Applied problems of nonlinear oscillation and wave theory

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DYNAMICS OF COUPLED GENERATORS OF QUASI-PERIODIC OSCILLATIONS WITH EQUILIBRIUM STATE

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Subject of the study. Recently, the problems of synchronization of systems demonstrating quasi-periodic oscillations arouse interest. In particular, it can be generators of quasi-periodic oscillations that allow a radiophysical realization. In this paper we consider the dynamics of two coupled oscillators of quasi-periodic oscillations with a single equilibrium state. **Novelty**. The difference from the already studied case of coupled modified Anishchenko–Astakhov generators consists in engaging of two-parameter analysis and analysis in a much wider range of parameter changes, as well as a more dimensionless equation for an individual generator. **Methods**. The method of charts of Lyapunov exponents is used, which reveals areas of various types of dynamics, up to four-frequency oscillations. The bifurcation mechanisms of complete synchronization are investigated. **Results**. The possibility of synchronous quasi-periodicity is demonstrated, when the phases of the generators are locked, but the dynamics of the system is generally quasi-periodic. The possibility of the effect of «death of oscillations» arising due to the dissipative character of coupling is revealed. The possibility of the effect of broadband quasi-periodicity is demonstrated. Its peculiarity consists in the fact that two-frequency oscillations arise in a certain range of variation of the coupling parameter and a wide range of frequency mismatch. The bifurcation mechanisms of this effect are presented. It is shown that a certain degeneracy is characteristic for it, which is removed when nonidentity is introduced along the control parameters of individual generators. A bifurcation analysis is presented for this case. Two-parameter analysis allowed us to identify points of quasi-periodic bifurcations of codimension two QSNF (Quasi-periodic saddle-node fan) on the parameter plane, associated with the synchronization of multi-frequency tori. These points are the tips of the tongues of the two-frequency regimes, which have a threshold for the coupling coefficient. In their vicinity, three- and four-frequency quasi-periodic regimes are also observed. **Discussion**. Synchronization of quasi-periodic generators has a number of new moments that are established in two-parameter analysis in a wide range of parametric changes.

Key words: quasi-periodic oscillations, coupled generators, synchronization.

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Introduction

Quasi-periodic oscillations are one of the well-known oscillation classes [1–4]. They occupy a kind of intermediate position between periodic and chaotic oscillations. Quasiperiodic oscillations can be characterized by a different number of incommensurable frequencies, so that there occur notions about two-, three-, four-frequency, etc. quasi-periodicity; in the phase space it corresponds to invariant tori of different dimensions. Recently, quasi-periodic oscillations have attracted a great research interest, as they significantly enrich fundamental understanding of the dynamics of self-oscillating systems [3–29]. The basic objectives of research were both nonautonomous systems in the form of various coupled oscillators with limit cycles and new examples of autonomous models with quasi-periodic behavior. Among the most notable phenomena are quasi-periodic bifurcations (bifurcations of invariant tori) [13, 14], Arnold resonance web [15], self-organized quasi-periodicity [17–19], the Landau-Hopf scenario [21], the effect of broadband synchronization [4, 20], etc. It is interesting that quasi-periodicity with different numbers of incommensurable frequencies and quasi-periodic bifurcations is possible not only for coupled limit cycle oscillators, but also for subsystems with chaotic dynamics [22]. Studies of small size ensembles of mappings, where multi-frequency quasi-periodic oscillations and quasi-periodic bifurcations also occur, are presented in [23–26].

If quasi-periodicity occurs in autonomous systems, we may speak of generators of quasi-periodic oscillations. The problem of their interaction is a natural problem within the framework of the approaches to the oscillation theory. Relevant studies were carried out in [11, 12, 27]. One of the first questions in the context of such a problem is the choice of a basic model. In [11, 12] a four-dimensional modified Anishchenko-Astakhov generator [10–12] was used; important and significant results were obtained, yet only the region of small frequency mismatch and small value of the coupling force was studied, and there was no two-parameter analysis. The lower-dimensional (three-dimensional) quasi-periodic oscillator proposed in [28] was used as the basic model in [27]. The disadvantages of this basic model include the fact that it does not have a state of equilibrium, which indicates its simplified nature and limits the application. These limitations have influenced the features of synchronization in coupled systems, in particular, researchers failed to observe the indicative effect of oscillation death associated with dissipatively coupled systems [1]. As part of this research, we study the dynamics of coupled generators of quasi-periodic oscillations, while as a basic model we choose another modification of the generator, which was proposed in [29] and has an equilibrium state. The dynamics of such a system is much more extensive; in particular, we are going to demonstrate that there is an effect of oscillation death and with a decrease in the coupling force there is a regime of two-frequency quasi-periodicity in a very wide range of frequency mismatch. The two-parameter analysis also reveals Arnold's resonance web and new quasi-periodic codimension two bifurcations.

This paper is structured as follows. Section 1 presents a brief overview of dynamics of autonomous generator, deals with arrangement of control parameters plane, which is interesting in terms of coupled generator system dynamics. Section 2 presents a study of coupled generator system. Section 2.1 deals with arrangement of parameter plane and possible types of regimes. Section 2.2 deals with regimes of oscillation death and broadband quasi-periodicity. Section 2.3 deals with quasi-periodic codimension two bifurcations. Section 2.4 is focused on the study of chaotic dynamics in a system of coupled generators.

1. Dynamics of model of a quasi-periodic oscillator with one equilibrium state

In [28] there is a generator of autonomous quasi-periodic oscillations described by equations:

$$\begin{aligned} &\ddot{x} - (\lambda + z + x^2 - \beta x^4)\dot{x} + \omega_0^2 x = 0, \\ &\dot{z} = \mu - x^2. \end{aligned} \tag{1}$$

Various aspects of dynamics of such a generator are described in [28, 16, 27], in particular, synchronization by external signal [16] and the dynamics of coupled generators [27]. However, model (1), as mentioned earlier, does not have a stable equilibrium state. The model proposed in [29] offers a more comprehensive view:

$$\begin{aligned} &\ddot{x} - (\lambda + z + x^2 - \beta x^4)\dot{x} + \omega_0^2 x = 0, \\ &\dot{z} = b(\varepsilon - z) - k\dot{x}^2. \end{aligned} \tag{2}$$

The model (2) can be interpreted as a hard-excitation oscillator with an inertial power supply circuit. Like the model (1) it is a three-dimensional dynamic system, where x, $y = \dot{x}$, z are dynamic variables of the system. Parameter ε indicates energy supply from the power source, parameter b is responsible for inertial properties of the power circuit. The term containing the coefficient k is responsible for power extraction to the oscillator. Model (2) has one equilibrium state:

$$x_0 = y_0 = 0, \quad z_0 = \varepsilon. \tag{3}$$

The equilibrium state may experience the Andronov–Hopf bifurcation which is the birth of limit cycles. To find it, we write the system (2) as a system of three first order differential equations

$$\begin{aligned} &\dot{x} = y \\ &\dot{y} = (\lambda + z + x^2 - \beta x^4)y - \omega_0^2 x, \\ &\dot{z} = b(\varepsilon - z) - ky^2. \end{aligned} \tag{4}$$

The linearization matrix at the equilibrium point is written as

$$M = \begin{pmatrix} 0 & 1 & 0 \\ -\omega_0^2 & -(\lambda + \varepsilon) & 0 \\ 0 & 0 & -b \end{pmatrix}. \tag{5}$$

The characteristic equation for determining the eigenvalues of the matrix L has the following form:

$$L^3 - L^2(\lambda + \varepsilon + b) + L(b(\lambda + \varepsilon) + \omega_0^2) - b\omega_0^2 = 0. \tag{6}$$

The invariants of this matrix are

$$S = \lambda + \varepsilon + b, \quad H = b(\lambda + \varepsilon) + \omega_0^2, \quad J = b\omega_0^2. \tag{7}$$

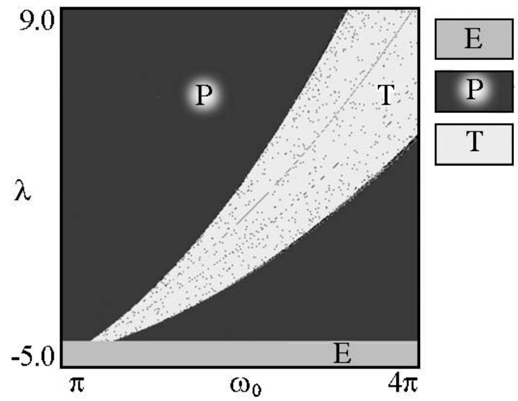

Fig. 1. Chart of the Lyapunov exponents for model of generator of quasi-periodic oscillations (2), ε=4, b=1, $k = 0.02$, $\beta = 1/18$. E – denotes stable equilibrium point, P – denotes periodic oscillations, T – denotes two-frequency quasi-periodic oscillations

The condition for the Andronov–Hopf bifurcation in a three-dimensional system has the form $J = SH$, $H > 0$ [30]. Then, from (7) we obtain

$$\lambda = -\varepsilon. \tag{8}$$

Figure 1 presents a chart of the Lyapunov exponents of system (2) in plane, where eigenfrequency is a control parameter (ω_0, λ), and other parameters are fixed: $\varepsilon = 4$, $b = 1$, $k = 0.02$, $\beta = 1/18$. The charts of Lyapunov exponents were constructed as follows: the full range of the Lyapunov exponents was calculated for each point of the parameter plane; in accordance with the spectral signature, a point in the parameter plane was marked in one or another color shade, where:

E denotes stable equilibrium, $\Lambda_1 < 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$;

P denotes periodic oscillation (cycle), $\Lambda_1 = 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$;

T denotes two-frequency quasi-periodic oscillations, $\Lambda_1 = 0$, $\Lambda_2 = 0$, $\Lambda_3 < 0$.

Figure on the right shows the correspondence of the color shade and literal notation. On all charts there is the Andronov-Hopf bifurcation line (8), which corresponds to the line $\lambda = -4$ for the selected parameter values. Below it is the region of stable equilibrium, and above is the region of a stable limit cycle. The latter has a fairly wide tongue of autonomous quasi-periodic oscillations (see Fig. 1), which will be used to select quasi-periodic dynamics of a separate generator. Note that within this tongue, inversely, there are very narrow regions of periodic regimes.

2. Dynamics of coupled generators

2.1. The structure of the parameter plane. Let us now consider the dynamics of coupled quasi-periodic oscillators (2). Two dissipatively coupled generators of this type are described by the following equations:

$$\begin{aligned}
&\ddot{x}_1 - (\lambda + z_1 + x_1^2 - \beta x_1^4)\dot{x}_1 + \omega_0^2 x_1 + M_c(\dot{x}_1 - \dot{x}_2) = 0,\\
&\dot{z}_1 = b(\varepsilon - z_1) - k\dot{x}_1^2,\\
&\ddot{x}_2 - (\lambda + z_2 + x_2^2 - \beta x_2^4)\dot{x}_2 + (\omega_0 + \Delta)^2 x_2 + M_c(\dot{x}_2 - \dot{x}_1) = 0,\\
&\dot{z}_2 = b(\varepsilon - z_2) - k\dot{x}_2^2,
\end{aligned} \tag{9}$$

where x_1, z_1 are variables determining the first generator; x_2, z_2 are variables of the second generator, Δ is frequency mismatch of the generators, M_c is coefficient of dissipative coupling. The system (9) is determined by four independent frequencies that are defined by the control parameters. As the control parameters we consider the eigenfrequency ω_0 and parameter λ, responsible for excitation of self-excited oscillations in each generator. It is important for us to consider the situation when subsystems demonstrate quasi-periodic oscillations. As seen in Fig. 1, the region of quasi-periodic oscillations is limited in the parameter space by the Neimark–Sacker bifurcation lines; at $\lambda = -1$ a two-frequency quasi-periodic dynamics is observed in the parameter interval ω_0 (6.201...8.45). If fundamental frequency ω_0 is fixed inside the region of two-frequency quasi-periodicity $\omega_0 = 2\pi$, then the first generator in the autonomous regime always demonstrates quasi-periodic oscillations. The change of frequency mismatch Δ into the positive region retains mainly autonomous quasi-periodic regime in the second subsystem, as well. If the second boundary of the tongue is reached, a limit cycle will emerge.

As the main research tool let us also use the method of the Lyapunov exponent charts, likewise for the autonomous generator, only in this case the system is six-dimensional, and the dynamics analysis will be based on the analysis of six Lyapunov exponents:

- E denotes stable equilibrium, $\Lambda_1 < 0, \Lambda_2 < 0, \Lambda_3 < 0, \Lambda_4 < 0, \Lambda_5 < 0, \Lambda_6 < 0$;
- P denotes periodic oscillation (cycle), $\Lambda_1 = 0, \Lambda_2 < 0, \Lambda_3 < 0, \Lambda_4 < 0, \Lambda_5 < 0, \Lambda_6 < 0$;
- T_2 denotes two-frequency quasi-periodic oscillations, $\Lambda_1 = 0, \Lambda_2 = 0, \Lambda_3 < 0, \Lambda_4 < 0, \Lambda_5 < 0, \Lambda_6 < 0$;
- T_3 denotes three-frequency quasi-periodic oscillations, $\Lambda_1 = 0, \Lambda_2 = 0, \Lambda_3 = 0, \Lambda_4 < 0, \Lambda_5 < 0, \Lambda_6 < 0$;
- T_4 denotes the four-frequency quasi-periodic oscillations, $\Lambda_1 = 0, \Lambda_2 = 0, \Lambda_3 = 0, \Lambda_4 = 0, \Lambda_5 < 0, \Lambda_6 < 0$;
- C denotes chaos, $\Lambda_1 > 0, \Lambda_2 = 0, \Lambda_3 < 0, \Lambda_4 < 0, \Lambda_5 < 0, \Lambda_6 < 0$;
- HC denotes hyperchaos, $\Lambda_1 > 0, \Lambda_2 > 0, \Lambda_3 = 0, \Lambda_4 < 0, \Lambda_5 < 0, \Lambda_6 < 0$.

Figure 2 shows arrangement of the parameter plane, with frequency mismatch of the coupling parameter (Δ, M_c) for a system of coupled generators (9) at $\lambda = -1$. Figure 2, *a* is a chart of Lyapunov exponents. Correspondence of the color shade to literal notation is given on the right. Figure 2, *b* shows the main bifurcation lines obtained by using the numerical bifurcation analysis package XPP AUTO.

At a small value of coupling force and frequency mismatch a four-frequency regime of quasi-periodicity T_4 is observed. With increase of the coupling force at low frequency mismatches a sufficiently large region of two-frequency quasi-periodic oscillations is observed (region T_2 in Fig. 2). This region corresponds to phase synchronization of two-frequency quasi-periodic oscillations.

Figure 3, *a* shows the graph of the winding number through variation of frequency mismatch. The winding number was defined as ratio of phases in each oscillator. The phases were calculated for two-dimensional projections in the plane (x, y). Note that the graph stops at a certain value of the winding number, since the phase trajectory projection of the first oscillator in the plane (x, y) moves into the region of the origin, and the phase ceases to be defined. As can be seen from the figure, at low frequency mismatches the winding number is equal to one; however, on the chart of the Lyapunov exponents (Fig. 3, *b*) we see that two-frequency quasi-periodic oscillations occur, that is, the generators are mutually captured, but their attractor is a two-dimensional torus. Thus, we may speak of the synchronous quasi-periodicity regime. In figure 3, *a* these regions are marked with letters PS. Figure 3, *c* gives an example of phase portrait (grey color) and

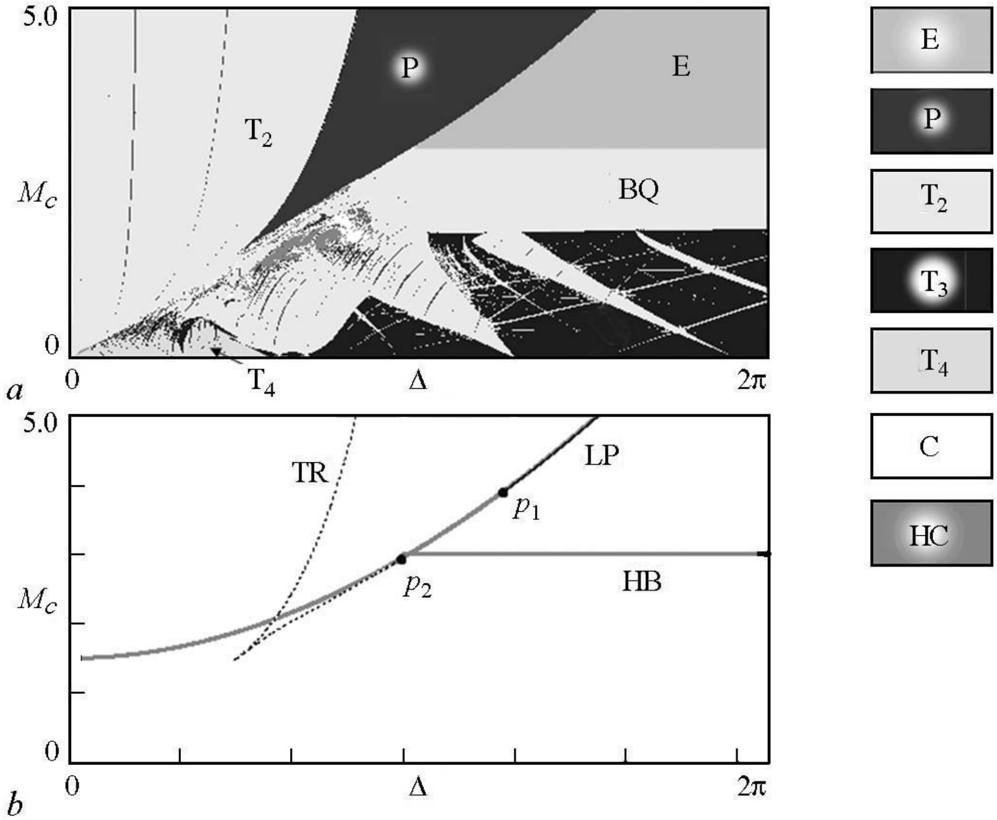

Fig. 2. Chart of Lyapunov exponents (a), bifurcation lines (b) in the parameter plane (Δ, M_c) for model of coupled quasi-periodic generators for $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\lambda = -1$, $\beta = 1/18$, $\omega_0 = 2\pi$

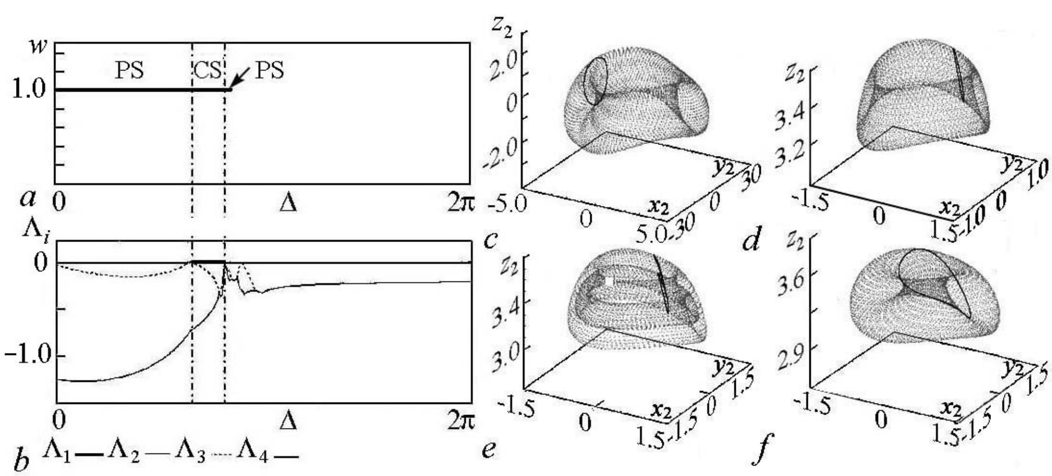

Fig. 3. Dependence of the wending number (a) and the largest four Lyapunov exponents (b) on the frequency mismatch for coupled oscillators (9), $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\lambda = -1$, $\beta = 1/18$, $\omega_0 = 2\pi$, $M_c = 2.5$. Phase portraits (grey color) and Poincaré section by plane $y_1 = 0$ (black color) for $\Delta = 0.5$ (c), 2.56 (d), 2.58 (e), $\Delta = \pi$ (f). PS denotes phase synchronized quasi-periodicity, CS is complete synchronization

Poincar? section (black color) in the synchronous quasi-periodicity region PS, where a smooth invariant curve is clearly visible.

The capturing phase region includes not only the tongue of synchronous quasi-periodicity, but also the tongue of full synchronization (CS on the graph of the winding number in Fig. 3, *a* and P on the chart of the Lyapunov exponents in Fig. 2, *a*). Note that in the charts of Lyapunov exponents we can see a bifurcation of the torus doubling at the transition of frequency mismatch through the value $\Delta = 2.57$ (transition from Fig. 3, *d* to Fig. 3, *e*). Figure 3, *f* corresponds to an even greater value of the mismatch $\Delta = \pi$; we are going to discuss the properties of this regime later.

With mismatch increasing at high coupling in the parameter plane (Δ, M_c) (see Fig. 2, *a*) there is a region of complete synchronization P, that forms the tongue, but it has its own features. First, there is a coupling value threshold and, secondly, this region has a frequency mismatch value threshold. The boundaries of this region form bifurcations of various types. In figure 2, *b* the dotted line represents the Neimark-Sacker bifurcation line (TR), the black solid line is the saddle-node bifurcation line (LP) and in grey color is the Andronov-Hopf bifurcation line (HB). The left border of the tongue fully corresponds to the Neimark-Sacker bifurcation, as a result of which the limit cycle becomes unstable and a two-dimensional torus occurs. The right border of the tongue with a small coupling force is the line of Neimark-Sacker bifurcation. The line of Neimark-Sacker bifurcation ends at point p_2 (3.034, 2.978), stopping on a line of the Andronov–Hopf bifurcation when $M_c = 3$. For large values of the coupling coefficient the boundary of full synchronization tongue is an Andronov-Hopf bifurcation, as a result of which the equilibrium state loses stability and simultaneously there occurs a saddle-node bifurcation, as a result of which a stable limit cycle occurs. Note that these bifurcations occur simultaneously only in a small region and the saddle-node bifurcation line ends at p_1 (4.303, 4.397) in the parameter plane.

2.2. Region of oscillation death and broadband quasi-periodicity. One of the essential features of the pattern observed at high frequency mismatch and strong coupling force is the region of oscillation death that corresponds to the regime of stable equilibrium and is denoted by the letter E. The boundary of the region of oscillation death has a coupling force threshold, in accordance with the ratio $M_c = \lambda + \varepsilon$, which corresponds to the equilibrium stabilization as a result of Andronov-Hopf bifurcation; for the nonce $M_c = 3$. Figure 2, *b* shows the bifurcation lines limiting the region of oscillation death. The lower boundary is the Andronov–Hopf bifurcation line. However, the left border has unique features: as mentioned above, at $M_c > 4.397$ the limit cycle occurs due to the saddle-node bifurcation, which occurs simultaneously with Hopf bifurcation. Also, a specific feature is that the Andronov–Hopf bifurcation line has a continuation for $M_c < 3$.

The next significant point is the presence of a region of two-frequency oscillations, forming a differential band between the region of three-frequency quasi-periodic oscillations and the region of oscillation death. The same regime that occurs below the region of the oscillation death was observed in [20, 21][1] and is called partial broadband synchronization. In our case, we may speak of a broadband quasi-periodicity. The corresponding region is denoted with letters BQ in Fig. 2, *a*. The regime?s main feature is that there

[1] see Fig. 7 in [20].

are two-frequency oscillations in a certain range of the coupling parameter and a wide range of frequency mismatch. Figure 3, f shows an example of a three-dimensional phase portrait and a two-dimensional Poincar? section of a two-dimensional torus corresponding to the broadband quasi-periodicity BQ.

The BQ region is represented by two-frequency quasi-periodic oscillations, therefore, with a decrease in the coupling force between the oscillators in Fig. 2, a there is an instant transition from a steady state of equilibrium to two-dimensional torus. This feature is due to the fact that related subsystems are identical by the excitation parameter of self-oscillations. Since the parameters λ are the same, a degenerate Andronov–Hopf bifurcation occurs, as a result of which the point becomes unstable and a two-dimensional torus occurs. This situation corresponds to the fact that two Lyapunov exponents become zero. Figure 4, *a* shows four largest Lyapunov exponents at variation of the coupling parameter M_c for $\Delta = 2\pi$. As seen in the figure, at a large coupling force all four exponents are negative, while the first and second are equal in absolute value. As the coupling force decreases, the absolute value of the first four exponents increases, and at $M_c = 3$, all four exponents are zero. With a further coupling force decrease the two largest exponents are zero, and the third and fourth become negative again.

This degeneration is removed with the introduction of non-identity for parameter λ between the coupled oscillators; an example is shown in Fig. 4 for $\lambda_1 = -1$ and $\lambda_2 = -0.5$. As in the previous case, with a strong coupling force all exponents are negative, which corresponds to the oscillation decay. At $M_c = 3.61$, a supercritical Hopf bifurcation (HB) occurs, after which the largest exponent becomes zero, and the second one becomes negative again, which corresponds to self-oscillations. Then, at $M_c = 2.946$, the Neimark–Sacker bifurcation (TR) occurs, as a result of which a two-dimensional torus occurs and the two Lyapunov exponents become zero.

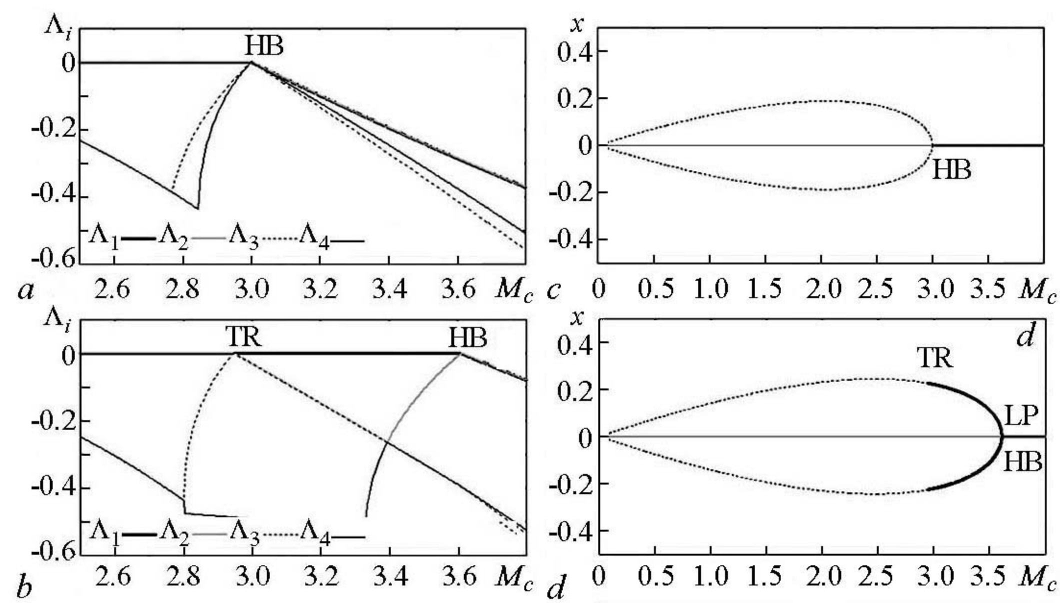

Fig. 4. *a, b* – dependence of the largest four Lyapunov exponents on the coupling strength, $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\beta = 1/18$, $\omega_0 = 2\pi$, $\lambda = -1$ и $\lambda_1 = -1$, $\lambda_2 = -0.5$; *c, d* – bifurcation diagrams for the same parameters

Figures 4, *c* and 4, *d* show bifurcation diagrams for variation of the parameter M_c, where bold and thin grey lines denote stable and unstable equilibrium states, respectively, solid and dotted black lines denote stable and unstable cycles. Bifurcation analysis shows that in the first case, as a result of the Andronov–Hopf bifurcation (HB), an unstable cycle occurs, which then collapses to zero in the coupling parameter. For the second case, the Andronov–Hopf bifurcation (HB) occurs simultaneously with a saddle-node bifurcation (LP); thus, the equilibrium state loses stability and simultaneously, as a result of saddle-node bifurcation, a limit cycle occurs, which at a weaker coupling force undergoes the Neimark–Sacker bifurcation (TR).

Note that in Fig. 2, *a* on the right side of the chart below the BQ region one may observe Arnold resonance web [15] based on the region of three-frequency oscillations.

2.3. Quasi-periodic bifurcation of codimension two. Now let us consider the features of system dynamics with a small coupling force. Figure 5 shows a zoomed fragment of the Lyapunov exponent chart. One may see the regions of four frequency quasi-periodic oscillations T_4, in which two tongues of three-frequency quasi-periodicity T_3 are embedded. The bases of these tongues are marked with values Δ_1 and Δ_2 and lie on the abscissa axis corresponding to the zero value of the coupling parameter M_c. These tongues are bounded by the lines of saddle-node bifurcations of four-dimensional tori [13], in the figure denoted by SNT. Two such lines SNT_1 and SNT_2 begin at the point of codimension two Δ_1 on the horizontal axis[2]. Note that this picture is similar to the classical one for the Arnold tongue, only instead of the region T_3 there is a periodic regime, and instead of T_4 there is a two-dimensional torus.

Whereas, two such lines SNT_2 and SNT_3, belonging to two different tongues, intersect and end at the point QSNF, which arises when two parameters are regulated and also has a codimension of two. Above this point, two-frequency quasi-periodic oscillations T_2 are observed in the overlap region of the three-frequency tongues, and this region also has a form of tongue. A similar structure is typical for systems with multi-frequency quasi-periodic oscillations [4]. Such points in [4] are called Quasi-periodic saddle-node fan QSNF.

A similar structure was described in [4] for four coupled Van der Pol oscillators, as well as a system of equations in phase approximation.

In figure 5 there are two points $QSNF_1$ and $QSNF_2$. The coupling amplitude threshold for each of them is approximately the same.

Also in figure 5 the point with coordinates $(\Delta_3, 0)$ is marked. To the right of it, a three-frequency quasi-periodicity occurs at an arbitrarily small coupling force. The reason for it is that mismatch value corresponds to transition of the second individual generator from quasi-periodic to periodic oscillations in accordance with figure 1. (The right boundary of the region of two-frequency quasi-periodicity in figure 1.)

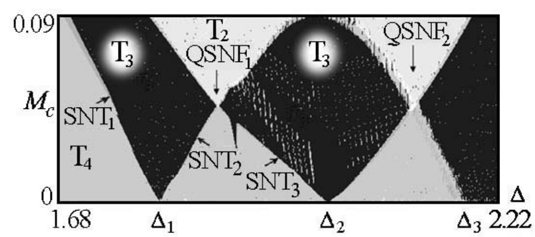

Fig. 5. Zoomed fragment of the chart of Lyapunov exponents of coupled oscillators (9) near special points QSNF. $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\lambda = -1$, $\beta = 1/18$, $\omega_0 = 2\pi$

[2]A similar picture was observed in [31], only the dimension of tori was reduced by one.

2.4. Chaotic dynamics. Another feature of the model (9) is the formation of chaotic dynamics. The regions of chaotic dynamics C are localized under the tongue of full synchronization, in the region where small tongues of high-order synchronization overlap. With large frequency mismatches and strong coupling force chaotic oscillations are not observed. Despite the fact that the region of chaos is small in the parameter space, there is not only chaos, but hyperchaos with two positive Lyapunov exponents HC.

Figure 6, *a* shows a zoomed fragment of the Lyapunov exponent chart near region of the main “island” for chaotic dynamics. As seen from the figure, most of the chaotic regimes represent chaos with one positive Lyapunov exponent, but within the island one

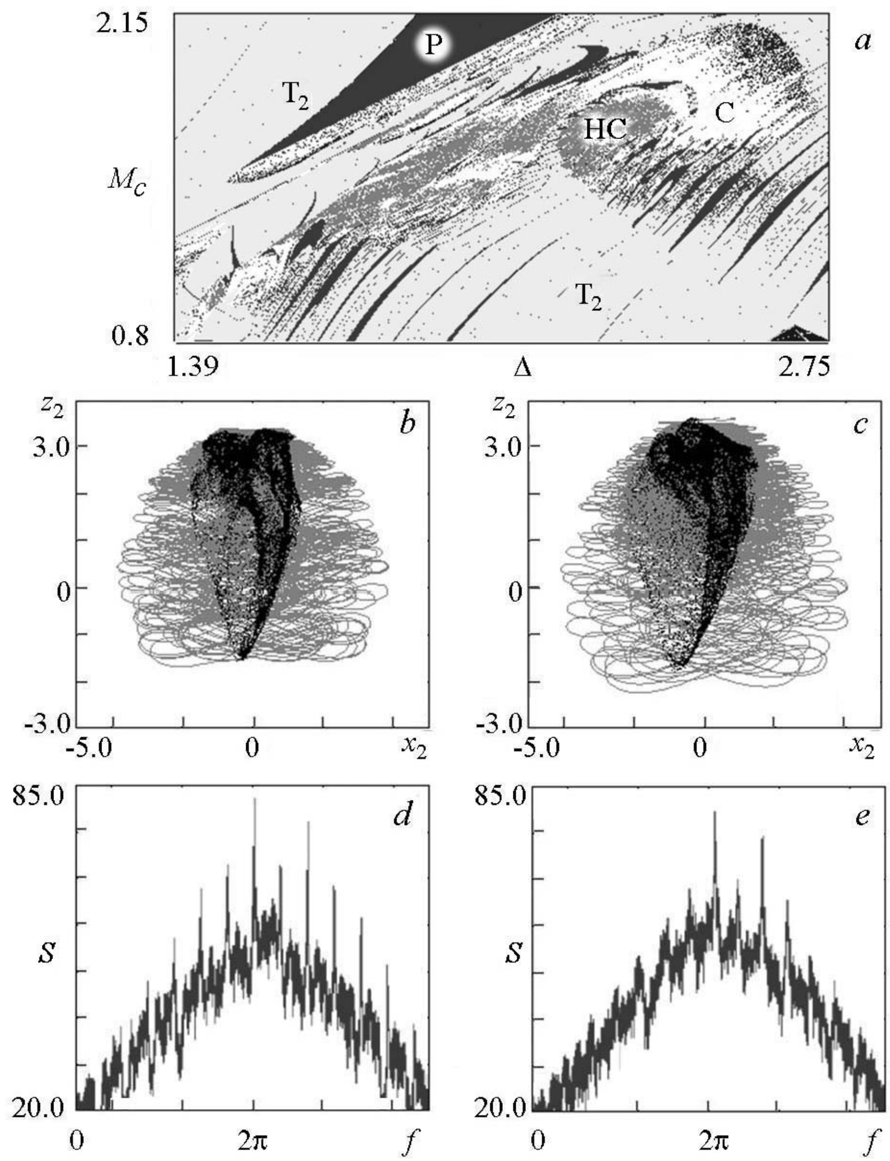

Fig. 6. *a* – zoomed fragment of the chart of Lyapunov exponents of coupled oscillators (9) near area of chaotic dynamics for $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\lambda = -1$, $\beta = 1/18$, $\omega_0 = 2\pi$; phase portraits, Poincare sections and Fourier spectrums for chaotic regime (*b*, *d*): $\Delta = 2.5$, $M_c = 1.7$; and hyperchaotic regime (*c*, *e*): $\Delta = 2.34$, $M_c = 1.69$

can distinguish regions of hyperhaos. Figures 6, b, d and 6, c, e illustrate the arrangement of phase space and Fourier spectra for the chaos and hyperchaos, respectively. For Fig. 6, b, d were recorded parameters $\Delta = 2.5$, $M_c = 1.7$, for which the largest Lyapunov exponents have values $\Lambda_1 = 0.088$, $\Lambda_2 = 0$, $\Lambda_3 = -0.025$. For Fig. 6, c, e we chose parameters $\Delta = 2.34$, $M_c = 1.69$, for which three largest Lyapunov exponents have values $\Lambda_1 = 0.106$, $\Lambda_2 = 0.018$, $\Lambda_3 = 0$.

As seen from the figure, the structure of attractors in a phase space is quite complicated both for chaos and hyperchaos. Fourier spectra have a base peak at frequency $\omega_0 = 2\pi$, and there is also a peak at frequencies $\omega_0 = 2\pi + \Delta$. Herewith, in the case of chaos in the Fourier spectrum, peaks at combinational frequencies are also observed, which is typical for quasi-periodic oscillations. Therefore, we may say that chaos arose as a result of the torus destruction. For hyperchaos the matching frequency is far less noticeable, the spectrum becomes smoother and more uniform, a small extension is observed.

Conclusion

Features of mutual synchronization of quasiperiodic oscillations have been studied on the example of a model with two coupled quasiperiodic oscillation generators in equilibrium state. Synchronization of generators in the plane (frequency mismatch ? coupling force) is described in detail. We have clearly demonstrated that in such a system both full synchronization corresponding to the periodic regime and synchronous quasi-periodicity regime is possible, if the oscillations are quasi-periodic and the phases of the generators are accurately captured. There was also a regime of oscillation death. The peculiarity of this system is also the possibility of broadband quasi-periodicity, which consists in the fact that there are two-frequency oscillations in a certain range of changes in the coupling force parameter and a wide range of frequency mismatch. This area is observed with a decrease of the coupling force parameter below the region of the oscillation death. Other regimes of two-frequency quasiperiodicity have the form of tongues embedded in the region of three-frequency oscillations, and forming a resonant Arnold web.

We have studied in full detail the transition from phase of full synchronization to broadband quasi-periodicity regime. At such a transition two-dimensional torus doubling bifurcations are found. The transition from four-frequency to three-frequency quasiperiodicity with weak coupling force has been studied. This transition is associated with points of quasi-periodic codimension two bifurcations QSNF (quasiperiodic saddle-node fan). These points are the tips of the tongues in two-frequency regimes, having a threshold value for the coupling coefficient. Near them there are also observed three- and four-frequency quasi-periodic regimes. We have discovered a possible application of chaotic dynamics with both one and two positive Lyapunov exponents for the proposed model.

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