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# Equations with the nonlinearities of dislocation and Fermi–Pasta–Ulam

*S. D. Glyzin*[1], *S. A. Kashchenko*[1,2], *A. O. Tolbey*[1]

[1]P.G. Demidov Yaroslavl State University, Russia,
14, Sovetskaya st., Yaroslavl 150003, Russia
[2]National Research Nuclear University «MEPhI», Russia
31, Kashirskoe sh., Moscow 115409, Russia
Email: glyzin.s@gmail.com, kasch@uniyar.ac.ru, a.tolbey@uniyar.ac.ru
Correspondence should be addressed to Anna O. Tolbey, a.tolbey@uniyar.ac.ru
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**Issue.** The class of Fermi–Pasta–Ulam equations and equations describing dislocations are investigated. Being a bright representative of integrable equations, they are of interest both in theoretical constructions and in applied research. **Investigation methods.** In the present work, a model combining these two equations is considered, and local dynamic properties of solutions are investigated. An important feature of the model is the fact that the infinite set of characteristic numbers of the equation linearized at zero consists of purely imaginary values. Thus, the critical case of infinite dimension is realized in the problem on the stability of the zero solution. In this case a special asymptotic method for construction of the so-called normalized equations is used. Using such equations, we determine the main part of the solutions of the original equation, after that we can investigate the asymptotic behavior using perturbation theory methods. **Results.** All solutions are naturally divided into two classes: regular solutions that smoothly depend on a small parameter entering the equation, and irregular ones, which are a superposition of functions that oscillate rapidly on a spatial variable. For each class of solutions, areas of such changes in the parameters of the equation are distinguished in which the main parts are described by different normalized equations. Sufficiently wide classes of such equations are presented, which include, for example, the families of the Schrödinger, Korteweg–de Vries, and other equations. The problem of determining such a set of parameters of the original equation for which the nonlinearity of dislocations and the nonlinearity of the FPU are comparable «in force» is considered, i.e. none of them can be neglected in the first approximation. **Discussion.** It is interesting to note that for regular and irregular solutions the areas of parameters in which nonlinearities are comparable are different. In the second case the corresponding region is much wider. The article consists of two chapters. In the first chapter, normalized equations for regular solutions are constructed, and in the second, for irregular ones. In turn, the first chapter is divided into three parts, in each part different normalized equations are constructed (depending on the values of the parameters).

*Key words*: bifurcations, stability, normal forms, singular perturbations, nonlinear dynamics.

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## Formulation of the problem

Let's take the simplest crystal structure consisting of layers of atoms located at some distance from each other. J.I.Frenkel and T.A.Kontrova [1] have suggested a model of behaviour of a point defect in solid crystal structure. The defects of this kind are sometimes called dislocations. In the same work [1] the mathematical model describing the behaviour of this point defect has been introduced into scientific use. This is the system like

$$m\ddot{y_n} + a\sin y_n = y_{n+1} - 2y_n + y_{n-1}, \ (n = 1, \dots, N), \tag{1}$$

where $m$ is the positive coefficient, $y_n = y_n(t, x_n)$ is the equilibrium deviation of the $n-$th atom. For $y_n(t, x_n)$ for extreme values of $n$ the conditions typical for one of the boundary value problems are satisfied: periodical $y_{N+1} = y_1, y_0 = y_N$; Dirichlet problem $y_{N+1} = y_0 = 0$; Neumann problem $y_{N+1} = y_N, y_0 = y_1$. The points $x_n$ lie on the section $[0, 2\pi]$ and $x_{n+1} = x_n + \varepsilon$, where $\varepsilon = 2\pi N^{-1}$. It is supposed that the number of interacting elements $N$ is large or, which is the same:

$$0 < \varepsilon \ll 1.$$

It should be noted that at present, a dislocation is understood to mean a more complex imperfection of the crystal structure than any of pointed defects [2]. Accounting of dislocations is based on the well-known Fermi–Pasta–Ulam problem.

$$m\frac{d^2y_n}{dt^2} = F_{n+1,n} - F_{n,n-1}, \ (n = 1, \dots, N), \tag{2}$$

where

$$F_{n+1,n} = y_{n+1} - y_n + \alpha(y_{n+1} - y_n)^2 + \beta(y_{n+1} - y_n)^3,$$

$\alpha$ and $\beta$ are positive coefficients.

The systems (1) and (2) differ from each other by lack of function $a\sin y_n$ in Fermi–Pasta–Ulam equation and presence of quadratic and cubic nonlinearity in the right part of (2). In order to take into account the more complex imperfections of point defects, let's consider the system containing as the elements of (1), so as the elements of the system (2). The model uniting both these equations has been considered in the work [3], in which a number of questions concerning Painleve integrability of its solutions, has been studied. One of the simplest methods of working with such equations is transition to continuous mass distribution (see for example [4]), whereby we get (after obvious renormalizations and replacing $a\sin y$ with a more general function $f(y)$)

$$\begin{aligned} &\ddot{y} + f(y) = y(t, x+\varepsilon) - 2y + y(t, x-\varepsilon) + \\ &\quad + \alpha\Big(y^2(t, x+\varepsilon) - 2y(t, x+\varepsilon)y(t,x) + 2y(t,x)\cdot y(t, x-\varepsilon) - y^2(t, x-\varepsilon)\Big) + \\ &\qquad\qquad + \beta\Big((y(t, x+\varepsilon) - y(t,x))^3 - (y(t,x) - y(t, x-\varepsilon))^3\Big). \end{aligned} \tag{3}$$

For definiteness, we assume that periodic boundary conditions are satisfied:

$$y(t, x + 2\pi) \equiv y(t, x). \tag{4}$$

The nonlinear function $f(y)$ is taken in the form

$$f(y) = ay + by^3 + \varphi(y), \quad \text{for} \ \ \varphi(y) = o(|y|^3) \ \ \text{as} \ \ y \to 0.$$

The work [1] describes the case when $f(y) = \varepsilon^2 \sin y$, i. e. $a = \varepsilon^2$, $b = \varepsilon^2/6$. Let's also note the works [3–10], in which the case $f(y) = 0$, i.e. the Fermi–Pasta–Ulam problem is considered.

The boundary problem (3), (4) can be considered in two fundamentally different cases. The first is related to the assumption of smoothness of solutions about $\varepsilon$ and decomposing of $y(t, x\pm\varepsilon)$ into asymptotic $\varepsilon$ series. We'll call this case "regular" and consider several variants of this work in dependence of relation between orders of smallness of the values $a, b$ towards $\varepsilon$. The second variant of the problem statement is an irregular case. In this case the solutions, rapidly oscillating by $x$, will be studied.

In the regular case, one of the main goals will be the selection of the orders of smallness of the coefficients of the $f(y)$ function included in the boundary value problem (3), (4), so that the contribution to dynamic properties of solutions of the nonlinearity $f(y)$ and FPU nonlinearity would be comparable. It's obvious that choosing $a$ and $b$ small enough, we'll come to the FPU problem and vice versa, if $a$ and $b$ are not small, the problem (3),(4) will come to the equation of sine–Gordon type [2, 7, 11] and the nonlinearities contained in the function $F_{n+1,n}$, will not play any role.

## 1. Regular solutions

Under the condition (2) we consider the behaviour of all solutions of the boundary problem (3), (4) with the initial conditions from some sufficiently small (and independent of $\varepsilon$) neighbourhood of the zero equilibrium state. In this case the main role is played by the solution of the boundary value problem linearized at zero

$$\ddot{y} + ay = y(t, x+\varepsilon) - 2y + y(t, x+\varepsilon), \quad y(t, x+2\pi) \equiv y(t, x). \tag{5}$$

The characteristic equation for (5) has the following form:

$$\lambda^2 + a = -4\sin^2\left(\frac{\varepsilon k}{2}\right), \quad k = 0, \pm 1, \pm 2, \ldots \tag{6}$$

Below we assume that the parameter $a$ is positive and for some fixed parameters $a_0$ and $a_1$ we have the relation

$$a = a_0 + \varepsilon a_1, \quad a_0 \geqslant 0 \quad \text{and} \quad a_1 > 0 \quad \text{for} \quad a_0 = 0.$$

All the roots of (6) are purely imaginary. Thus in the problem of stability of zero in (3), (4) the critical case of infinite dimension is realized. Using the methods suggested in [10, 12–14], below we'll construct families of special boundary problems to describe the various groups of solutions of the initial boundary value problem (3), (4). We must mark that from an applied point of view [1, 3, 7], the case when the parameter $a$ is small, is particularly important. In this regard, the case when $a_0 = 0$ will be considered.

In this section, we explore regular solutions, i.e. the solutions for which the asymptotic representation takes place

$$y(t, x+\varepsilon) = y(t, x) + \varepsilon y'(t, x) + \frac{1}{2}\varepsilon^2 y''(t, x) + \ldots \tag{7}$$

In terms of the location of the characteristic equation roots (6) we consider the solutions, basically formed on the modes with "finite" numbers $k$ (i.e. independent from $\varepsilon$ parameter).

Here we stop on studying regular solutions of boundary problem (3), (4). Let's substitute (7) into (3). For determining $y(t,x)$ we obtain the boundary problem with an accuracy of $o(\varepsilon^6)$:

$$\frac{\partial^2 y}{\partial t^2} = \varepsilon^2\Big[\frac{\partial^2 y}{\partial x^2} + \frac{\varepsilon^2}{12}\frac{\partial^4 y}{\partial y^4} + \frac{2\varepsilon^4}{6!}\frac{\partial^6 y}{\partial x^4}\Big] - f(y) +$$

$$+\alpha\varepsilon^3\left[2\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x^2} + \frac{\partial}{\partial x}\Big(\frac{1}{6}\varepsilon^2\Big(\frac{\partial y}{\partial x}\frac{\partial^3 y}{\partial x^3} + \frac{1}{2}\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2\Big)\Big)\right] + \tag{8}$$

$$+\beta\varepsilon^4\frac{\partial}{\partial x}\left[\Big(\frac{\partial y}{\partial x}\Big)^3 + \frac{1}{4}\varepsilon^2\Big(\Big(\frac{\partial y}{\partial x}\Big)^2\frac{\partial^3 y}{\partial x^3} + \frac{\partial y}{\partial x}\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2\Big)\right],$$

$$y(t, x+2\pi) \equiv y(t,x). \tag{9}$$

Let's mark that by this way in [1] the well-known sine-Gordon equation has been obtained, and in this case in the equation (8) the terms down to second order of $\varepsilon$ have been left. It's interesting to know what changes in the dynamics of boundary problem (8), (9) take place when taking into account the terms of a higher order of smallness. Let's pay attention that without taking into account the term $2\varepsilon^4(6!)^{-1}\partial^6 y/\partial x^6$ the boundary problem (8), (9) is incorrect.

In particular, the main local existence and uniqueness theorem is not satisfied. Based on these considerations, it is advisable to write the decomposition of the right parts of (3) with accuracy up to $o(\varepsilon^6)$. We also note here that the periodic conditions (9) are chosen for definiteness, since, in the case of the Dirichlet or Neumann conditions, the method for studying the corresponding boundary value problem remains the same.

So, for all sufficiently small values of $\varepsilon$, we consider the question of local — in a neighbourhood of a zero equilibrium state, sufficiently small and independent of $\varepsilon$ — behaviour of the boundary problem solutions (8), (9).

An important role in the analysis of solutions from a small neighbourhood of zero is played by the linearized equation

$$\frac{\partial^2 y}{\partial t^2} = \varepsilon^2\left[\frac{\partial^2 y}{\partial x^2} + \frac{\varepsilon^2}{12}\frac{\partial^4 y}{\partial x^4} + 2\frac{\varepsilon^4}{6!}\frac{\partial^6 y}{\partial x^6}\right] - ay \tag{10}$$

with boundary conditions (9). Characteristic equation for (10), (9) is the following:

$$\lambda^2 = -\varepsilon^2\left((2\pi k)^2 - \frac{(2\pi k)^4}{12}\varepsilon^2 + \frac{2(2\pi k)^6}{6!}\varepsilon^4\right) - a, \quad (k = 0, \pm1, \pm2, \pm3, \ldots). \tag{11}$$

All the roots (11) have zero real parts. In the case $f(y) = 0$ the boundary problem (8), (9) was studied with the use of asymptotic methods in the works [10,15]. The study is based on normalization method. Let's speak about it briefly. The boundary problem solutions (10), (9) can be formally written as a series:

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \xi_k \exp(\varepsilon\lambda_k(\varepsilon)t),$$

where $\lambda_k$ are the roots of the equation (11). Then the solution $u(t,x,\varepsilon)$ of the boundary problem (8), (9) is sought as a formal asymptotic series

$$u(t,x,\varepsilon)=\varepsilon\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\xi_k(\tau)\exp(i2\pi kx)+\varepsilon^3u_3(t,\tau,x)+\ldots,\tag{12}$$

where $\tau=\varepsilon^2t$, and $\xi_k(\tau)$ are the unknown functions slowly changing by $\tau$. Let's substitute the series (12) into the boundary problem (8), (9), as a result, we come to an infinite system of ordinary differential equations for $\xi_k(\tau)$. This system is called the normalized equation for the boundary value problem (8), (9). Sometimes this infinite system can be written in a compact form as a nonlinear equation in partial derivatives. As a rule, in all the cases considered below it is possible to present a normalized equation in compact form. In this case, normalized equations are constructed, which determine the dynamics of solutions for a sufficiently small $\varepsilon$. In our work we consider the effect of function $f(y)$ upon the structure of normal forms of boundary problem (8), (9) for different orders of smallness of the values $a$ and $b$. Of particular interest is the situation in which the contribution of the function $f(y)$ and the FPU equation turns out to be close in order. In this connection, we begin the consideration of the normal form of the problem with the case $a_0=0$ and $b$ having the order of unity, then we study the case of $a_0=0$ and small $b$, and at last, we examine the situation, when $a_0\neq 0$, within which we allocate the subcases when $b$ is of the order of unity and $b$ of the order of $O(\varepsilon^4)$.

**1.1. Normal form for $a_0=0,\ b\neq 0$.** Let $a_0=a_1=0$, and $b$ as $\varepsilon\to 0$ have order equal to unity. Then after renormalizing the "time" $\varepsilon t\to t$ and replacing $y(t,x)=\varepsilon u(t,x)$ the problem (8), (9) is simplified, since we can omit the terms of the order of $\varepsilon^3$ and $\varepsilon^4$

$$\frac{\partial^2u}{\partial t^2}=\frac{\partial^2u}{\partial x^2}+\frac{\varepsilon^2}{12}\frac{\partial^4u}{\partial x^4}+\frac{2\varepsilon^4}{6!}\frac{\partial^6u}{\partial x^6}-bu^3,\tag{13}$$

$$u(t,x+2\pi)\equiv u(t,x).\tag{14}$$

The linear equation

$$\frac{\partial^2u}{\partial t^2}=\frac{\partial^2u}{\partial x^2}+\frac{\varepsilon^2}{12}\frac{\partial^4u}{\partial x^4}+\frac{2\varepsilon^4}{6!}\frac{\partial^6u}{\partial x^6}\tag{15}$$

with periodical boundary conditions (14) for $\varepsilon=0$ has a set of periodic solutions $u_0$, $\xi_k\exp(i2\pi k(x+t))$, $\eta_k\exp(i2\pi k(x-t))$, $k=\pm1,\pm2,\ldots$, where $u_0$ is the real and $\xi_k$, $\eta_k$ are the complex constants. According to the algorithm of the works [14, 16] for studying of the nonlinear boundary problem solutions (13), (14), we introduce the formal series

$$u=\varepsilon\Big(u_0(\tau)+\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\xi_k(\tau)\exp(i2\pi k(x+t))+\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\eta_k(\tau)\exp(i2\pi k(x-t))\Big)+\\+\varepsilon^3u_3(\tau,t,x)+\ldots,\tag{16}$$

$$\xi_{-k}(\tau)=\bar{\xi}_k(\tau),\quad \eta_{-k}(\tau)=\bar{\eta}_k(\tau).$$

Here $\tau=\varepsilon^2t$ is the slow time and the dependance of the second and third arguments of the functions $u_j(\tau,t,x)$ is periodic. Once more we mark that the series like (16) are formal, i. e. the question about their convergence is not considered. From the results below it follows that they are asymptotic. The formula (16) can be made simpler. For this we set

$$\xi(\tau,x)=\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\xi_k(\tau)\exp(i2\pi kx),\quad \eta(\tau,x)=\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\eta_k(\tau)\exp(i2\pi kx).\tag{17}$$

Let's substitute (16) into the original equation (13). In the resulting formal identity we equate the coefficients with the same powers of $\varepsilon$. We recall that the transition from the equation for the function $u$ to the equations for the Fourier coefficients $u_0, \xi_k$ and $\eta_k$ is called "normalization", and the equations for new slow variables — the "normal form".

For the first degree of $\varepsilon$ (in a formal identity) we get the true equality, and collecting the coefficients for $\varepsilon^3$, we come to a boundary value problem for $u_3$:

$$\frac{\partial^2 u_3}{\partial t^2} - \frac{\partial^2 u_3}{\partial x^2} = F(u_0, \xi, \eta), \quad u_3(\tau, t, x+1) \equiv u_3(\tau, t, x), \tag{18}$$

where

$$F(u_0, \xi, \eta) = -b(u_0 + \xi + \eta)^3 + 2\frac{\partial^2 \xi}{\partial\tau\partial x} - 2\frac{\partial^2 \eta}{\partial\tau\partial x} + \frac{1}{12}\Big(\frac{\partial^4 \xi}{\partial x^4} + \frac{\partial^4 \eta}{\partial x^4}\Big),$$

$$\xi = \xi(\tau, x+t), \quad \eta = \eta(\tau, x-t).$$

For solvability of the equation (18) in the specified class of functions, it is necessary and sufficient that the function $F(u_0, \xi, \eta)$ doesn't contain the harmonics $\exp(ik(x+t))$ and $\exp(ik(x-t))$ for $k = 0, \pm 1, \pm 2, \ldots$. Let's apply this criterion. As a result, we come to three ratios. Firstly, from the condition of zero-equality of the Fourier coefficient of the function $F(u_0, \xi, \eta)$ for the zero harmonic, i. e. from the condition $M(F(u_0, \xi, \eta)) = 0$, where is the designation

$$M(\varphi(x)) = \int_0^1 \varphi(x)dx,$$

we get that

$$u_0^3 + u_0 A + B = 0. \tag{19}$$

Here

$$A = 3(M(\xi^2) + M(\eta^2)),\ B = 3(M(\xi^3) + M(\eta^3)). \tag{20}$$

Secondly, from the equality to zero of the Fourier coefficients of the function $F$ with harmonics $\exp(i2\pi k(x+t))$ $k = \pm 1, \pm 2, \ldots$ we conclude that

$$2\frac{\partial^2 \xi}{\partial\tau\partial x} = \frac{1}{12}\frac{\partial^4 \xi}{\partial x^4} + F_1(u_0, \xi, \eta), \tag{21}$$

where $F_1(u_0, \xi, \eta) = 3b\big[u_0^2\xi + 3u_0(\xi^2 - M(\xi^2)) + \xi^3 - M(\xi^3) + 3M(\eta^2)\xi\big]$. The third relation is obtained by equating to zero the Fourier coefficients of the function $F$ at harmonics $\exp(i2\pi k(x-t))$ $(k = \pm 1, \pm 2, \ldots)$ :

$$-2\frac{\partial^2 \eta}{\partial\tau\partial x} = \frac{1}{12}\frac{\partial^4 \eta}{\partial x^4} + F_1(u_0, \eta, \xi). \tag{22}$$

We note that for $\xi$ and $\eta$ the following periodical boundary conditions are fulfilled

$$\xi(\tau, x+1) \equiv \xi(\tau, x), \quad \eta(\tau, x+1) \equiv \eta(\tau, x). \tag{23}$$

Since the value of $A$ in (20) is positive, the equation (19) with respect to $u_0$ has a unique solution $u_0 = u_0(\xi, \eta)$. Note that $u_0$ can be written as

$$u_0 = v_0(\xi + \eta), \tag{24}$$

while the function $v_0(s)$ has the property $v_0(cs) = c^2 v_0(s)$, because

$$M(\xi^2) + M(\eta^2) = M((\xi + \eta)^2), \quad M(\xi^3) + M(\eta^3) = M((\xi + \eta)^3).$$

Because of this, the nonlinearity in (21), (22) is cubic.

It is necessary to mark that “interaction” of the equations (21) and (22) is very specific. In (21) the influence of the variable $\eta$ is carried out through the averaged values $M(\eta^2)$ and $M(\eta^3)$, while in (22) — respectively, through $M(\xi^2)$ and $M(\xi^3)$.

Let’s formulate the result on the connection of solutions of the boundary value problem (13), (14) and solutions (21)–(23). Its rationale follows from the constructions above.

Let $u_0, \xi_0(\tau, x+t)$ and $\eta_0(\tau, x-t)$ be the solutions of the boundary problem (19), (21)–(23). Using them one can find the function $u_3$ from (18). Just checking that

$$\begin{aligned}u_3(\tau,t,x) = &\frac{3b}{4}\Big[\Big(2u_0\eta_0 + 3(\eta_0^2 - M(\eta_0^2))\Big)\int_0^{x+t}\xi_0(\tau,s)ds+\\ &+(2u_0\xi_0 + 3(\xi_0^2 - M(\xi_0^2)))\int_0^{x-t}\eta_0(\tau,s)ds+\\ &+\eta_0\int_0^{x+t}(\xi_0^2(\tau,s) - M(\xi_0^2))ds + \xi_0\int_0^{x-t}(\eta_0^2(\tau,s) - M(\eta_0^2))ds\Big].\end{aligned} \quad (25)$$

**Theorem 1.** *Let the condition $a_0 = a_1 = 0$ be fulfilled. Then the boundary problem (13), (14) has an asymptotic residual with a precision up to $o(\varepsilon^3)$ solution $u_0(t,x,\varepsilon)$, for which*

$$u_0(t,x,\varepsilon) = \varepsilon(v_0(\xi_0+\eta_0) + \xi_0(\tau,x+t) + \eta_0(\tau,x-t)) + \varepsilon^3 u_3(\tau,t,x).$$

Thus, the boundary problem (21)–(23) plays the role of a normal form for the boundary problem (13), (14). It is much simpler than the initial boundary value problem, since it is not singularly perturbed. Note that in this case, the normal form of the problem is determined by the dislocation equation, while the FPU nonlinearity is not used in its definition.

**1.2. Normal form for $a_0 = 0$ and small $b$.** In the case when the parameter $b$ has the order of 1, the square and cubic nonlinearities contained in the function $F_{n+1,n}$ do not fall into the normal form that determines the dynamics of the boundary value problem (3), (4). In order to the contribution of nonlinearity to be comparable, the parameter $b$ must be small. It turns out that for this it is necessary to put $b = \varepsilon^4 b_0$, where $b_0$ is fixed. In addition, we assume that, as in the previous section, $a_0 = a_1 = 0$. The boundary problem (8), (9) after renormalizing the “time” $t \to \varepsilon t$ takes the form

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + \frac{\varepsilon^2}{12}\frac{\partial^4 y}{\partial x^4} + \frac{2\varepsilon^4}{6!}\frac{\partial^6 y}{\partial x^6} + \alpha\varepsilon\Big[2\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x^2} + \frac{\partial}{\partial x}\Big(\frac{1}{6}\varepsilon^2\Big(\frac{\partial y}{\partial x}\frac{\partial^3 y}{\partial x^3} + \frac{1}{2}\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2\Big)\Big)\Big]+\\ +\beta\varepsilon^2\frac{\partial}{\partial x}\Big[\Big(\frac{\partial y}{\partial x}\Big)^3 + \frac{1}{4}\varepsilon^2\Big(\Big(\frac{\partial y}{\partial x}\Big)^2\frac{\partial^3 y}{\partial x^3} + \frac{\partial y}{\partial x}\Big(\frac{\partial^2 y}{\partial x^2}\Big)^2\Big)\Big] + \varepsilon^2 b_0 y^3,\end{aligned} \quad (26)$$

$$y(t, x+2\pi) \equiv y(t,x). \quad (27)$$

The solution of the boundary value problem linearized at zero (26), (27) can be decomposed into a formal Fourier series in elementary solutions $\xi_k\exp(i\pi k(x+t))$ and $\eta_k\exp(i\pi k(x-t))$ $(k = \pm1, \pm2, \ldots)$. Therefore, we can say that in studying the local dynamics of the problem (26), (27), the critical (in the problem of the stability of the zero equilibrium state) case of infinite dimension is realized. Normalization of the boundary value problem (26), (27) differs significantly in cases $b_0 = 0$ and $b_0 \neq 0$. We first state the situation for $b_0 = 0$. Moreover, the problem (26), (27) is actually a transformation of the original problem for the Fermi–Pasta–Ulam equation. The research methodology is based on the assumption that the solutions $y(t,x,\varepsilon)$ in (26), (27) can be represented as a formal expression (see [10, 14–16])

$$y(t,x,\tau,\varepsilon)=\varepsilon\xi(\tau,x+t,\varepsilon)+\varepsilon\eta(\tau,x-t,\varepsilon)+\varepsilon^2 y_2(t,x,\tau,\varepsilon)+\varepsilon^3 y_3(t,x,\tau,\varepsilon)+\ldots, \qquad (28)$$

where

$$\xi_{-k}(\tau,\varepsilon)=\overline{\xi}_k(\tau,\varepsilon), \qquad \eta_{-k}(\tau,\varepsilon)=\overline{\eta}_k(\tau,\varepsilon),$$

$$\xi(\tau,z,\varepsilon)=\textstyle\sum_{k=-\infty}^{+\infty}\xi_k\exp(ikz), \quad \eta(\tau,z,\varepsilon)=\sum_{k=-\infty}^{+\infty}\eta_k\exp(ikz).$$

Here $\tau=\varepsilon^2 t$ is the "slow" time, dependence of the second and third arguments of the functions $y_j(\tau,t,x)$ is periodic. Substitute (28) into (26). To determine the function

$$U=\varepsilon y_2(t,x,\tau,\varepsilon)+\varepsilon^2 y_3(t,x,\tau,\varepsilon)$$

we get the equation

$$\frac{\partial^2 U}{\partial t^2}=\frac{\partial^2 U}{\partial x^2}+R_1(t,x,\tau,\varepsilon)+R_2(t,x,\tau,\varepsilon), \qquad (29)$$

where the function $R_1(t,x,\tau,\varepsilon)$ contains all the terms which decomposition into a Fourier series is performed only according to the system of functions $\exp(ik(x+t))$ or $\exp(ik(x-t))$ $(k=0,\pm1,\pm2,\ldots)$, and $R_2(t,x,\tau,\varepsilon)$ contains all the other terms. Let's note that the equation (29) is solvable in the specified class of functions, which are $2\pi$-periodical in $t$ and $x$ within the condition $R_1(t,x,\tau,\varepsilon)\equiv 0$. The function $R_2(t,x,\tau,\varepsilon)$ has the form

$$R_2(t,x,\tau,\varepsilon)=2\alpha\frac{\partial}{\partial x}\Big(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\Big)+O(\varepsilon),$$

thus for $y_2(t,x,\tau,\varepsilon)$ we come to the equation

$$\frac{\partial^2 y_2}{\partial t^2}=\frac{\partial^2 y_2}{\partial x^2}+2\alpha\frac{\partial}{\partial x}\Big(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\Big).$$

From here we get that

$$y_2=-\frac{\alpha}{2}\frac{\partial}{\partial x}\Big(\xi\eta\Big).$$

Taking account of this equality in (29), we conclude that the condition for the solvability of the equation (29) for $y_2(t,x,\tau,\varepsilon)$ in the specified class of functions consists in the fulfilment of the relations

$$\varepsilon^2\frac{\partial^2\xi}{\partial\tau^2}+2\frac{\partial^2\xi}{\partial\tau\partial x}=\frac{1}{12}\frac{\partial^4\xi}{\partial x^4}+\frac{2}{6!}\varepsilon^2\frac{\partial^6\xi}{\partial x^6}+2\alpha\frac{\partial\xi}{\partial x}\frac{\partial^2\xi}{\partial x^2}+\frac{\varepsilon^2\alpha}{6}\frac{\partial}{\partial x}\left[\frac{\partial\xi}{\partial x}\frac{\partial^3\xi}{\partial x^3}+\frac{1}{2}\left(\frac{\partial^2\xi}{\partial x^2}\right)^2\right]+$$
$$+\varepsilon^2\frac{\partial}{\partial x}\left[\beta\left(\frac{\partial\xi}{\partial x}\right)^3+(3\beta-2\alpha^2)\,M\left(\left(\frac{\partial\eta}{\partial x}\right)^2\right)\frac{\partial\xi}{\partial x}\right]+\varepsilon^2 b_0\Big(\xi^3+3\xi M(\eta^2)\Big), \qquad (30)$$

$$\varepsilon^2\frac{\partial^2\eta}{\partial\tau^2}-2\frac{\partial^2\eta}{\partial\tau\partial x}=\frac{1}{12}\frac{\partial^4\eta}{\partial x^4}+\frac{2}{6!}\varepsilon^2\frac{\partial^6\eta}{\partial x^6}+2\alpha\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}+\frac{\varepsilon^2\alpha}{6}\frac{\partial}{\partial x}\left[\frac{\partial\eta}{\partial x}\frac{\partial^3\eta}{\partial x^3}+\frac{1}{2}\left(\frac{\partial^2\eta}{\partial x^2}\right)^2\right]+$$
$$+\varepsilon^2\frac{\partial}{\partial x}\left[\beta\left(\frac{\partial\eta}{\partial x}\right)^3+(3\beta-2\alpha^2)\,M\left(\left(\frac{\partial\xi}{\partial x}\right)^2\right)\frac{\partial\eta}{\partial x}\right]+\varepsilon^2 b_0\Big(\eta^3+3\eta M(\xi^2)\Big), \qquad (31)$$

$$\xi(\tau,x+2\pi,\varepsilon)\equiv\xi(\tau,x,\varepsilon), \quad \eta(\tau,x+2\pi,\varepsilon)\equiv\eta(\tau,x,\varepsilon). \qquad (32)$$

As before, we believe that $M(\varphi(x))=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\varphi(x)dx$.

Let's formulate several conclusions about the interaction of waves $\xi(\tau, x+t, \varepsilon)$ and $\eta(\tau, x-t, \varepsilon)$, moving in different directions. Firstly, this interaction is carried out through the components

$$\begin{gathered}\varepsilon^2\Big[(3\beta-2\alpha^2)M\left(\left(\frac{\partial\eta}{\partial x}\right)^2\right)\frac{\partial^2\xi}{\partial x^2}+3b_0\xi M(\eta^2)\Big] \text{ and}\\ \varepsilon^2\Big[(3\beta-2\alpha^2)M\left(\left(\frac{\partial\xi}{\partial x}\right)^2\right)\frac{\partial^2\eta}{\partial x^2}+3b_0\eta M(\xi^2)\Big],\end{gathered} \tag{33}$$

respectively. Secondly, it is relatively weak, since it has the order of $\varepsilon^2$.

Note that the larger is the "average" of one wave, the greater is the change in speed of the other wave. Note that the phenomenon when the waves pass through each other unchanged, but only with a small time shift, is well known in the theory of solitons [9, 11, 17, 18].

It is interesting to note that in case $b_0 = 0$, the Fermi–Pasta–Ulam equation is analyzed. In this case the boundary problem (30)–(32) allows to lower the order. Let's make several transformations in the equations (30) and (31). We take into account that $\partial^2\xi/\partial t^2$ and $\partial^2\eta/\partial t^2$ in (30) and (31) are expressed through the derivative of the spatial variable from some expression. For regular solutions, we have the relations:

$$\frac{\partial\xi}{\partial\tau}=\frac{1}{24}\frac{\partial^3\xi}{\partial x^3}+\frac{\alpha}{2}\left(\frac{\partial\xi}{\partial x}\right)^2+O(\varepsilon),\quad \frac{\partial\eta}{\partial\tau}=-\frac{1}{24}\frac{\partial^3\eta}{\partial x^3}-\frac{\alpha}{2}\left(\frac{\partial\eta}{\partial x}\right)^2+O(\varepsilon).$$

From here we get that

$$2\frac{\partial^2\xi}{\partial\tau^2}=\frac{1}{288}\frac{\partial^6\xi}{\partial x^6}+\frac{\alpha}{24}\frac{\partial^3}{\partial x^3}\left(\left(\frac{\partial\xi}{\partial x}\right)^2\right)+\alpha\frac{\partial\xi}{\partial x}\left(\frac{1}{12}\frac{\partial^4\xi}{\partial x^4}+2\alpha\frac{\partial\xi}{\partial x}\frac{\partial^2\xi}{\partial x^2}\right)+O(\varepsilon)$$

and

$$2\frac{\partial^2\eta}{\partial\tau^2}=\frac{1}{288}\frac{\partial^6\eta}{\partial x^6}+\frac{\alpha}{24}\frac{\partial^3}{\partial x^3}\left(\left(\frac{\partial\eta}{\partial x}\right)^2\right)+\alpha\frac{\partial\eta}{\partial x}\left(\frac{1}{12}\frac{\partial^4\eta}{\partial x^4}+2\alpha\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}\right)+O(\varepsilon).$$

These equations allow to write the boundary problems (30), (32) and (31), (32) for the functions

$$u=\frac{\partial\xi}{\partial x},\qquad v=\frac{\partial\eta}{\partial x} \tag{34}$$

in the following form:

$$2\frac{\partial u}{\partial\tau}=\frac{1}{12}\frac{\partial^3 u}{\partial x^3}+\frac{\varepsilon^2}{960}\frac{\partial^5 u}{\partial x^5}+2\alpha\frac{\partial u}{\partial x}u+\frac{\varepsilon^2\alpha}{6}\frac{\partial}{\partial x}\left[u\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right]+\varepsilon^2\left[-\frac{\alpha}{48}\frac{\partial^3(u^2)}{\partial x^3}-\right.$$
$$\left.-\frac{\alpha}{24}u\frac{\partial^3 u}{\partial x^3}-\alpha^2u^2\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(\beta u^3+(3\beta-2\alpha^2)M(v^2)u\right)\right], \tag{35}$$

$$2\frac{\partial v}{\partial\tau}=\frac{1}{12}\frac{\partial^3 v}{\partial x^3}+\frac{\varepsilon^2}{960}\frac{\partial^5 v}{\partial x^5}+2\alpha\frac{\partial v}{\partial x}v+\frac{\varepsilon^2\alpha}{6}\frac{\partial}{\partial x}\left[v\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right]+\varepsilon^2\left[-\frac{\alpha}{48}\frac{\partial^3(v^2)}{\partial x^3}-\right.$$
$$\left.-\frac{\alpha}{24}v\frac{\partial^3 v}{\partial x^3}-\alpha^2v^2\frac{\partial v}{\partial x}+\frac{\partial}{\partial x}\left(\beta v^3+(3\beta-2\alpha^2)M(u^2)v\right)\right], \tag{36}$$

$$u(\tau, x+2\pi, \varepsilon)\equiv u(\tau, x, \varepsilon),\quad v(\tau, x+2\pi, \varepsilon)\equiv v(\tau, x, \varepsilon). \tag{37}$$

It is important to emphasize that it is possible to calculate the explicit values of the expressions $M(u^2)$ and $M(v^2)$ up to $O(\varepsilon)$ through the initial conditions for solutions of the original boundary value problem (26) and (27). Let $y(0,x)=a(x)$, $\left.\dfrac{\partial y}{\partial t}\right|_{t=0}=b(x)$, where $a(x)$ и $b(x)$ are some continuously differentiable $2\pi$–periodic functions.

Then

$$M(u^2)=\frac{1}{4}M\Big(\Big(b(x)+\frac{da}{dx}\Big)^2\Big),\quad M(v^2)=\frac{1}{4}M\Big(\Big(b(x)-\frac{da}{dx}\Big)^2\Big).$$

It follows from (34) that for the functions $u$ and $v$ the following conditions are true:

$$M(u)=M(v)=0. \tag{38}$$

We also mark that the zero approximation for the boundary problems (35), (37), (38) and (36), (37), (38) would be the Korteweg–de Vries equation

$$\frac{\partial w}{\partial \tau}=\frac{1}{6}\frac{\partial^3 w}{\partial x^3}+\alpha w\frac{\partial w}{\partial x},\quad w(\tau,x+2\pi)\equiv w(\tau,x). \tag{39}$$

Let's formulate the main result.

**Theorem 2.** *Let $u(\tau,x)$ and $v(\tau,x)$ be the bounded as $\tau\to\infty$ (together with derivatives in $x$ up to the 5th order inclusive) solutions of the boundary problem (35)–(38). Then the boundary problem (26), (27) has an asymptotic residual with a precision up to $o(\varepsilon^3)$ solution $y(t,x,\varepsilon)$, for which*

$$y(t,x,\varepsilon)=\varepsilon(\xi(\tau,x+t)+\eta(\tau,x-t))+\varepsilon^2y_2(t,x,\tau,\varepsilon)+\varepsilon^3y_3(t,x,\tau,\varepsilon),$$

*where $\tau=\varepsilon^2t$ and the ratios (34) are satisfied.*

The modified Korteweg–de Vries equation and Korteweg–de Vries–Burgers equation have been studied by many researches [7, 19–22]. The problems of integrability and of building (at certain values of the coefficients) exact solutions have been examined [22–25]. In this paper we use the approach of the works [12, 13, 26–30], in which a method for studying local dynamics for infinite-dimensional critical cases has been developed. Thus, special partial differential equations have been investigated, describing the asymptotic behaviour of the so-called regular solutions in the continuous Fermi–Pasta–Ulam model. At the same time, methods of the local (in the neighbourhood of the equilibrium state) analysis of the solutions dynamics have been used and developed. These methods are based on the well-known formalism of normalization method. The question of the interaction of waves moving in different directions has been studied. It is shown that, firstly, this interaction is relatively weak, since it is described by terms of the order of $\varepsilon^2$. Secondly, the interaction only leads to a phase velocity shift. The magnitude of the corresponding shift is determined explicitly through some integral characteristics of the initial conditions.

In conclusion of this section, we note that in the case of smallness of the parameter $b$ ($b=O(\varepsilon^4)$) it's interesting to compare the properties of normal forms (30)–(32) and (35)–(37). The main conclusion concerns the resulting waves moving in different directions, while their interaction turns out to be weak.

### 1.3. Normal Form for $a_0>0$.

In this section we consider the problem for

$$a_0>0. \tag{40}$$

In this case when $\varepsilon=0$ the characteristic equation (6) has infinitely many pairs of pure imaginary roots, asymptotically close to $\pm ia_0$, which means that (5) has periodical solutions $\xi\exp(\pm ia_0t)$ for $\varepsilon=0$. To construct asymptotic expansions of solutions (9), (10) we introduce (following the method from [10, 12–14]), a formal series

$$y=\varepsilon[\xi(\tau,x)\exp(iat)+\bar{\xi}(\tau,x)\exp(-iat)]+\varepsilon^3y_3(\tau,t,x)+\ldots, \tag{41}$$

where $\tau = \varepsilon^2 t$, and the functions $y_j(\tau, t, x)$ are periodic on the second and third arguments. Substituting (41) into (5), we come to the equation for determining $\xi(\tau, x)$

$$2ia_0 \frac{\partial \xi}{\partial \tau} = \frac{\partial^2 \xi}{\partial x^2} - (a_1 + 3b|\xi|^2)\xi \tag{42}$$

with periodical boundary conditions

$$\xi(\tau, x + 2\pi) \equiv \xi(\tau, x). \tag{43}$$

Let's mark that the equation (42) is classic Schroedinger equation. Its solutions give an opportunity to obtain asymptotic residual solutions of the equation (10) according to the formula (41). In particular, the periodic solution in (42), (43) corresponds to a torus in (10), (9). Of great interest there are the data about the behaviour of equations like (42) for large time. The results concerning integrability and the existence of an exact solution of the equation (42) are given in [7, 31, 32]. For values of $b$ of order 1, the replacement of (41) in our boundary value problem leads to a normal form, independent of the FPU nonlinearity ($F_{n+1,n}$).

Due to the fact that in the previous case the nonlinearity $F_{n+1,n}$ was not taken into account, let's take $a$ and $b$ so, that in the resultant normal form the terms responsible for the dislocation and the terms of FPU equation would have the same orders of smallness. For this we suppose $a = \varepsilon^4 a_1$ and $b = \varepsilon^4 b_0$, then for the boundary problem (26)–(27) the solvability conditions for (29) are the following

$$a_1\xi + \varepsilon^2 \frac{\partial^2 \xi}{\partial \tau^2} + 2\frac{\partial^2 \xi}{\partial \tau \partial x} = \frac{1}{12}\frac{\partial^4 \xi}{\partial x^4} + \frac{2}{6!}\varepsilon^2 \frac{\partial^6 \xi}{\partial x^6} + 2\alpha \frac{\partial \xi}{\partial x}\frac{\partial^2 \xi}{\partial x^2} + \frac{\varepsilon^2 \alpha}{6}\frac{\partial}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial^3 \xi}{\partial x^3} + \frac{1}{2}\left(\frac{\partial^2 \xi}{\partial x^2}\right)^2\right] +$$
$$+\varepsilon^2 \frac{\partial}{\partial x}\left[\beta\left(\frac{\partial \xi}{\partial x}\right)^3 + (3\beta - 2\alpha^2)\, M\left(\left(\frac{\partial \eta}{\partial x}\right)^2\right)\frac{\partial \xi}{\partial x}\right] + \varepsilon^2 b_0\left(\xi^3 + 3\xi M(\eta^2)\right), \tag{44}$$

$$a_1\eta + \varepsilon^2 \frac{\partial^2 \eta}{\partial \tau^2} - 2\frac{\partial^2 \eta}{\partial \tau \partial x} = \frac{1}{12}\frac{\partial^4 \eta}{\partial x^4} + \frac{2}{6!}\varepsilon^2 \frac{\partial^6 \eta}{\partial x^6} + 2\alpha \frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} + \frac{\varepsilon^2 \alpha}{6}\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial x}\frac{\partial^3 \eta}{\partial x^3} + \frac{1}{2}\left(\frac{\partial^2 \eta}{\partial x^2}\right)^2\right] +$$
$$+\varepsilon^2 \frac{\partial}{\partial x}\left[\beta\left(\frac{\partial \eta}{\partial x}\right)^3 + (3\beta - 2\alpha^2)\, M\left(\left(\frac{\partial \xi}{\partial x}\right)^2\right)\frac{\partial \eta}{\partial x}\right] + \varepsilon^2 b_0\left(\eta^3 + 3\eta M(\xi^2)\right), \tag{45}$$

$$\xi(\tau, x + 2\pi, \varepsilon) \equiv \xi(\tau, x, \varepsilon), \qquad \eta(\tau, x + 2\pi, \varepsilon) \equiv \eta(\tau, x, \varepsilon). \tag{46}$$

The terms on right part of (44), (45)

$$\varepsilon^2\left[(3\beta - 2\alpha^2) M\left(\left(\frac{\partial \eta}{\partial x}\right)^2\right)\frac{\partial^2 \xi}{\partial x^2} + 3b_0 \xi M(\eta^2)\right], \quad \varepsilon^2\left[(3\beta - 2\alpha^2) M\left(\left(\frac{\partial \xi}{\partial x}\right)^2\right)\frac{\partial^2 \eta}{\partial x^2} + 3b_0 \eta M(\xi^2)\right]$$

are similar to the terms (33), as in section 1.2, they define the interaction of waves moving in different directions, given by the values $\xi$ and $\eta$.

For $\varepsilon = 0$ in (44) we get the equation

$$a_1\xi + 2\frac{\partial^2 \xi}{\partial \tau \partial x} = \frac{1}{12}\frac{\partial^4 \xi}{\partial x^4} + 2\alpha \frac{\partial \xi}{\partial x}\frac{\partial^2 \xi}{\partial x^2}. \tag{47}$$

Note that for $a_1 = 0$ this equation reduces to the classical Korteweg–de Vries equation. As above, a statement of conformity can be formulated here, which is the main result of this section.

**Theorem 3.** *Let $\xi(\tau, x)$ and $\eta(\tau, x)$ be the solutions of the boundary problem (44)–(46), bounded as $\tau \to +\infty$ together with the derivatives of $x$ up to the 5th order inclusive. Then the boundary problem (26), (27) has an asymptotic residual with a precision up to $o(\varepsilon^3)$ solution $y(t, x, \varepsilon)$, for which*

$$y(t, x, \varepsilon) = \varepsilon(\xi(\tau, x+t) + \eta(\tau, x-t)) + \varepsilon^2 y_2(t, x, \tau, \varepsilon) + \varepsilon^3 y_3(t, x, \tau, \varepsilon),$$

*where $\tau = \varepsilon^2 t$.*

The proved statement allows us to study the local dynamics of the boundary value problem (26), (27) to proceed to the normalized boundary problem with respect to generalized complex amplitudes $\xi(\tau, x), \eta(\tau, x)$ for studying the local dynamics of the boundary value problem (26), (27). We note that both the nonlinearity of dislocations and the Fermi–Pasta–Ulam nonlinearity significantly effect on the properties of the constructed normal form.

## 2. Fast Oscillating Solutions

In this section we consider the question concerning irregular solutions (3), (4), forming on asymptotically high (as $\varepsilon \to 0$) modes. The orders of smallness of the quantities $a, b, \alpha, \beta$ will be considered identical. It's interesting to mark that just in this case the nonlinearities $f(y)$ and $F_{n+1,n}$ make a comparable contribution to the normal form of the problem.

We fix arbitrarily the parameter $\delta \neq 0$ and investigate the solutions (3), (4), which are formed on the modes with numbers

$$k = \pm(2\delta\varepsilon^{-1} + \theta + m), \quad m = 0, \pm 1, \pm 2, \ldots, \tag{48}$$

where $\theta = \theta(\varepsilon) \in [0, 1]$ complements the component $2\delta\varepsilon^{-1}$ to the integer.

Let's denote $\gamma^2(\delta, a_0) = 4\sin^2(\delta) + a_0$. Then

$$\lambda_m = \pm i\Big[\gamma(\delta)\Big(1 + \frac{\varepsilon(\theta + m)\sin(2\delta)}{\gamma^2(\delta)}\Big) + o(\varepsilon^2)\Big]. \tag{49}$$

We introduce the formal series

$$y = \varepsilon y_1 + \varepsilon^2 y_2(\tau, t, x) + \varepsilon^3 y_3(\tau, x, t) + \ldots, \tag{50}$$

$$y_1 = \sum_{m=-\infty}^{+\infty} \xi_m(\tau) \exp\Big(i\Big(\frac{2\delta}{\varepsilon} + \theta + m\Big)x + i\gamma(\delta)\Big(1 + \frac{\varepsilon(\theta+m)\sin 2\delta}{2\gamma^2(\delta)} + o(\varepsilon^2)\Big)t\Big) +$$
$$+ \sum_{m=-\infty}^{+\infty} \eta_m(\tau) \exp\Big(i\Big(\frac{2\delta}{\varepsilon} + \theta + m\Big)x - i\gamma(\delta)\Big(1 + \frac{\varepsilon(\theta+m)\sin 2\delta}{2\gamma^2(\delta)} + o(\varepsilon^2)\Big)t\Big) + \overline{cc}.$$

Here $\tau = \varepsilon^2 t$, $\overline{cc}$ denotes terms that are conjugate to those contained in the same bracket, and $y_i$ depend periodically on $x$ and on $t$. Let's denote

$$\xi(\tau, x) = \sum_{m=-\infty}^{+\infty} \xi_m(\tau)\exp(imx), \quad \eta(\tau, x) = \sum_{m=-\infty}^{+\infty} \eta_m(\tau)\exp(imx).$$

Then the formula (50) takes the form

$$y(\tau, t, x, \varepsilon) = \varepsilon y_1(\tau, t, x) + \varepsilon^2 y_2(\tau, t, x) + \varepsilon^3 y_3(\tau, t, x) + \ldots \tag{51}$$

Substituting (51) into (3) at each step of the algorithm gives the corresponding boundary value problems for determining $y_i(\tau, t, x)$. For $\varepsilon^2$, a boundary value problem for $y_2(\tau, t, x)$ arises

$$\ddot{y}_2 = \triangle y_2 + 2i\alpha(\sin(4\delta) - 2\sin(2\delta)) \cdot (\xi^2 \exp(2i\varphi) + \eta^2(2i\psi) + 2\xi\eta \exp(i(\varphi + \psi)) + \overline{cc}, \quad (52)$$

$$y(t, x + 2\pi) \equiv y(t, x), \quad (53)$$

where

$$\triangle y_2 = y_2(t, x + \varepsilon) - (2 + a_0)y_2 + y_2(t, x - \varepsilon),$$
$$\varphi = \Big(\frac{2\delta}{\varepsilon} + \theta\Big)x + \gamma(\delta)\Big(1 + \varepsilon\theta\gamma^{-2}(\delta)\frac{1}{2}\sin(2\delta)\Big)t,$$
$$\psi = \Big(\frac{2\delta}{\varepsilon} + \theta\Big)x - \gamma(\delta)\Big(1 + \varepsilon\theta\gamma^{-2}(\delta)\frac{1}{2}\sin(2\delta)\Big)t,$$
$$z_\pm = x \pm \gamma^{-1}(\delta)\frac{\sin(2\delta)}{2}t.$$

The solution of the problem (52), (53) has the form

$$y_2(\tau, t, x, \varepsilon) = A_{11}\xi^2 \exp(2i\varphi) + A_{22}\eta^2 \exp(2i\psi) + A_{12}\xi\eta \exp(i(\varphi + \psi)) + \overline{cc},$$

where $A_{11} = A_{22} = \dfrac{8i\alpha \sin(2\delta)\sin^2(\delta)}{3(a_0 + 4\sin^2(\delta))}$, $A_{12} = -\dfrac{16i\alpha \sin(2\delta)\sin^2(\delta)}{a_0 + 4\sin^2(\delta)}$.

Let's introduce the notation for operators $L^+(\delta)\xi$ and $L^-(\delta)\eta$

$$L^+(\delta)\xi \overset{\text{def}}{=} 2i\gamma(\delta)\frac{\partial\xi}{\partial\tau} - a_1\xi - R(\delta)\cdot\Big[\frac{\partial^2\xi}{\partial z_+^2} + 2i\theta\frac{\partial\xi}{\partial z_+} - \theta^2\xi\Big],$$
$$L^-(\delta)\eta \overset{\text{def}}{=} -2i\gamma(\delta)\frac{\partial\eta}{\partial\tau} - a_1\eta - R(\delta)\cdot\Big[\frac{\partial^2\eta}{\partial z_-^2} + 2i\theta\frac{\partial\eta}{\partial z_-} - \theta^2\eta\Big],$$

where $R(\delta) = \cos(2\delta) - \dfrac{1}{4}\gamma^{-2}(\delta)\sin^2(2\delta)$ or $R(\delta) = \dfrac{7}{4}\cos^2(\delta) - 1$.

For $\varepsilon^3$ we have the problem for $y_3(\tau, t, x)$. The right part of the corresponding equation contains the third and first harmonics for $\varphi$ and $\psi$. Taking into account that the first harmonics are resonant, for this problem the conditions of existence of bounded solutions are expressed by the following system of equations for $\xi$ and $\eta$

$$\begin{aligned} L^+(\delta)\xi &= 3\xi(|\xi|^2 + 2|\eta|^2)\cdot\Big(\Big[-6 + 2\cos(2\delta) - \frac{\cos(4\delta)}{2}\Big]\cdot\beta + b\Big),\\ L^-(\delta)\eta &= 3\eta(2|\xi|^2 + |\eta|^2)\cdot\Big(\Big[-6 + 2\cos(2\delta) - \frac{\cos(4\delta)}{2}\Big]\cdot\beta + b\Big) \end{aligned} \quad (54)$$

with periodic boundary conditions

$$\xi(\tau, z_+ + 2\pi) \equiv \xi(\tau, z),\ \ \eta(\tau, z_- + 2\pi) \equiv \eta(\tau, z). \quad (55)$$

The connection between the solutions of the boundary value problem (54), (55) and the initial boundary value problem (3), (4) is established by the following statement. It will contain the sequence $\varepsilon_n \to 0$, which is determined by the condition $\theta(\varepsilon) = \theta_0$.

**Theorem 4.** *Let's fix arbitrarily the parameters $\delta$ and $\theta_0 \in [0, 1)$. Let $\xi(\tau, z_+)$ and $\eta(\tau, z_-)$ be the solutions of the problem (54), (55) for $\theta = \theta_0$, bounded as $\tau \to \infty$, $x \in [0, 2\pi]$. Then there exists a sequence $\varepsilon_n \to 0$, determined by the condition $\theta(\varepsilon) = \theta_0$, so that with $\varepsilon = \varepsilon_n$ the boundary problem (3), (4) has an asymptotic residual with a precision up to $o(\varepsilon^3)$ solution $\widetilde{y}(t, x, \varepsilon_n)$, for which the presentation (51) takes place*

$$\widetilde{y}(t, x, \varepsilon) = \varepsilon y_1(\tau, t, x) + \varepsilon^2 y_2(\tau, t, x) + \varepsilon^3 y_3(\tau, t, x).$$

Thus, the boundary problem (54), (55) plays the role of the normal form for the initial boundary problem (3), (4) and determines its solutions that rapidly oscillate in space.

## Conclusion

The work considers systems with nonlinearities of dislocations and Fermi–Pasta–Ulam and the corresponding boundary value problems. For these problems we have built the special systems of nonlinear boundary problems, which play the roles of normal forms; their solutions are asymptotic in the discrepancy close to solutions of the original boundary value problems. Special attention is paid to the situation when the nonlinearities of dislocations and Fermi–Pasta–Ulam make a comparable contribution to the resulting normalized equations.

Especially we have considered the behaviour of so-called regular and irregular solutions (quickly oscillating by spatial variable). When describing such classes of solutions, various normalized boundary value problems arise. Besides, there is the noticeable difference between the values of coefficients, for which the influences of nonlinearity of dislocations and the FPU nonlinearity are comparable. At first sight, it may seem that the resulting normal forms are not simpler and even more complicated than the original equations. But it is not so. The essence is in the theorems 1–4. They say that the main part of the solutions of the equation (3) is the solution just of the normalized equations. Thus, solving the original equations, we have not only to find the main parts of the solutions, but also to calculate several complex functions. Besides, it's important to take into account that normal forms contain significantly less Fourier harmonics than the original problem. All this allow to suggest that normal forms can significantly help in studying the original problem.

Let's mark that only the normal forms from section 1 are these one in the traditional sense of the term: system of equations for determining slowly varying amplitudes. The normal forms from section 2 are not normal forms in the exact sense, but their solutions can effectively help to construct the slowly varying amplitudes and the solutions of the initial problem.

From the constructed normal forms there follow the important conclusions about the interaction of waves moving in opposite directions, i.e. about the influence of the variables $\xi$ and $\eta$ on each other. The most important is the integral influence: the effect of one variable upon other is determined by the average on spatial variable of the square of amplitudes $\xi$ or $\eta$. It's interesting to note that in formally close Fermi—Pasta—Ulam problem the situation is different. There, the interaction only leads to a shift of phase velocities [10, 15].

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