}. \tag{30}$$

We can see that (30) is a strongly nonlinear equation, because there is no limit transition to linear problem with $\varsigma \to 0$. The equation (30) has been used by W. Heisenberg in his version of nonlinear quantum field theory [62]. For strongly nonlinear waves the model (30) has been analysed in [10, 60]. It has been shown that the equation (30) can be matched by the evolutional first-order equation

$$\frac{\partial \varsigma}{\partial t} = \pm\sqrt{\frac{3}{2}}\,|\varsigma|\,\frac{\partial \varsigma}{\partial y}. \tag{31}$$

This equation contains QC-nonlinearity $|\varsigma|\,\varsigma$, discussed in the sections 2–4.

The anizothropic properties described by the equation (29), are evident in the presence for homogeneous statical deformation. Suggesting in (29) $\varsigma = \varsigma_0 + \varsigma'$ and keeping the term ς' linear for small perturbations, we derive:

$$\frac{\partial^2 \varsigma'}{\partial t^2} - c^2 \frac{\partial^2 \varsigma'}{\partial x^2} = \frac{3}{2} c^2 \varsigma_0^2 \frac{\partial^2 \varsigma'}{\partial y^2} + \frac{1}{12}\, c^2 a^2 \frac{\partial^4 \varsigma'}{\partial x^4}. \tag{32}$$

Searching for the solution of (32) as a plane running wave, we find the dispersion law:

$$\frac{\omega^2}{c^2} = k^2 \left(1 - \frac{k^2 a^2}{12} \cos^2\theta\right) \cos^2\theta + \frac{3}{2}\varsigma_0^2 k^2 \sin^2\theta. \tag{33}$$

Here θ is the angle between wave vector $\vec{k}$ and the axis x. When $\theta = 0$ or $\theta = \pi/2$ we have pure longitudinal or transverse wave with the velocities

$$c_{\|}^2 = c^2 - \frac{1}{12}\omega^2 a^2, \quad c_{\perp}^2 = \frac{3}{2}^2 \varsigma_0^2. \tag{34}$$

For small static deformation $c_{\perp}^2 \ll c_{\|}^2 \sim c^2$, i.e. the transverse wave is much «slower» than the longitudinal wave. If there is no static deformation and $\varsigma_0 = 0$, the transverse oscillations can not propagate at all, i.e. the wave process doesn't appear. This situation is typical for muscles, where the speed of shear waves has the order of m/sec, while the longitudinal waves move with the speed near 1.5 km/sec (near the sound velocity in water). Besides the velocities depend on the direction of wave propagation towards the orientation of muscle fibers and muscle tension [63]. These properties are in the basis of the work of modern medical elastographs used for diagnosis of state and pathology in muscles and other soft tissues [64].

For arbitrary angles $\theta \neq 0$, $\theta \neq \pi/2$ the equations (29) and (32) describe the waves which are nor pure longitudinal nor transversal. The equation (29) has also solitary and other interesting solutions.

6. Nonlinear integro-differential equations

The equations like [65]

$$\frac{\partial}{\partial \theta}\left[\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} - \frac{\partial^2}{\partial \theta^2}\int_0^{\infty} K(s) V(\theta - s)\, ds\right] = \Delta_{\perp} V \tag{35}$$

have rather general sense. For degenerate kernel $K(s)$, which may be Delta-function or combination of its derivatives, the following equations could be obtained from [35]: Khokhlov–Zabolotskaya (17), Kadomtsev–Petviashvili, Khokhlov–Zabolotskaya–Kuznetsov and other beam equations; in the case of one-dimensional waves – equation of type (1).

Differential equations result from (35) for some other kernels, too [66]. The most well-known are the equations with exponential kernel, which is predicted by relaxation model of Mandelstam–Leontovitch [57]. In this case for plane waves (35) follows:

$$\left(1 + \theta_{rel}\frac{\partial}{\partial\theta}\right)\left[\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial\theta}\right] = D\frac{\partial^2 V}{\partial\theta^2}. \quad (36)$$

The equation like (36) has been derived in [67] and in integral form – in [68].

Different forms of kernels useful for application are discussed in [69]. In particular, the dependence of wave attenuation coefficient on frequency is described by power law with fractional exponent, which is typical for biological tissues and geophysical structures, and it needs integro-differential description. The same is for media with complex internal dynamics of relaxation type.

How can the kernel $K(s)$ be found in any concrete case? The frequency laws of dispersion and absorption (real and imaginary parts of wave number $k'(\omega)$, $k''(\omega)$) are measured or determined from the physical model of Mandelstam–Leontovitch type. Then the inverse problem is solved and the kernel is reconstructed by standard methods, which use the principle of causality and the Kramers–Kronig relations.

For example, the exponent index in frequency dependence of attenuation in biological tissues changes from 2.1 (skull bones) and 1.7 (testicle tissue) to 1.1 (skeleton) and 0.6 (skin) [5]. Most of all in MHz diapason (medical ultrasound) $k'' \sim \omega^{2-\nu}$, $0 < \nu < 1$. For this law it's not hard to show that $K(s) \sim s^{\nu-1}$. The peculiarity with $s = 0$ is mostly insufficient, because the equation contains «convolution» of singular kernel with oscillation function describing the wave field.

Only few explicit solutions for equations (35) are known. Stationary solutions for the Q- [67] and QC-type [70] media are found.

If we are interested not only in concrete medium but in common properties of nonlinear waves, there is a convenient method reducing integro-differential equation to differential-difference equation or even to simple mapping. This method [69, 70] is effective for kernels differing from zero in finite intervals. The simplest case accords to the medium with constant «memory»: $K(s) = 1$, $0 < s < 1$; $K(s) = 0$, $s > 1$. In other words, before $s = 1$ the medium «remembers everything» and in this moment «forgets everything». For such kernel the equation (36) looks like

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial\theta} = D\frac{\partial}{\partial\theta}\left[V(z, \theta - 1) - V(z, \theta)\right]. \quad (37)$$

For the stationary wave (37) it transforms into difference relation

$$V(\theta - 1) = V(\theta) + \frac{1}{2D}\left[1 - V^2(\theta)\right]. \quad (38)$$

The mapping (38) $V(\theta) \to V(\theta - 1)$ describes the shock wave front, which width grows with the increasing of D.

The following fact may seem surprising to physicists. According to statistical data from «Thomson Reuters», the most frequently cited mathematical works in 2013 were the articles devoted to nonlinear differential equations with fractional derivatives. There are equations including integral term, as (35), but with special form of kernel. Fractional derivative is usually understood in the sense Riemann–Liouville

$$(D_{0+}^{\alpha} u) = \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(s,x)\, ds}{(t-s)^{\alpha+1-n}}, \quad n-1 < \alpha < n, \quad n \in N. \quad (39)$$

Such kernels just are «exotic». More often we can see oscillating kernels (for example, in optics) or relaxation-type kernels (in acoustic or mechanics of hereditary media).

Thus, the border between «exotic» and «popular» models is conditional, flexible and depends on collective activity of a large group of scientists and active cross-citation.

Conclusion

Thus, in this brief overview we discussed several equations, not well known in the «nonlinear society», some analytical solutions and physical results. Our knowledge have been mostly formed under the influence of the classical works in the area of nonlinear physics and mathematics. Today it possibly have sense to refer to «exotic» and to construct models with new physical content. The experience having been accumulated during the past years, is certainly very important in their analysis. From mathematical point of view it seems to be useful to develop the group methods of detecting symmetries for models, which contain generalized functions and singularities. It would be interesting to apply the inverse scattering problem method for conservative systems, for example, with QC- and M-nonlinearities. For physics the primary interest seems to lie in study of strongly nonlinear wave processes and also their nonlinear-corpuscular properties. We look to readers' interest and support for new works in this direction.

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