Review of actual problems of nonlinear dynamics

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Fractional models in hydromechanics

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Topic and purpose. The last two decades are marked by wide spreading fractional calculus in theoretical description of the natural processes. Replacement of the integer-order operators by their fractional (and even of complex order) counterparts opens up a continuous field of new differential equations in which the standard equations of theoretical physics (wave-, diffusion-, etc.) are is represented by separate spikelets at points with integer coordinates. But what do the fractional-order derivatives mean physically? What are the common reasons for the appearance of fractional derivatives in the equations? Is it possible to predict in advance the appearance of fractional operators in a particular problem? These questions are not yet removed from the agenda and remain to be at the focus of attention of numerous conferences devoted to the theory and application of fractional calculus. This topic is developing in this review as well. **Models investigated.** There are considered the hydromechanical problems, where fractional operators emerge in natural ways: motion of a body in the viscous flud, hydromechanics of turbulence, turbulent diffusion. No exotic structures and processes, fractals and quantum-mechanical paradoxes. **Results.** The review shows how naturally fractional operators appear on the classical field of hydrodynamic problems under the pen of Heisenberg, Weizsacker, Kolmogorov, Obukhov, Monin – theoreticians who can not be suspected of being unscrupulous in choice of mathematical tools. **Discussion.** Actually speaking, the whole review is a continuous discussion of the «inevitability of the strange world» of fractional calculus (Uchaikin V.V. The method of fractional derivatives. Ulyanovsk: «Artishok», 2008), and the fact that this is done within the framework of classical hydromechanics only strengthens the convincing conclusions.

Key words: fractional Laplacian, nonlocality, turbulent diffusion, projection operators, open systems.

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Introduction

Fractional equations of mathematical physics can be divided into three classes: equations with fractional space-derivatives, with fractional time-derivatives, and mixed equations containing both kinds of operators. Many "users" (pardon for slang) link the fractional order of spatial derivatives with fractality (self-similar heterogeneity) of the medium. A possible reason for this point of view may be the linguistic factor, because the English words "fractal" and "fractional" are consonant, but the first word refers to highly inomogeneous geometric structures, with numerous breaks and tears whereas while the second one refers to differential equations solutions of which are not just continuous, but even differentiable. A number of works by various authors (see my reviews [2–4], books [5, 6]) are devoted to discussions on the legitimacy of identifying these properties, opened in 1995 by Rutman [1]. The problem is that all fractals are heterogeneous structures, while the kernel included in fractional differential operators is translationally invariant, which implies the homogeneity of medium, at least at small scales. However, media that are homogeneous on a small scale and processes in them are well described by integer-order equations integer orders with variable coefficients. In other words, dividing a non-homogeneous into elementary cells involves small-sized elements which can be treated as homogeneous a regular nonhomogeneous continuum. In the fractal case such partition produces the same thing on a new scale. Nothing becomes simpler is; a part looks like a whole. But fractional operators are based on the concept of continuity, while fractal structures are discontinuous on all scales (one cannot reach the asymptotic behavior in which the fractal would be seen as continuous object). Continuity can be returned only by averaging over the ensemble of fractals, therefore, the systems and processes under consideration should be random (stochastic). Fractality of random realizations is reflected in "long" power-type correlations, which in turn generate fractional integral operators acting on spatial variables. Due to the finite propagation velocity of disturbances, long space correlations are accompanied by long-time correlations, related to fractional time-derivatives. In a certain sense, such ensemble averaging can be considered as validation of known attempts to represent fractional integrals as fractal integrals, but what an ensemble is, and how far the average characteristics from the individual are, still remains unclear. The similar situation is with the meaning of the fractional time derivative. Some authors even talk on fractal properties of time itself, although there are no physical reasons for this.

Alternative interpretation of fractional integro-differentiation can be made on the basis of the concept of an open system (OS) as a part of a closed system. Having written the Liouville equation for the motion of a closed system and using the technique of projection operators to extract the part describing the motion of the OS from it, we get the Lindblad equation containing integral term with delay. In statistical mechanics, the concept of delay was introduced by L. Boltzmann and then had the following adjectives - integral (Boltzmann, [7]), hereditary (Volterra, [8, 9]), hereditary (Rabotnov, [10]), endochronic (Valanis and Lee , [11]). In the latter version, developed in the theory of plasticity, the delay effect is associated with the interpretation of the time itself. The concept of "time" is used in continuum mechanics as "ordinary" physical (laboratory) time and as some functional, characterizing the intensity of the process, endochronous (intrinsic, internal "time", convenient for describing non-trivial nonlinear processes of deformation (creeping and relaxation) and limiting., The concept of "structural" ("incubation") time is used in the mechanics of dynamic fracture. That implies the existence of a structural (scale) level of fracture, depending on the type of medium and loading, and, consequently, a certain characteristic fracture time at a given stage. The additional requirement of self-similarity, considered at the level of the general principle of statistical thermodynamics, turns heredity into a power law, and the Volterra type heredity integrals into fractional integrals, the inversion of which leads to differential operators of fractional order. The most popular derivatives in applications are the

Riemann – Liouville ones:

$$ {}_a\mathsf{D}_t^{\nu} f(t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int\limits_a^t (t-\tau)^{-\nu} f(\tau) d\tau $$

and Gerasimov-Caputo

$$ {}_a^{\nu}\mathsf{D}_t f(t) = \frac{1}{\Gamma(1-\nu)} \int\limits_a^t (t-\tau)^{-\nu} f'(\tau) d\tau. $$

In these formulas, $t > a$, $\nu \in (0,1)$, and the transition to higher fractional orders is carried out by additional differentiation of an integer order.

1. Hidden variables

Describing dynamics of a mechanical system as continuous medium, in the first approximation we ignore presence of an internal structure and its participation in the process under consideration. In this case, initial dynamical equations are written for macroscopic characteristics of the system that do not contain information about its structure. However, in some cases (for example, in problems of thermomechanics or polymer dynamics), the internal structure can play a significant role and it is taken into account by including (or rather, keeping in the process of transition from micro- to macro-variables) an appropriate set of variables called *internal variables*, which can also be interpreted (with reference to a common set of external variables) as *hidden variables*. Modern ideas about the evolution of hidden variables in dynamic problems can be found in review papers [13–15].

There exist two approaches to implication of the internal variables: thermodynamical approach, based on traditional concepts of thermodynamics (thermodynamical forces, entropy, etc.), and mechanical approach, which is characterized by an additional set of internal degrees of freedom and takes into account mechanical inertia in contrast to thermodynamic approach .

The fractional time derivative in mechanical problems appears for the first time while approximating experimental data on the behavior of viscoelastic bodies under load [16, 17]. R. Nigmatullin associated the fractional nature of the time-derivative with diffusion in media with fractal geometry [18].

The Liouville equation with a fractional time derivative was first written by Hilfer [19] while analyzing the dynamics of phase transitions using integrated “coarse-grained” (block) variables

$$ X_{iN} = \sum_{j=1}^{M} X_{iN}(j), $$

where $X_{iN}(j)$ denotes the scalar observable X at the node j ($j \in \{1, ..., M\}$) belonging to the block i ($i \in \{1, ..., N\}$). By introducing a centered and normalized random variable

$$ X_N = \left(\sum_{i=1}^{N} X_{iN} - C_N \right) \Big/ D_N \tag{1} $$

and assuming translational invariance, independence, and identity of distributions of these variables, Hilfer uses the generalized Levy limit theorem, which states that, with a suitable choice of the C_N and D_N sequences, the random variable (1) has a limit distribution belonging to the class of stable Levy distributions. These distributions, in turn, are closely related to fractional order derivatives (see [2]). The divergence of the mathematical expectation of the enlarged energies during the phase transition

leads to a fractional-differential modification of the Liouville equation for the macroscopic variable $X(t)$.

Using this analysis, Hilfer came to conclusion about existence and some specific properties of phase transitions of order less than 1, which he called nonequilibrium transitions. The most important feature of such systems is the nonexponential decay of stationary states, which occurs according to the power law $X(t) \propto t^{\nu-1}$.

Later, Hilfer's studies in this field led him to the postulate that *the temporal evolution of all physical systems is noninvertible* [20]. Supposing that this law of noninvertibility is no more and no less than empirical law of nature at the same level as the law of conservation of energy, Hilfer writes that the invertible behavior of the system is idealization, that the invertibility property is apparent when the system under consideration is separated (isolated) from its past and from its surrounding.

Basing on numerous facts of non-Debye charge relaxation in insulators, some authors write the equations of diffusion

$$ {}_0\mathsf{D}_t^{\nu}\rho = \Theta_{\nu}\triangle\rho, $$

continuity

$$ {}_0\mathsf{D}_t^{\nu}\rho + \mathrm{div}(\rho\mathbf{v}_{\nu}) = 0, $$

and finally the Liouville

$$ {}_0\mathsf{D}_t^{\nu}\rho = \mathcal{L}\rho, $$

noting that its particular case is the equation of "fractional equilibrium state"

$$ {}_0\mathsf{D}_t^{\nu}\rho = 0. $$

However, the reader should be warned about the incomplete adequacy of this term, the gap is obvious: the solution ρ in the above-written equations does not obey the normalizing condition because its integral over the phase space is not constant, but decreases with time.

In [21], the classical stationary (with time-independent Liouvillian) Liouville equation

$$ \frac{\partial\rho}{\partial t} = -i\mathcal{L}\rho \tag{2} $$

into the equation with a fractional degree of the Liouville operator was formally transformed

$$ {}_0\mathsf{D}_t^{\nu}\rho = (-i\mathcal{L})^{\nu}\,\rho + O(t^{-1-\nu}). \tag{3} $$

Equation (3) constructed in above-cited work (without any special, however, physical reasons) is not equivalent to equation (2) (due to the omitted term) its solution depends on ν. For $\nu = 1$, it coincides with the solution of the classical equation, for $\nu = 0$ – with the equilibrium solution of the stationary equation:

$$ \mathcal{L}\rho = 0. $$

Later we will return to the discussion of the fractional differential version of the Liouville equation, but in a slightly different key.

2. Generalization of Newton's equations

In [22], the problem of a body fall in the atmosphere is considered with the use of a fractional differential analogue of the Newton equation. Not hindering himself with the physical justification

for using fractional calculus to solve this problem, the author referred to the dissipation process, and ultimately compared the obtained solutions with classical results. Starting with the simplest case –

$$m_\nu \, {}_0^\nu\mathsf{D}_t V = mg, \tag{4}$$

he finds the velocity

$$V(t) = V_0 + \frac{mgt^\nu}{m_\nu \Gamma(1+\nu)},$$

and then the coordinate

$$x(t) = \int\limits_0^t V(\tau)d\tau = x_0 + V_0 t + \frac{mgt^{\nu+1}}{m_\nu \Gamma(2+\nu)}.$$

Here, V_0 is the initial velocity, x_0 is the initial coordinate along the axis going downward. Note, that at short times ($t^\nu > t$ for $\nu < 1$), fractional model of fall shows a larger velocity than the ordinary (Newton's) model while at large times the situation is inverted, and this fact is to agree with the influence of dissipation. One more problem arises: the dimension ality of the coefficient m_ν does not coincide with the dimension of the mass, which makes us look for new formulas for momentum, kinetic energy, and related dynamic variables. To ensure agreement with the dimension of energy, it was proposed in [23, 24] to express the momentum in the form

$$p_\nu = m_\nu \, {}_0^\alpha\mathsf{D}_t x(t), \quad \alpha = (1+\nu)/2.$$

As a result, the expression for the total energy takes the form

$$E = \frac{p_\nu^2}{2m_\nu} + U(x) = \frac{m_\nu}{2}\left[{}_0^\alpha\mathsf{D}_t x(t)\right]^2 - mgx.$$

No additional arguments to justify these constructions are given in the paper; there is only the fact that when $\nu \to 1$ all these formulas take the form which is conventional for classical mechanics. In addition, the solution to this problem is given taking into account the resistance of the medium,

$$m_\nu \, {}_0^\nu\mathsf{D}_t V + bV = mg, \tag{5}$$

and there is even a reference to the experiment (although not very distinct): the free fall of six bodies (in the original "men") in an atmosphere with the average weight of one body 261.2 pounds from the height of 31400 to 2100 feet is satisfactorily described by a fractional differential solution with $\nu = 0.998$ and $m/m_\nu = 1.457$. The only trouble is that the classical solution describes this process very well.

The above given formulation of fractional dynamics is not unique. For example, Baleanu et al. [25] constructed a different version of such generalization by introducing fractional velocity and fractional momentum on the interval $[a, b]$ with the help of relations

$$V(t) = (1/2)(A \, {}_a^\alpha\mathsf{D}_t + B \, {}_t^\beta\mathsf{D}_b)x(t)$$

and

$$p(t) = (m/2)(A \, {}_a^\alpha\mathsf{D}_t + B \, {}_t^\beta\mathsf{D}_b)x(t) = p_\alpha + p_\beta,$$

respectively, where $0 < \alpha, \beta \leq 1$, and A and B are constants with dimensions $T^{\alpha-1}$ and $T^{\beta-1}$. As a result, a fractional analogue of Newton's second law was obtained as an equation

$$(1/2)(\kappa_\alpha \, {}_t\mathsf{D}_b^\alpha p_\alpha + \kappa_\beta \, {}_a\mathsf{D}_b^\beta p_\beta) = F, \tag{6}$$

supplemented by a transverse condition

$$\left[{}_t\mathsf{D}_b^{\alpha-1} p_\alpha - {}_a\mathsf{D}_t^{\beta-1} p_\beta \right]_a^b = 0. \tag{7}$$

The authors note that when $\alpha = 1,\ \beta = 1$

$${}_a\mathsf{D}_t^\alpha = {}_a^\alpha\mathsf{D}_t = {}_t\mathsf{D}_b^\alpha = {}_t^\alpha\mathsf{D}_b = \frac{d}{dt}$$

and equations (6)–(7) are reduced to the standard Newtonian equation. It is really so. But the following remark: "If the generalized force in equation (6) is equal to zero, then the generalized Newton's law can be written as: $\kappa_\alpha\ {}_t\mathsf{D}_b^\alpha p_\alpha + \kappa_\beta\ {}_a\mathsf{D}_t^\beta p_\beta = 0$" causes some bewilderment. Newton's first law is not simply a consequence of the second (otherwise it would not be part of the Newtonian axiom system). From all possible frames of reference the first law distinguishes a family of inertial systems in which the second law "does its work". The nonlocal in time definitions of momenta used in the previous formulations cannot satisfy the Galilean transformations and could make sense for particles in a certain medium that provides disturbance of the translational properties of the Lagrangian of the system in time and space.

3. Body in a viscous fluid

Let us recall the problem of the motion of a body on the surface of an incompressible viscous fluid. On a horizontal surface $z = 0$ (the z axis is directed upward) of an infinitely deep layer $(-\infty < z < 0)$ of such a liquid, there is a large thin plate, to which a horizontal force $F(t)$ is applied, which carries it along with the adjacent layers of liquid into movement along the x axis. The motion of the plate is described by Newton's equation

$$m\frac{dV}{dt} = F(t) + Q(t), \tag{8}$$

where $Q(t) = -S\eta\partial v(z,t)/\partial z|_{z=0}$ is the resistance force acting on the plate on the part of the liquid, $v(z,t)$ is the x-component of the liquid velocity at depth z (the rest of its components are equal to zero). According to adhesion condition, $V(t) = v(0,t)$, and the field of velocities $v(z,t)$ satisfies the Navier – Stokes equation

$$\rho\frac{\partial v}{\partial t} = \eta\frac{\partial^2 v}{\partial z^2}. \tag{9}$$

In the system of equations (8)–(9), which describes a mechanical system as a body plus liquid, all derivatives of integer order and all the operators are differential. Infinitesimal evolution at any time t depends only on the state $(V(t), v(z,t))$ at that very moment, and for this reason the further evolution of the system at a given state does not depend on the prehistory (according to probability terminology, the process is the Markov's one).

But, following the well-known procedures [26], we exclude the variable $v(z,t)$ from this system and obtain the equation for the remaining variable – the plate velocity. If at $t < 0$ the liquid together with the plate were at the state of the rest, and then the applied force is limited in magnitude, the remaining equation looks as

$$m\frac{dV}{dt} = F - S\sqrt{\eta\rho}\left[\frac{1}{\sqrt{\pi}}\int\limits_0^t \frac{1}{\sqrt{t-\tau}}\frac{dV(\tau)}{d\tau}d\tau\right].$$

Now it is an integro-differential Volterra-type equation with delayed argument of an unknown function under the integral. Moreover, the term enclosed in square brackets is a fractional derivative of the order $\nu = 1/2$ (when the conditions are indicated, the difference between the two types of derivatives disappears).

Let us rewrite this equation as following

$$m\frac{dV}{dt} + S\sqrt{\eta\rho}\ {}_0^{\nu}\mathsf{D}_t V = F(t). \tag{10}$$

Physical interpretation of this result is that the voltage observed at the point (x, z) at time t is determined by the distribution of velocities of liquid particles coming from the neighborhood of another point of this layer (x', z), where they were, let us say, at the moment $t' < t$. Due to translational invariance of the solution with respect to x, the same velocity distribution at this moment (t') also took place at the observation point (x, z). This is the simplest mechanism of heredity – "mechanical" memory.

A similar equation for the motion of a ball with mass m and radius a in a viscous medium looks as

$$m\frac{dV}{dt} = F(t) + Q(t),$$

где сила сопротивления $Q(t)$ дается формулой

$$Q(t) = -6\pi\eta a V(t) - \frac{2}{3}\pi\rho a^3\frac{dV(t)}{dt} - 6\pi\eta a^2\sqrt{\frac{\rho}{\pi\eta}}\int\limits_{t_0}^{t}\frac{1}{\sqrt{t-\tau}}\frac{dV(\tau)}{d\tau}d\tau,$$

where the resistance force $Q(t)$ is given by the formula

$$Q(t) = -6\pi\eta a V(t) - \frac{2}{3}\pi\rho a^3\frac{dV(t)}{dt} - 6\pi\eta a^2\sqrt{\frac{\rho}{\pi\eta}}\int\limits_{t_0}^{t}\frac{1}{\sqrt{t-\tau}}\frac{dV(\tau)}{d\tau}d\tau,$$

derived in the works of Boussinesq [27] and Basset [28]. The first term here is the Stokes force, the second one is the inertial component of the resistance, corresponding to the presence of the attached ball mass; the third one is proportional to the fractional derivative of the order $\nu = 1/2$ [29]. If the body was at rest until the initial moment $t = 0$, the lower limit in the last integral is replaced by zero:

$$\left(m + \frac{2}{3}\pi\rho a^3\right)\frac{dV(t)}{dt} + 6\pi a^2\sqrt{\rho\eta}\ {}_0^{\nu}\mathsf{D}_t V + 6\pi\eta a V(t) = F(t). \tag{11}$$

As it follows from the Tauberian theorem, the main asymptotic (as $t \to \infty$) part of $V^{\mathrm{as}}(t)$ of the solutions of equations (10) and (11) satisfies the shortened fractional order equations $\nu = 1/2$

$$S\sqrt{\eta\rho}\ \ {}_0^{\nu}\mathsf{D}_t V^{\mathrm{as}}(t) = F(t) \tag{12}$$

and

$$6\pi a^2\sqrt{\rho\eta}\ {}_0^{\nu}\mathsf{D}_t V^{\mathrm{as}}(t) + 6\pi\eta a V^{\mathrm{as}}(t) = F(t), \tag{13}$$

respectively. It is noticeable that the authors of the papers cited above are mistaken regarding the interpretation of fractional derivatives in equations (4)–(5): these are not analogues of the accelerating terms of the Newton equation, but the influence of the external surrounding, which determines the body behavior as asymptotic at $t \to \infty$ when the inertial force is exhausted and the initial conditions are no matter any longer (the body "forgot" about them).

4. Damper – spring system

Let us consider another example. A piston with the mass m_1 with a coefficient of friction against its walls η connected by a spring with the length l and stiffness $k > (\eta/2)^2$ with a ball of the mass m_2 moving in the tube without friction is located in a horizontal tube open on both sides. The force $F(t)$ bounded in absolute value is applied to the ball from the moment $t = 0$. We deal with a dynamical system with two degrees of freedom described by differential equations

$$m_1\ddot{x}_1 = -\eta\dot{x}_1 + k(x_2 - x_1 - l),$$

$$m_2\ddot{x}_2 = F(t) - k(x_2 - x_1 - l).$$

Let us supplement this system of equations with the conditions

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0, \quad x_2(0) = l, \quad \dot{x}_2(0) = 0,$$

assuming that at the initial moment of time the system is stationary and the piston is at the origin of coordinates.

The solution of the first equation with respect to x_1 (assuming that $x_2(t)$ is known) under given conditions is expressed through its Green's function

$$G(t) = \frac{1}{m_1\omega_1}\exp\left(-\frac{\eta t}{2m_1}\right)\sin(\omega_1 t), \quad \omega_1 = \sqrt{\frac{k}{m_1} - \left(\frac{\eta}{2m_1}\right)^2}$$

by the relation

$$x_1(t) = \int_0^t G(t-\tau)k[x_2(\tau) - l]d\tau.$$

Substituting this solution into the second equation of the system, we obtain a closed equation for part 1 of the system under consideration, which acquires the integro-differential form

$$m_2\ddot{x}_2 + kx_2 = k^2\int_0^t G(t-\tau)x_2(\tau)d\tau + F_2(t), \tag{14}$$

with a free term

$$F_2(t) = F(t) + kl\left[1 - k\int_0^t G(\tau)d\tau\right].$$

Note that we got the same result again: excluding from consideration one of the interacting parts of the system described by the Newton equations (in other words, the Markov system), we find that the rest part is controlled by the integro-differential equation. Now it is the non-Markov process, or the process with memory.

Let us imagine for clearness that we covered the part of the tube, containing a piston with a spring, with screen and we only see a ball and its motion obeys the equation (14).The influence of pre-history $x_2(t-\tau)$ of the ball's motion on its behavior at time t is realized through the invisible (hidden) variable $x_1(t)$. Does not this suggest that the presence of such integrals with delay may indicate the presence of hidden variables? Naturally, it does. Even Zener, while commenting on the integral (hereditary, according to Volterra terminology) term in the constitutive equation of viscoelasticity, suggested that this heredity may serve as a sign of the existence of hidden parameters, to which, as an

example, he attributed the temperature [29]. Meanwhile, heredity is a clear step towards the fractional-integral operator. It is enough to find an argument in favour of the specific form of the kernel of the integral operator $K(t,t')$ and suppose, let us say, its invariance with respect to time shift

$$K(t,t') = K_0(t-t')$$

(which in the two above-given examples turned out to be automatic), and then require uniformity in the Euler sense:

$$K_0(a\tau) = a^{\alpha} K_0(\tau).$$

The last requirement, of course, does not arise from any of the "first principles" and can be introduced only under the pressure of some obvious, for example, experimental facts. We used this approach to substantiate the fractional differential kinetics of dispersion transport [30] (see also the detailed information about this approach in the book [31]). Principally, Euler homogeneity is associated with self-similarity of the system [32]. Self-similar systems present a special class of open systems that are not subsystems of closed systems (we call it class A). The above described examples belonged to another class of open systems, which are subsystems of closed systems (let us denote this class by the symbol B). We will postpone the discussion of systems of this class until the end of the review.

5. Generalized gas dynamics

The traditions and principles of theoretical physics triggered the search of the origins of hereditarity in the molecular-kinetic nature of matter. The most suitable object for this purpose turned out to be rarefied gases. The basis of gas dynamics is the Boltzmann equation; which in the spatially homogeneous case has the form

$$\frac{\partial \varphi(\mathbf{v},t)}{\partial t} = \mathsf{B}_0[\mathbf{v}, \varphi(\,\cdot\,,t)]. \tag{15}$$

where B_0 is the Boltzmann bilinear *collision operator* acting on $\varphi(\mathbf{v},t)$ which is the one-particle distribution density of molecules in the velocity space:

$$\mathsf{B}_0[\mathbf{v}, \varphi(\,\cdot\,,t)] \equiv \int d\mathbf{u} \int \sigma(\Theta; g)[\varphi(\mathbf{v}',t)\varphi(\mathbf{u}',t) - \varphi(\mathbf{v},t)\varphi(\mathbf{u},t)]d\Omega.$$

In this expression, $\mathbf{v}$ and $\mathbf{u}$ are the velocities of the colliding molecules before or after (primed) collision, $g = |\mathbf{v}-\mathbf{u}|$ are their relative velocity, Θ is the scattering angle, $\sigma(\Theta; g)d\Omega$ is the differential scattering cross section into the solid angle $d\Omega$ (the laws of conservation of energy and momentum definitely connect $\mathbf{v}'$ and $\mathbf{u}'$ with $\mathbf{v}$, $\mathbf{u}$ and Θ).

The assumptions which underlie the derivation of this equation greatly simplify the solution of gas-dynamic problems, but are still not considered sufficiently substantiated for dense gases and liquids. *Generalized kinetic equations* associated with the names of L. van Hove, I. Prigogine, R. Braut, P. Resibois, R. Zwanzig, E. Montroll (see [33–38]) are. Derived under weaker restrictions, these equations preserve the most important property of the Boltzmann equation – they describe the irreversible process from relaxation to equilibrium. The main mathematical difference of the generalized equations is the inclusion of the delay in the collision integral by additional integration (over the shifted time argument) of the one-particle distribution function in the nonlinear collision operator B_τ acting on it with respect to phase variables

$$\frac{\partial \varphi(\mathbf{v},t)}{\partial t} = \int_0^t \mathsf{B}_\tau[\mathbf{v}, \varphi(\,\cdot\,,t-\tau)]d\tau + f(\mathbf{v},t). \tag{16}$$

In the case of an equilibrium state, the solution of the Boltzmann equation does not depend on time, $\varphi(\mathbf{v}, t) = \varphi_{\mathrm{eq}}(\mathbf{v})$ and the influence of delay disappears.

The sequence of the equations obtained in the process of the derivation of these equations forms a new hierarchy. Let us recall the main stages of the derivation.

1. From the canonical Hamilton equations for a closed system N of interacting molecules, without any additional assumptions, the Liouville equation for the phase density $\rho^N \equiv \rho^N(\{\mathbf{r}, \mathbf{v}\}; t)$ is derived:
$$\frac{\partial \rho^N}{\partial t} = \mathsf{L}^N \rho^N,$$
where L^N is the *Liouville operator*. Due to "astronomically" large number of variables N, it is unsuitable for immediate description of the motion of macroscopic systems, and the assumption about closureness limits its practical application on the part of mesoscopic systems and nano-systems.
2. Introducing the projection operator P, this averages the phase density over the space of configurations,
$$\mathsf{P}\rho^N = \frac{1}{V^N} \int\limits_V \cdots \int\limits_V \rho^N(\{\mathbf{r}, \mathbf{v}\}; t) \prod_{j=1}^{N} d\mathbf{r}_j \equiv \varphi^N(\{\mathbf{v}\}; t),$$
the Liouville equation is transformed into the equivalent *formal kinetic equation*
$$\frac{\partial \varphi^N}{\partial t} = \int\limits_0^t \mathsf{C}_\tau^N[\varphi^N(\{\mathbf{v}\}|t - \tau)] d\tau + f^N(\{\mathbf{v}\}, t),$$
where C_τ^N is the operator of interaction between colliding molecules. The time-integral appears due to decomposition of the solution ρ^N into two components: *relevant* $\varphi^N = \mathsf{P}\rho^N$ and *irrelevant* $(1 - \mathsf{P})\rho^N$, and elimination of the latter from the pair of equation. Neither new processes, nor restrictions on old ones are introduced. Any Markov process has this property: averaging over a part of the variables gives rise to integration over time, indicating a non-Markov character of the *reduced* process (the opposite idea is also true: a non-Markov process can be transformed into a Markov process by suitable expansion of the phase space). The free term of the equation f^N depends on the initial value of only the irrelevant component. The number of independent variables, although decreased in a half, is still extremely large, and the equation itself has now become open.
3. In order to reduce the N-particle set of variables to a single-particle one and at the same time finish the equation, the **hypothesis of molecular chaos** is introduced: *the random velocities of the molecules entering collision are mutually independent*:
$$\varphi^N(\{\mathbf{v}\}; t - \tau)) = \prod_{j=1}^{N} \varphi(\mathbf{v}_j; t - \tau),$$
where $\varphi(\mathbf{v}_j; t - \tau)$ are single-particle densities in the velocity space. As a result, we obtain generalized kinetic equation
$$\frac{\partial \varphi(\mathbf{v}, t)}{\partial t} = \int\limits_0^t \mathsf{B}_\tau[\mathbf{v}, \varphi(\,\cdot\,, t - \tau)] d\tau + f(\mathbf{v}, t),$$

where

$$\mathsf{B}_\tau[\mathbf{v}, \varphi(\,\cdot\,|t-\tau)] \equiv \mathsf{TL} \int \cdots \int \mathsf{C}_\tau^N \left[\prod_{j=1}^{N} \varphi(\mathbf{v}_j|t-\tau) \right] \prod_{j=2}^{N} d\mathbf{v}_j,$$

integration is assumed at all numbered velocities (index 1 is omitted), and TL means the thermodynamic limit.

4. The gas is sufficiently rarefied so that the interaction of molecules can be considered in the model of binary instantaneous collisions. In this case

$$\mathsf{B}_\tau[\mathbf{v}, \varphi(\,\cdot\,, t-\tau)] = \mathsf{B}_0[\mathbf{v}, \varphi(\,\cdot\,, t-\tau)]\delta(\tau)$$

and we get the Boltzmann equation

Having rejected the last restriction, we approximate the dependence of the collision operator on the delay time by the factor $b(\tau)$ normalized to 1:

$$\mathsf{B}_\tau[\mathbf{v}, \varphi(\,\cdot\,, t-\tau)] = \mathsf{B}_0[\mathbf{v}, \varphi(\,\cdot\,, t-\tau)]b(\tau), \qquad \int_0^\infty b(\tau)d\tau = 1.$$

This delay is the memory effect, or the non-Markov effect, which reflects real processes occurring in the medium. The resulting equation

$$\frac{\partial \varphi(\mathbf{v}, t)}{\partial t} = \int_0^t \mathsf{B}_0[\mathbf{v}, \varphi(\,\cdot\,, t-\tau)]d\tau + f(\mathbf{v}, t)$$

forms the mathematical basis of hereditary, or, as it is also called, *generalized hydrodynamics* [39].

6. Generalized hydrodynamics

For a long time, it has been considered that hydrodynamics is applicable at times which are much larger than "kinetic" times: the collision time ($\sim 10^{-15}$ s) and time between successive collisions (for ordinary liquids, for example, water, $\sim 10^{-14}$ s). Numerical simulation showed, however, that in some cases conventional hydrodynamics is also applicable on much smaller time scales – approximately up to 30 collisions, that is, up to times of 10^{-13} s. Let us point out that in applications of mathematical statistics the number 30 also plays a noticeable role: it is believed that for typical samples of such a volume the central limit theorem, which is equivalent to the diffusion approximation in hydrodynamics, is already applicable.

Generalized hydrodynamics was the result of the spreading of the hydrodynamic approach from macroscopic to microscopic scales. The Navier–Stokes equations retained their form, but the transfer coefficients began to depend on the radius of action of the gradients and on the duration of their application. Thus, scale factors of length and time were included, which made it possible to go beyond the limits of the model of a continuous medium and take into account its molecular structure. The fluid characterized by the generalized viscosity $\eta(\mathbf{r}-\mathbf{r}', t-t')$ has two properties – nonlocality and memory: the signal generated at time t' at the point $\mathbf{r}'$ will arrive at the point $\mathbf{r}'$ at a much later moment t. This delay is caused not so much by the inertial properties of the liquid but by the diffusion nature of the propagation of fluctuations. In any case, it takes some time to change the arrangement of molecules. In addition, it also became clear that correlations with partners should be taken into account.

In addition, it also became clear that correlations with partners in previous collisions should be taken into account. Previously, the decay of correlations was supposed to be exponential and on hydrodynamic scales correlations seemed insignificant (the hypothesis of molecular chaos). Numerical simulation showed the presence of a long-term tail of the autocorrelation function for the fluid velocity at intermediate densities [40]: even after several hundreds of collisions, the particle velocity exhibits correlation with its initial value, which decreases according to a power law. In macroscopic hydrodynamics, this fact has been known since the time when Stokes defined it while examining the motion of a ball in a continuous medium. No one, however, thought that this conclusion would remain valid for balls of atomic size.

The fact that under certain conditions we can no longer consider collisions to be instantaneous and must take into account their duration played the important role in the interpretation of the memory effect. The characteristic collision time $\tau_{\rm col}$ becomes now comparable with the relaxation time $\tau_{\rm rel}$, which must be taken into account at high densities.

It is easy to understand from the following considerations. The characteristic time of binary collisions at low densities is estimated by the ratio of the radius of interactions r_0 to the average molecular velocity $\overline{v}$:

$$\tau_{\rm col} \approx r_0/\overline{v}.$$

The relaxation time is estimated by reducing the kinetic equation to the relaxation type

$$\frac{\partial(\varphi - \varphi_{\rm eq})}{\partial t} \approx -nr_0^2\overline{v}(\varphi - \varphi_{\rm eq})$$

and has the form

$$\tau_{\rm rel} \approx 1/(nr_0^2\overline{v}).$$

Thus, we see that

$$\tau_{\rm col}/\tau_{\rm rel} \approx nr_0^3,$$

and at low densities n, the collision time $\tau_{\rm col}$ is negligible compared to the relaxation time.

The authors of the book [36] come to a similar conclusion taking into account the well-known expansion of the inverse relaxation time in powers of the density of gas molecules (concentration) n:

$$\tau_{\rm rel}^{-1} = r_0^2 n\overline{v}[1 + r_0^3 n + (r_0^3 n)^2 + \dots].$$

The ratio of the considered times is

$$\tau_{\rm col}/\tau_{\rm rel} = r_0^3 n + (r_0^3 n)^2 + \dots,$$

so, if we want to keep the correction $(r_0^3 n)^2$, we can't consider the collisions to be instantaneous.

The second term on the right side of the generalized kinetic equation (16) describes the decay of spatial correlations that can exist at the initial time. It can be shown that if there are no spatial correlations at the initial time, then $f(\mathbf{v};t) = 0$ for all $t > 0$. Now let us suppose that at the initial time the correlations extend over distances of about the interaction radius r_0. The chaotically directed velocities of the particles in the correlation cluster lead to their mutual scattering, cluster spreading, and after a time $\tau_{\rm col} = r_0/\overline{v}$, the particles stop interacting with each other. Therefore, we should expect that

$$f(\mathbf{v};t) \approx 0, \quad t \gg \tau_{coll}.$$

Calculations confirm this in the case of a weakly interacting gas with exponential repulsion of molecules, but no strong evidence of this property has been found in the general case.

A similar situation takes place with the operator term. It is considered that

$$\mathsf{B}_\tau[\varphi] \simeq 0, \quad t \gg \tau_{\rm col}.$$

The important role of the diffusion model in hydrodynamic problems has already been emphasized above. The presence of long-term correlations modifies this model greately: included into the diffusion equation

$$\frac{\partial f}{\partial t} = K\frac{\partial^2 f}{\partial x^2}$$

diffusion coefficient K is proportional to the integral of the autocorrelation function $\int_0^\infty \rho(t)dt$ and therefore diverges if the correlations decrease according to a power law. In the modified diffusion equation, which approximately takes into account nonlocality of the process with the help of an additional term with the 4th derivative (Burnett correction),

$$\frac{\partial f}{\partial t} = K\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^4 f}{\partial x^4},$$

both coefficients diverge. "The simplest possible way leading to the redefinition of the coefficients", says the work of B. Alder and W. Alley [40], is generalization of Fick's law by introducing nonlocal (in time) "memory function":

$$\frac{\partial f}{\partial t} = K_0 \int\limits_0^t \rho(t-t')\frac{\partial^2 f}{\partial x^2}dt' + B_0 \int\limits_0^t \rho(t-t')\frac{\partial^4 f}{\partial x^4}dt'.$$

"This equation not only gives a well-defined Burnet coefficient, but also quantitatively predicts the singlet distribution function at large times."

7. Turbulent diffusion

The turbulent nature gives this non-stationary and heterogeneous environment an additional quality of randomness. Immediately, a whole "bunch" of problems arises. They are determination of the average characteristics of the flow, its fluctuations, correlations, emission probabilities (large deviations), etc. The most important task in this case is the establishment (choice) of the statistical assembly over which averaging will be performed. The next most important task is the choice of the procedure of decoupling of correlations, that is, the transformation of an equation averaged over an ensemble

$$\frac{\partial \langle G\rangle}{\partial t} = \nabla\langle [D(\mathbf{x},t)\nabla G(\mathbf{x},t;\mathbf{x}_0,t_0)]\rangle,$$

to the equation for the average tracer propagator

$$\langle G(\mathbf{x},t;\mathbf{x}_0,t_0)\rangle = g(\mathbf{x}-\mathbf{x}_0, t-t_0)$$

$$\frac{\partial g}{\partial t} = \widehat{T} g(\mathbf{x},t),$$

containing the *operator of turbulent diffusion* $\widehat{T}$ (for simplicity we assume that the turbulence is homogeneous and stationary). It turns out to be nonlocal.

The specificity of turbulent diffusion is due to the action on the particle of vortices of various sizes existing in the turbulent medium. The distance between two tracers can change significantly in a short time only under the influence of vortex, the dimensions of which are comparable with this

distance. The farther these particles are from each other, the larger are the sizes of the vortices that carry them from each other, the greater the distance l between them grows. According to the classical diffusion theory this effect can be achieved by introducing the dependence of the diffusion coefficient D on the relative coordinates, i.e., on the distance $D = D(r)$. This approach was used in the pioneering work of Richardson [41], who wrote the equation for the probability distribution density $p(r,t)$ of the random distance between the pair of impurity particles that were at one point at the time $t = 0$, in the form of

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial r}\left[D\left(r\right)\frac{\partial p}{\partial r}\right]$$

with a diffusion coefficient $D\left(r\right) \propto r^{4/3}$, corresponding to an increase in the width of the diffusion packet $\Delta(t)$ according to the law

$$\Delta(t) \propto t^{3/2},$$

which is significantly different from the normal diffusion law $\Delta(t) \propto t^{1/2}$. Theoretically, such behavior was substantiated in the famous works of A.N. Kolmogorov and A.M. Obukhov as a consequence of the hypothesis of self-similarity of locally isotropic turbulence, determined by a single dimensional parameter which is the dissipation rate of turbulent energy ε [42, 43].

The main difficulty of this approach is substantiating the spatial dependence of the diffusion coefficient. Richardson associated the growth of this coefficient with distance r with the action of vortices i.e. when r increases, vortices of larger sizes are involved in the process of relative motion, and particle displacements increase. According to Kolmogorov concept of turbulence, 4/3 is direct consequence of dimensionality: it is enough to assume that both the initial and final distances are much smaller than the typical size of the largest vortices L and much larger than the Kolmogorov length $\eta = \nu^3/\varepsilon)^{1/4}$, where ν is viscosity, and ε is the average dissipation rate of the turbulent component of kinetic energy [44].

The emergence of nonlocal nature of the equation of turbulent diffusion can be found in the work of Tchen [45], based on the results of Heisenberg and Kolmogorov. Using the well-known transformations of a hydrodynamic turbulent system in this work, he reduced the equation for the Fourier transform of the mean propagator of tracers (or labeled molecules) to the form

$$\frac{\partial^2 \tilde{g}(\mathbf{k},t)}{\partial t^2} = [k^2\nu(k)]^2\tilde{g}(\mathbf{k},t),$$

where the turbulent viscosity $\nu(k)$ is related to the spectral density of turbulent dissipation per unit time $F(k)$ by the Heisenberg formula [46]

$$\nu(k) = \kappa\int\limits_k^\infty \sqrt{F(q)/q^3}dq, \tag{17}$$

in which κ is an abstract number. The family of solutions of the equation (17) also contains solutions of the standard relaxation equation

$$\frac{\partial \tilde{g}(\mathbf{k},t)}{\partial t} = -R(k)\tilde{g}(\mathbf{k},t)$$

with inverse relaxation time

$$R(k) = k^2\nu(k).$$

The inverse relaxation time with a negative sign is the Fourier transform of the turbulent diffusion operator, $T(\mathbf{x}) = \mathcal{F}^{-1}[-k^2\nu(k)]$:

$$\frac{\partial g(\mathbf{x},t)}{\partial t} = Tg(\mathbf{x},t).$$

If we ignore the dependence of ν on k, suggesting that $\nu(k) = \nu_0$ in the entire range of wave numbers, this equation becomes the standard diffusion equation

$$\frac{\partial g(\mathbf{x},t)}{\partial t} = \nu_0 \Delta g(\mathbf{x},t).$$

In fact, ν is not constant; its dependence on k is determined by the shape of the spectral function (17). According to the 5 /3 *Kolmogorov–Obukhov law*, the latter one is presented in a power-law form

$$F(k) = k^{-5/3}\psi(\lambda k), \tag{18}$$

where $\psi(\lambda k)$ is the window-function ("filter"), which vanishes outside the inertial region. The expression resulting from this substitution

$$R(k) = (k^2)^\gamma \psi(k\lambda), \quad \gamma = 1/3,$$

is (up to a constant factor) the Fourier transform of some nonlocal operator acting with respect to the spatial variable. If we ignore the filter again, taking the function $\psi(z)$ as identically equal to one in the entire range of wave numbers, as it was done in the Monin–Yaglom book [47], then we get the equation

$$\frac{\partial \tilde{g}(\mathbf{k},t)}{\partial t} = -(k^2)^\gamma \tilde{g}(\mathbf{k},t),$$

which in natural space-time variables takes the form of an equation with a fractional power γ of the Laplace operator

$$\frac{\partial g(\mathbf{x},t)}{\partial t} = -(-\Delta)^\gamma g(\mathbf{x},t),$$

again turns into a diffusion equation for $\gamma = 1$. It should be emphasized that neglecting the boundedness of the power representation in the wave number region is somewhat discredits this equation, and we have to look for a more adequate representation of the nonlocal version.

The corresponding propagator, i.e. the solution of the equation for a point instant source

$$\frac{\partial f(\mathbf{x},t)}{\partial t} = -B(-\Delta)^{\alpha/2} f(\mathbf{x},t) + \delta(\mathbf{x})\delta(t) \tag{19}$$

is expressed in terms of the 3-dimensional density of the isotropic stable distribution (Levy–Feldheim distribution) $g_3(\mathbf{x};\alpha)$ by the relation

$$f(\mathbf{x},t) = (Bt)^{-3/\alpha} g_3((Bt)^{-1/\alpha}\mathbf{x};\alpha), \quad \alpha \in (0,2].$$

The dispersion of this distribution is infinite,

$$\langle \mathbf{X}^2 \rangle = \infty,$$

and to characterize the spreading of the diffusion packet spreading over time, we should choose another measure of its width, for example, the width at the height h Δ_h, or the radius of the sphere Rp which contains a given probability p. Being proportional to each other, these measures grow proportionally with time $t^{1/\alpha}$, which corresponds to Richardson's law for $\alpha = 2/3$.[1]

[1]Because of this classifying diffusion types according to the spreading speed of a diffusion package, we should not use dispersion, which does not always exist, but the one of the above given measures that always exist and can be chosen so that in the limit $\alpha \to 2$ coincides with dispersion. This is applied to $\langle r^2 \rangle$, $\langle (\Delta \mathbf{u})^2 \rangle$ and other characteristics.

8. From Brownian motion to Levy flights

Brownian motion is a mathematical model of molecular diffusion, which assumes the continuous motion of a diffusing particle under the action of continuously incoming uncorrelated delta pulses from the surrounding space. The result of these assumptions is the non-differentiability of the Brownian trajectory, which excludes the concept of instantaneous velocity and, for this reason, eliminates the presence of a front (a surface that separates a region of space accessible at a given moment of time from an inaccessible one in the case of a point instant source). The absence of such a surface leads to the conclusion about the instantaneous expansion of the initially compact packet of particles over the entire space, which contradicting physics. Though, if we characterize this process with one parameter, the mean square of the distribution $\langle R^2(t)\rangle$, we obtain

$$\langle R^2(t)\rangle \propto \quad t \to \infty. \tag{20}$$

This law is easier to test experimentally than the evolution of the spatial distribution in the packet of diffusing particles and can be considered the first sign of the molecular nature of the diffusion process, also called the Gaussian process due to the Gaussian distribution of particles in the packet. If we assume that the vector $\mathbf{R}(t)$ connecting the initial position of the particle with its position at the moment of observation t consists of a large number of identically distributed and statistically independent small vector terms, and using the central limit theorem, we immediately get the Gaussian probability distribution in the packet and the linear law of increasing dispersion (20). The probability density $f(\mathbf{x}, t)$ itself obeys the diffusion equation

$$\frac{\partial f}{\partial t} = D\Delta f(\mathbf{x}, t),$$

derived by Einstein under the assumption of independence of the change of the coordinates of a wandering particle in successive time intervals. Einstein got it as the limit of the Kolmogorov–Chapman integral equation

$$f(\mathbf{x}, t+\tau) = \int d\mathbf{x}' f(\mathbf{x}-\mathbf{x}', t) f(\mathbf{x}', \tau),\ t, \tau > 0, \tag{21}$$

which in physical terms is precisely the result of successive collisions of molecules, let us say, of an ideal gas with a relatively heavy Brownian particle. Both of these equations are limited by the condition of homogeneity of the medium, otherwise the functions f in equation (21) cease to be translationally invariant and the diffusion coefficient becomes dependent on the coordinates, and the diffusion equation takes the form

$$\frac{\partial f}{\partial t} = \nabla[D(\mathbf{x})\nabla f(\mathbf{x}, t)]. \tag{22}$$

In 1909, Jean Baptiste Perrin substantiated the law (20) experimentally, proving that atoms exist. The description of this experiment can be found in any textbook on molecular physics. Let us note, however, that Einstein derived his equation for particles suspended in gas (momentum transfer occurred during collisions separated by rather short time intervals), while Perrin worked with a suspension of gamboge (chewing resin) in liquid (water). The difference, by the way, is not so fundamental: the diffusion equation is derived both in gasdynamics and in hydrodynamics (even in the dynamics of neutron fluxes in nuclear facilities).

Almost two decades later (in 1926), Richardson conducted similar experiments, but on a different scale: neither in test tubes nor in Petri dishes, but in the air or, more precisely, in a turbulent atmosphere, and the objects of observation were not dust particles or particles of suspension, but puffs of smoke from the factory pipe and specially sent to the atmosphere air balloons. Having performed many

measurements, he found that the average square of the distance between two particles varies according to the law

$$\langle R^2(t)\rangle \propto t^3, \tag{23}$$

which is named after him. Of course, as soon as we come to a random process, the image of Brownian motion immediately arises in our imagination. In fact, this is a very specific case. The motes thrown into the turbulent flow behave differently than the Brownian particle. Their trajectories are smooth, and when filmed on a video camera, they allow us to determine the particle velocities at the moments of time that we want. We see that the motion is autocorrelated: the movement velocity during the next time interval differs little from the velocity in the previous one and inertia saves the memory of this event. But how can one explain physically the accelerated nature (22) of the diffusion of the diffusion packet in a turbulent medium in comparison with the Brownian law (21)? The main reason is that the process ceased to be a process with independent increments, such as the Brownian motion, thereby violating the conditions of the central limit theorem, the consequence of which is law (20), the particles are no longer under the influence of independent shocks of different gas molecules, but are transported by gas – liquid as a continuously (albeit irregularly) moving substance. The motions of different test particles ceased to be independent, that is why we consider the expansion of the cloud as a whole, and not the probability density of an individual test particle, although in principle one can pose the problem of such a distribution, for example, with respect to the center of mass of such a cloud, actually speaking the relative distribution of two particles, or, equivalently, the distribution of the distance R between two randomly selected particles of a diffusion cloud. Arguing that turbulent diffusion is carried out by the combined action of a combination of vortices of different sizes, Richardson, as we can say, artificially applied the diffusion equation to the description of its evolution

$$\frac{\partial p(R,t)}{\partial t} = \frac{\partial[D(R)\partial p(R,t)/\partial R]}{\partial R}$$

with diffusion dependent on the coefficient $D(R) \propto |R|^{4/3}$. The self-similar solution of such an equation with $D(R) = R^{4/3}$ has the form

$$p(R,t) = \frac{9}{4}\frac{1}{\sqrt{\pi}t^{3/2}}\exp\left\{-\frac{9}{4}\frac{R^{2/3}}{t}\right\}, \quad 0 < R < \infty,$$

the average square of which

$$\langle R^2\rangle = (280/243)t^3$$

corresponds with empirical law in dimensions (23). However, the diffusion nature of the evolution of this quantity, prescribed by this equation, leaves a feeling of inadequacy of the vortex picture of turbulence: a particle captured by a large vortex is quickly carried away from its neighbor, remaining in the zone of action of the small vortex, and only after some time leaving its vortex, being carried away over a long distance by a new whirlwind. If we draw a graph of such a movement, we will see long jumps interspersed with relatively short intervals of the Brownian type.

Such considerations could probably serve as a motivation for the authors of [48] to introduce Levy flights (distributed according to the power law of jumps) into the model, depending on the length of the jump, but not depending on the time speed $V(t)$. The time distribution of such a flight is given by the formula (9) of the cited work which has the form

$$\psi(t|R) = \delta(R - V(R)t),$$

where $V(R)$ is the speed at which the particle travels the distance R, so that the joint distribution of the displacement vector $\mathbf{R}$, replacing the relative position vector in their model, and the time spent on

this movement is characterized by the factum

$$\Psi(\mathbf{R}, t) = \psi(t|\mathbf{R})p(\mathbf{R}),$$

the second factor $p(\mathbf{R})$ is defined through the characteristic function $\exp\{-C|\mathbf{k}|^{\alpha}\}$, $0 < \alpha < 2$ of the stable Levi–Feldheim density. The conditional density represented by the first factor, the authors presented as

$$\psi(t|\mathbf{R}) = \delta(R - V(R)t),$$

although it would be more logical to write it in the form $\psi(t|\mathbf{R}) = \delta(t - R/V(R))$, which, however, mathematically does not change anything. The trajectories of such particles are approximated by independent broken curves composed of independent random vectors obeying the isotropic stable Levi – Feldheim distribution with exponent $\alpha \in (0, 2)$. The velocity $V(|\mathbf{R}|)$ is estimated based on the Kolmogorov scaling hypothesis for the inertial region of turbulent dissipation. The average kinetic energy of vortices is characterized by the scale R, $E_R \sim V_R^2$. If the speed of its transmission through this scale $\varepsilon_R \sim E_R/t_R$ is constant, then"$\varepsilon \sim V_R^3/R$ and $V(R) \sim R^{1/3}$. In the hierarchical turbulence model with fractal dimension d_f $E_R \sim V_R^2 q_R$, where $q_R = (R/R_0)^{\mu}$, R_0 is the external length scale, E is the Euclidean dimension and $\mu = E - d_f$.

From the following formulas asymptotic expressions for the velocity

$$V(R) \sim R^{1/3+\mu/6}$$

and mean square displacement are derived

$$\langle R^2(t)\rangle \sim \begin{cases} t^3 + 3\mu/(4-\mu), & \alpha \le (1-\mu)/3; \\ t^2 + 6(1-\alpha)/(4-\mu), & (1-\mu)/3 \le \alpha \le (10-\mu)/6; \\ t, & \alpha \ge (10-\mu)/6. \end{cases}$$

Although the authors note that for $\mu = 0$, the first of the cases cited corresponds with Richardson's law, there remains a feeling of inadequacy both in determining the speed using the distance and in understanding the nature of the trajectories themselves. In my opinion, the case examined by Schönfeld [49] is clearer in this regard.

9. Schönfeld vortex model

Arguing his point of view, Schönfeld considered flat vortex with a characteristic size ϱ centered at x, y of the coordinate plane, which generates velocity w at the origin; the velocity being directed at an angle of φ to the x axis. This vortex adds to the change in impurity concentration

$$n(x, y) \mapsto n(x - \varrho\cos\varphi, y - \varrho\sin\varphi).$$

and he estimated it through the x-component of the diffusion flux

$$\delta j_x = n(x - \varrho\cos\varphi, y - \varrho\sin\varphi)w\cos\varphi$$

and summed it over all the distances ϱ from the axes of the vortices, characterized by axisymmetric (independent on the angle φ) distribution $W d\varrho/\varrho$:

$$j_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\chi \int_0^{\infty} \frac{d\varrho}{\varrho} W(\rho) n(x - \varrho\cos\chi, y - \varrho\sin\chi)\cos\chi.$$

The Fourier transform in both spatial variables with subsequent substitution of the continuity equation into the Fourier transform

$$\frac{\partial \widetilde{n}}{\partial t} = ik_x \widetilde{j}_x + ik_y \widetilde{j}_y$$

gives us

$$\frac{\partial \widetilde{n}}{\partial t} + k^2 \widetilde{K}(k) \widetilde{n}(\mathbf{k}, t) = 0,$$

where

$$\widetilde{K}(k) = \frac{1}{2\pi k} \int\limits_0^\infty \frac{d\varrho}{\varrho} W(\varrho) J_1(2\pi k \varrho)$$

In the case of intermolecular scales, as the author notes, it is reasonable to assume that the density $W(\varrho)$ is substantially different from zero only for small values of the argument. On the other hand for the values which are several times higher than the average molecular path, it can be considered equal to zero (with this we turn it into the delta function). Assuming that $J_1(2\pi k\varrho) \approx \pi k \varrho$ in the region of small values of the argument, we get the Fourier transform of the molecular diffusion equation:

$$\frac{\partial \widetilde{n}}{\partial t} + k^2 \widetilde{n}(\mathbf{k}, t) = 0,$$

At the end of the article, Schönfeld tried to return to the equation of turbulent diffusion, trying different model images for $\widetilde{K}(k)$, but did not find anything suitable.

10. Chen's statistical theory

Continuous transfer of kinetic energy on a various scale is the most characteristic feature of the phenomenon of turbulence is. Originally turbulence appears in the form of large-scale formations such as vortices, it is distributed between an increasing number of successively decaying derived vortices. That increases the local gradient of the turbulent field and the rate of energy dissipation. In general, this process is characterized by the spectral function $F(k)$, where $F(k)dk$ is the kinetic energy transmitted in the range of wave numbers $(k, k{+}dk)$. Due to the difficulties in the analytical description of the interactions of vortices with each other, the spectral function is estimated taking dimensions into consideration. Using these notions, Heisenberg derived the transfer function W_k (equal to the energy transmitted by harmonics with wave numbers fewer than k, harmonics with wave numbers greater than k) in the form [46]

$$W_k = 2\nu_k \int\limits_0^k F(q) q^2 dq, \tag{24}$$

where $\nu_k = \kappa \int_k^\infty \sqrt{F(q) q^{-3}} dq$ is the turbulent viscosity, and κ is the numerical constant. In this form, energy transfer is represented similar to viscous dissipation by the product of turbulent viscosity per vorticity square. Without changing the principle of dimensionality, Obukhov presented the transfer function in the form [43].

$$W_k = \kappa \int\limits_k^\infty \left[2 \int\limits_0^k F(q) q^2 dq \right]^{1/2} F(q') dq'. \tag{25}$$

This form represents energy transfer as the product of Reynolds shear stresses (first integral) and vorticity (second integral). Forms (24) and (25) correspond to two different approaches of the description

of turbulence, based on the consideration of turbulent dissipation and shear stresses in a turbulent fluid, respectively. In the inertial region, the representations of Heisenberg and Obukhov are consistent with the Kolmogorov theory (with the law $F(k) \propto k^{-5/3}$ in particular), but outside it give different results.

A consistent theory of turbulent diffusion, based on the ideas of Heisenberg, Weizsacker, Kolmogorov and Obukhov, implemented within the mathematical apparatus of statistical mechanics in combination with the Reynolds separation of statistical fluctuations and turbulent pulsations, was constructed in a series of papers by Chen [45, 50]. Recalling that Richardson had considered a pair of particles, Chen using the Liouville equation for a system M of particles (molecules) in volume V equations for the one-particle$F_a(t, \mathbf{x}_a, \mathbf{p}_a)$ and two-particle $F_{ab}(t, \mathbf{x}_a - \mathbf{x}_b, \mathbf{p}_a, \mathbf{p}_b)$ distribution densities in the phase space of a dynamical system with a given interaction potential $\phi_{ab} = \phi(\mathbf{x}_a - \mathbf{x}_b, \mathbf{p}_a, \mathbf{p}_b)$:

$$\frac{\partial F_a}{\partial t} + \frac{\mathbf{p}_a}{m_a}\frac{\partial F_a}{\partial \mathbf{x}_a} = \sum_b \frac{M_b}{V} \iint \left(\frac{\partial \phi_{ab}}{\partial \mathbf{x}_a}\frac{\partial F_{ab}}{\partial \mathbf{p}_a} - \frac{\partial \phi_{ab}}{\partial \mathbf{p}_a}\frac{\partial F_{ab}}{\partial \mathbf{x}_a} \right) d\mathbf{x}_b d\mathbf{p}_b$$

and

$$\frac{\partial F_{ab}}{\partial t} + \frac{\mathbf{p}_a}{m_a}\frac{\partial F_{ab}}{\partial \mathbf{x}_a} + \frac{\mathbf{p}_b}{m_b}\frac{\partial F_{ab}}{\partial \mathbf{x}_b} = \frac{\partial \phi_{ab}}{\partial \mathbf{x}_a}\frac{\partial F_{ab}}{\partial \mathbf{p}_a} + \frac{\partial \phi_{ab}}{\partial \mathbf{x}_b}\frac{\partial F_{ab}}{\partial \mathbf{p}_b} +$$
$$+ \sum_b \frac{M_c}{V} \iint \left(\frac{\partial \phi_{ac}}{\partial \mathbf{x}_a}\frac{\partial F_{abc}}{\partial \mathbf{p}_a} + \frac{\partial \phi_{bc}}{\partial \mathbf{x}_b}\frac{\partial F_{ab}}{\partial \mathbf{p}_b} - \frac{\partial \phi_{ac}}{\partial \mathbf{p}_a}\frac{\partial F_{abc}}{\partial \mathbf{x}_a} - \frac{\partial \phi_{bc}}{\partial \mathbf{p}_b}\frac{\partial F_{abc}}{\partial \mathbf{x}_b} \right) d\mathbf{x}_b d\mathbf{p}_b.$$

Noting that for systems of weakly interacting particles one can prove the equation

$$F_{abc} = F_a F_b F_c + F_a F'_{bc} + F_b F'_{ac} + F_c F'_{ab},$$

where

$$F'_{ab} = F_{ab} - F_a F_b \tag{26}$$

etc. Chen used it as a rough tool to close the system, or hypothesis. Going on from the probability densities to the densities of the first and second factorial moments (corresponding to the average number of particles and the average number of pairs of particles),

$$N_a(t, x_a) = \int F_a(t, \mathbf{x}_a, \mathbf{p}_a) d\mathbf{p}_a$$

and

$$G_{ab}(t, \mathbf{x}_a - \mathbf{x}_b) = \iint F_{ab}(t, \mathbf{x}_a - \mathbf{x}_b, \mathbf{p}_a, \mathbf{p}_b) d\mathbf{p}_a d\mathbf{p}_b,$$

he worked out the equation

$$\frac{\partial N_a}{\partial t} = \frac{\partial}{\partial x_{ai}} \left(K_{ij} \frac{\partial N_a}{\partial x_{aj}} \right),$$

where

$$K_{ij} = \int_0^t d\tau \sum_{bc} \iint \left(\frac{M_b}{V}\frac{\partial \phi_{ab}}{\partial p_{ai}} \right) \left(\frac{M_c}{V}\frac{\partial \phi_{ab}}{\partial p_{aj}} \right) G'_{bc}(t - \tau, \mathbf{x}_b - \mathbf{x}_c) d\mathbf{x}_b d\mathbf{x}_c,$$

and the prime mark plays the same role as in formula (26). The pair function G'_{bc} reflects the space-time correlations and indirectly indicates the sizes of the turbulent vortices and the duration of their period.

Following Reynolds, Chen divided the observed concentration of diffusing particles c into a statistical mean $N = \langle c \rangle$ and fluctuations n: n: $c = N + n$. Then he presented the velocity $\mathbf{v} = \mathbf{V} + \mathbf{u}$ in a similar way, taking $\langle \mathbf{u} \rangle = 0$, and substituting these partitions into the continuity equations and

Navier–Stokes equations, and averaged them over the statistical assembly. In economical tensorial notations, the equations obtained by him take the form:

$$N_{,t} + \langle u_k n_{,k} \rangle = 0,$$

$$n_{,t} + u_k n_{,k} - \langle u_k n_{,k} \rangle + u_k N_{,k} = 0,$$

$$u_{i,t} + u_j u_{i,j} - \langle u_j u_{i,j} \rangle = -p_{,i}/\rho + \nu u_{i,jj}.$$

The next stage is the Fourier transform (transformants are marked with a tilde over the symbols). Taking into account irretractability conditions according to which which Chen worked,

$$ik_j \tilde{u}_j(\mathbf{k}) = 0,$$

$$\tilde{N}_{,t} = -i \int q_j \langle \tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{n}(\mathbf{q}) \rangle d\mathbf{q},$$

$$\tilde{N}_{,tt} = i \int k_j \langle \tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{n}(\mathbf{q}) \rangle_{,t} d\mathbf{q}, \tag{27}$$

$$\tilde{n}_{,t} = -iq_j \int \left[\tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{n}(\mathbf{q}) - \langle \tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{n}(\mathbf{q}) \rangle \right] d\mathbf{q} - iq_j \int \tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{N}(\mathbf{q}) d\mathbf{q}, \tag{28}$$

$$\tilde{u}_{i,t} = -iq_j \int \left[\tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{u}_s(\mathbf{q}) - \langle \tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{u}_s(\mathbf{q}) \rangle \right] (\delta_{is} - k_i k_s / k^2) d\mathbf{q}. \tag{29}$$

It follows from (28)-(29)

$$[\tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{n}(\mathbf{q})]_{,t} = iq_l \tilde{u}_j(\mathbf{k}-\mathbf{q}) \int \tilde{u}_l(\mathbf{q}-\mathbf{q}') \tilde{N}(\mathbf{q}') d\mathbf{q}'. \tag{30}$$

Using the phase exchange formula [50]

$$\int \tilde{u}_l(\mathbf{k}-\mathbf{q}) \tilde{N}(\mathbf{q}) d\mathbf{q} = -ik_l N(\mathbf{k}) \nu_k$$

and spectral representation of the diffusion coefficient [46]

$$\nu_q = \chi \int_q^\infty \sqrt{F(k)/k^3} dk,$$

formula (30) is represented as

$$[\tilde{u}_j(\mathbf{k}-\mathbf{q}) \tilde{n}(\mathbf{q})]_{,t} = q^2 \tilde{N} \nu_q \tilde{u}_j(\mathbf{k}-\mathbf{q}). \tag{31}$$

Substituting (31) into (27), we obtain

$$\frac{\partial^2 \tilde{N}}{\partial t^2} = (\nu_k k^2)^2 \tilde{N},$$

$$\frac{\partial \tilde{N}}{\partial t} = -R(k) \tilde{N}(\mathbf{k}) \tag{32}$$

with the inverse relaxation time

$$R(k) = \nu_k k^2 = \kappa k^2 \int_k^\infty \sqrt{F(q)/q^3} dq.$$

Finally, using the spectral function F, we obtain the expression for $R(k)$

$$R(k) = (3\kappa/4)\sqrt{K_0}(\varepsilon k^2)^{1/3}, \qquad \Lambda^{-1} \ll k \ll \lambda^{-1}.$$

But for the latter restriction, then the coefficient $(k^2)^{1/3}$ would correspond to a third power of the Laplace operator: $\Delta^{1/3}$.

The solution of equation (32) has the form

$$\tilde{N}(\mathbf{k}, t) = e^{-t/R(k)}\tilde{N}(\mathbf{k}, 0),$$

where

$$R(k) = R(1) \cdot k^{\alpha},$$

if we take ψ as 1. Chen [45] turned the Fourier transform at the initial condition $\tilde{N}(\mathbf{k}, 0) = 1$, corresponding to a point instantaneous source for one-, two- and three-dimensional diffusion processes. In our notation, it looks as:

$$\frac{\partial \bar{G}(\mathbf{x}, t)}{\partial t} = -R(1)(-\Delta)^{\alpha/2}\bar{G}(\mathbf{x}, t).$$

Let us remind that his solution takes a self-similar form with a scaling variable $\xi = rt^{-1/\alpha}$ α ($\alpha \in (0, 2]$).

That is, given synoptically, "arithmetic" of Chen's theory.

11. Alternative way: open system model

While thinking about the reasons for the nonlocality of the "diffusion" operator in the theory of turbulent diffusion, we again turn to the fact that in the case of molecular diffusion in a rarefied gas, a tracer (for example, a molecule of another gas) collides with individual molecules that are not connected in any way with the others. For this reason, the process is described by a local equation. However, in the case of turbulent diffusion, the molecules which surround the tracer are connected into whole entity (though not rigid, but, in the probability language, correlated), and the momentum exchange upon collision with such a formation (for example, a vortex) is already a nonlocal process itself, the equation of which can be written, in particular, in the form

$$\frac{\partial n(\mathbf{x}, t)}{\partial t} = \triangle \int d\mathbf{x}' K(\mathbf{x} - \mathbf{x}')n(\mathbf{x}', t).$$

The very separation (although not completely definite) of the local characteristics of the medium (density, velocity) in a turbulent state into two components, called statistical fluctuations and turbulent pulsations, opens up the possibility of representing such a medium as the coexistence of two media exchanging energy and momentum between each other... Let us look at this situation taking into account modern theory of open systems [51, 52].

11.1. Open systems. Perhaps the simplest and most general interpretation of the hereditary phenomenon is in the concept of open systems, which can be regarded as some kind of validation of hidden variables. Speaking of a closed (conservative) system, we usually mean a mechanical system that is completely isolated from the outside world and does not exchange with it either mass, energy, momentum, or other additive characteristics, parts of which interact with each other through non-dissipative forces, and therefore, the indicated values ??are preserved during the motion of the system. The potential nature of the forces makes it possible to put into operation the standard apparatus of

analytical dynamics, in particular, the Liouville equation. So it is natural to call the systems open ones when they do not satisfy at least one of the indicated conditions.[2]

These concepts, which are convenient in a purely mechanical sense, are not sufficient in ther modynamically. Thus, the exchange of small uncorrelated portions of energy, forming a canonical ensemble, practically does not affect the thermodynamic properties of the system, without distinguishing it from a closed one in a narrow sense, which is characterized by a microcanonical ensemble. For this reason, along with this classification (we will call such systems closed or open systems in a narrow sense), their analogues are used in a broad sense. In this study we will consider a model that is open only in the narrow sense. The reader will find the details of dynamics of open systems in a broad sense in books [53, 54].

11.2. An open system as a subsystem of a closed one. Let us consider a closed Hamiltonian system characterized by a set of phase variables $\{q_1, p_1; \ldots; q_n, p_n\}$. We will divide it into two parts: subsystem 1 with coordinates $\mathbf{x} \equiv \{q_1, p_1; \ldots; q_{n_1}, p_{n_1}\}$ and subsystem 2 with the rest coordinates, the set of which is denoted by $\mathbf{y}$. The Hamiltonian of the original system appears as the sum

$$\mathcal{H}(\mathbf{x}, \mathbf{y}) = \mathcal{H}_1(\mathbf{x}) + \mathcal{H}_2(\mathbf{y}) + \mathcal{H}_{12}(\mathbf{x}, \mathbf{y}), \tag{33}$$

the first term in which is the Hamiltonian of subsystem 1 in the absence of subsystem 2, the second one is the Hamiltonian of subsystem 2 in the absence of subsystem 1, and the third one is the interaction Hamiltonian of these subsystems. If $(\mathbf{x}_0, \mathbf{y}_0)$ is the determinate initial state of the system, and $\mathbf{X}(t; \mathbf{x}_0, \mathbf{y}_0)$ and $\mathbf{Y}(t; \mathbf{x}_0, \mathbf{y}_0)$ are the phase trajectories of each of the subsystems parameterized by the total physical time t, then the phase probability density

$$f_{n_1+n_2}(q_1, p_1; \ldots; q_{n_1+n_2}, p_{n_1+n_2}; t) \equiv f(\mathbf{x}, \mathbf{y}, t)$$

will be written as

$$f(\mathbf{x}, \mathbf{y}, t) = \delta\left[\mathbf{x} - \mathbf{X}(t; \mathbf{x}_0, \mathbf{y}_0)\right]\delta[\mathbf{y} - \mathbf{Y}(t; \mathbf{x}_0, \mathbf{y}_0)].$$

However, it is more natural to keep in mind a certain assembly of random initial states $\{\mathbf{X}_0, \mathbf{Y}_0\}$, characterized by the density

$$f(\mathbf{x}, \mathbf{y}, t_0) = \langle \delta(\mathbf{x} - \mathbf{X}_0)\delta(\mathbf{y} - \mathbf{Y}_0) \rangle . \tag{34}$$

This function satisfies the Liouville equation, which we present as

$$\frac{\partial f}{\partial t} = \mathsf{L} f(\mathbf{x}, \mathbf{y}, t), \tag{35}$$

and the initial condition is assumed to be distributed according to (34).

The Liouville operator defined by the formula

$$\mathsf{L} f = -\{\mathcal{H}, f\},$$

also splits into three terms

$$\mathsf{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12}, \tag{36}$$

containing $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_{12}$ respectively. In the absence of interaction between the subsystems, the operator $\mathcal{H}_{12}$ disappears, and the correlations between them are determined only by the initial state. If there are no correlations

$$f(\mathbf{x}, \mathbf{y}, t_0) = f_1(\mathbf{x}, t_0) f_2(\mathbf{y}, t_0),$$

[2]By the way, there exist other definitions of open systems.

then the solution of equation (35), marked in this case by the superscript 0, has the form of a product of functions

$$f^0(\mathbf{x},\mathbf{y},t) = f_1(\mathbf{x},t)f_2(\mathbf{y},t), \tag{37}$$

each of which satisfies its equation,

$$\frac{\partial f_1}{\partial t} = \mathcal{L}_1 f_1(\mathbf{x},t),$$
$$\frac{\partial f_2}{\partial t} = \mathcal{L}_2 f_2(\mathbf{y},t),$$

with its initial condition:

$$f_1(\mathbf{x},t_0) = \langle \delta(\mathbf{x}-\mathbf{X}_0)\rangle,$$
$$f_2(\mathbf{y},t_0) = \langle \delta(\mathbf{y}-\mathbf{Y}_0)\rangle.$$

Let us note that

$$\int f_1(\mathbf{x},t)d\mathbf{x} = \int f_2(\mathbf{y},t)d\mathbf{y} = \iint f(\mathbf{x},\mathbf{y},t)d\mathbf{x}d\mathbf{y} = 1$$

for all $t \geq t_0$.

11.3. Zwanzig–Mori projection operators. At presence of interaction between the subsystems, decomposition of solution into relevant and irrelevant projections is performed by means of Zwanzig–Mori formalism:

$$\mathsf{P}f(\mathbf{x},\mathbf{y},t) = \int d\mathbf{y}' f(\mathbf{x},\mathbf{y}',t)\cdot f_2(\mathbf{y},t_0) = f_1(\mathbf{x},t)f_2(\mathbf{y},t_0) \equiv \phi_1(\mathbf{x},\mathbf{y},t). \tag{38}$$

Let us again act on both sides of this equation by the operator P:

$$\mathsf{P}^2 f(\mathbf{x},\mathbf{y},t) = \mathsf{P}\left(\int d\mathbf{y}' f(\mathbf{x},\mathbf{y}',t)\cdot f_2(\mathbf{y},t_0)\right) = \int d\mathbf{y}'\left(\int d\mathbf{y}'' f(\mathbf{x},\mathbf{y}'',t)f_2(\mathbf{y}',t_0)\right)\cdot f_2(\mathbf{y},t_0) =$$
$$= \int d\mathbf{y}'' f(\mathbf{x},\mathbf{y}'',t)\left(\int f_2(\mathbf{y}',t_0)d\mathbf{y}'\right)\cdot f_2(\mathbf{y},t_0) = \int d\mathbf{y}'' f(\mathbf{x},\mathbf{y}'',t)\cdot f_2(\mathbf{y},t_0) = \phi_1(\mathbf{x},\mathbf{y},t).$$

Subsequent use of the operator P did not change the result. For this reason, the operator P is called projection. We will define the operator P complementary to P' by the formula

$$\mathsf{P}' = \mathbf{1} - \mathsf{P} \tag{39}$$

(here 1 is the identity operator). It is easy to verify that P' is also a projection operator and is orthogonal to the operator P:

$$\mathsf{P}\mathsf{P}' = \mathsf{P}'\mathsf{P} = 0 \tag{40}$$

In addition, it can be shown that P and P' commute with $\mathcal{L}_1$ and $\mathcal{L}_2$,

$$\mathsf{P}\mathcal{L}_1 - \mathcal{L}_1\mathsf{P} = \mathsf{P}'\mathcal{L}_1 - \mathcal{L}_1\mathsf{P}' = 0, \quad \mathsf{P}\mathcal{L}_2 - \mathcal{L}_2\mathsf{P} = \mathsf{P}'\mathcal{L}_2 - \mathcal{L}_2\mathsf{P}' = 0, \tag{41}$$

the operators P and $\mathcal{L}_2$ are mutually orthogonal,

$$\mathsf{P}\mathcal{L}_2 = \mathcal{L}_2\mathsf{P} = 0, \tag{42}$$

and the identity is fair

$$\mathsf{P}\mathcal{L}_{12}\mathsf{P} = 0 \tag{43}$$

(see the book of Maximilian Di Ventra [55]).

11.4. Splitting of the Liouville equation. According to (38), the phase density of the total system can be represented as the sum of two terms

$$f(\mathbf{x},\mathbf{y},t) = \mathsf{P}f(\mathbf{x},\mathbf{y},t) + \mathsf{P}'f(\mathbf{x},\mathbf{y},t) \equiv \phi_1(\mathbf{x},\mathbf{y},t) + \phi_2(\mathbf{x},\mathbf{y},t), \tag{44}$$

having the properties from (39):

$$\mathsf{P}\phi_1 = \phi_1, \quad \mathsf{P}\phi_2 = 0, \quad \mathsf{P}'\phi_1 = 0, \quad \mathsf{P}'\phi_2 = \phi_2. \tag{45}$$

Substituting the decompositions (36) and (44) into the Liouville equation (35), we obtain

$$\frac{\partial}{\partial t}(\phi_1 + \phi_2) = (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12})(\phi_1 + \phi_2). \tag{46}$$

Having removed the parentheses, we act on both sides of this equation by the operator P:

$$\frac{\partial}{\partial t}\mathsf{P}\phi_1 + \frac{\partial}{\partial t}\mathsf{P}\phi_2 = \mathsf{P}\mathcal{L}_1\phi_1 + \mathsf{P}\mathcal{L}_1\phi_2 + \mathsf{P}\mathcal{L}_2\phi_1 + \mathsf{P}\mathcal{L}_2\phi_2 + \mathsf{P}\mathcal{L}_{12}\phi_1 + \mathsf{P}\mathcal{L}_{12}\phi_2.$$

Regarding the relations (41)–(43) and (45) allows us to significantly reduce this equation without any losses:

$$\frac{\partial\phi_1}{\partial t} = \mathcal{L}_1\phi_1 + \mathsf{P}\mathcal{L}_{12}\phi_2. \tag{47}$$

Let us return to equation (46) and act on its both parts with the additional operator P':

$$\frac{\partial}{\partial t}\mathsf{P}'\phi_1 + \frac{\partial}{\partial t}\mathsf{P}'\phi_2 = \mathsf{P}'\mathcal{L}_1\phi_1 + \mathsf{P}'\mathcal{L}_2\phi_1 + \mathsf{P}'\mathcal{L}_{12}\phi_1 + \mathsf{P}'(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12})\phi_2.$$

The first term on the left side disappears due to the third property in the list (45), the first and the second terms on the right are transformed using relations (40) and (41):

$$\mathsf{P}'\mathcal{L}_1\phi_1 = \mathsf{P}'\mathcal{L}_1\mathsf{P}f = \mathsf{P}'\mathsf{P}\mathcal{L}_1 f = 0,$$
$$\mathsf{P}'\mathcal{L}_2\phi_1 = \mathsf{P}'\mathcal{L}_2\mathsf{P}f = \mathsf{P}'\mathsf{P}\mathcal{L}_2 f = 0.$$

Considering that

$$\mathsf{P}'\frac{\partial\phi_2}{\partial t} = \frac{\partial(\mathsf{P}'\phi_2)}{\partial t} \equiv \frac{\partial\phi_2}{\partial t},$$

and returning to the symbol of the complete Liouvillian (36), we represent the second equation of the system as

$$\frac{\partial\phi_2}{\partial t} = \mathsf{P}'\mathsf{L}\phi_2 + \mathsf{P}'\mathcal{L}_{12}\phi_1.$$

11.5. The Lindblad equation. Recalling that in general case the initial condition is assumed to be given at t_0, we present the formal solution of the second equation of the system as

$$\phi_2(\mathbf{x},\mathbf{y},t) = \int_{t_0}^{t} d\tau \exp[(t-\tau)\mathsf{P}'\mathsf{L}]\mathcal{L}_{12}\phi_1(\mathbf{x},\mathbf{y},\tau) + \exp[(t-t_0)\mathsf{P}'\mathsf{L}]\phi_2(\mathbf{x},\mathbf{y},t_0). \tag{48}$$

Considering the property (43), the factor $\mathsf{P}'\mathcal{L}_{12}\phi_1$, which appeared under the integral, is slightly simplified:

$$\mathsf{P}'\mathcal{L}_{12}\phi_1 = (1-\mathsf{P})\mathcal{L}_{12}\mathsf{P}f = \mathcal{L}_{12}\phi_1.$$

We substitute the solution (48) of the second equation into the right side of the first equation (47) of the system

$$\frac{\partial}{\partial t}\phi_1(\mathbf{x},\mathbf{y},t) = \mathsf{PL}\phi_1(\mathbf{x},\mathbf{y},t) + \mathsf{PL}\int\limits_{t_0}^{t} d\tau \exp[(t-\tau)\mathsf{P'L}]\mathcal{L}_{12}\phi_1(\mathbf{x},\mathbf{y},\tau)+$$
$$+\mathsf{PL}\exp[(t-t_0)\mathsf{P'L}]\phi_1(\mathbf{x},\mathbf{y},t_0).$$

According to (38), we turn to the phase density of subsystem 1 and bring the last equation to the form

$$\frac{\partial}{\partial t}f_1(\mathbf{x},t) = \mathsf{L}_1 f_1(\mathbf{x},t) + \int\limits_{t_0}^{t} d\tau \mathsf{Q}_1(t-\tau)f_1(\mathbf{x},\tau) + M_1(\mathbf{x},\mathbf{y},t_0).$$

where

$$\mathsf{L}_1 f_1(\mathbf{x},t) = [f_2(\mathbf{y},t_0)]^{-1}\mathsf{PL}f_2(\mathbf{y},t_0)f_1(\mathbf{x},t),$$
$$\mathsf{Q}_1(t-\tau)f_1(\mathbf{x},\tau) = [f_2(\mathbf{y},t_0)]^{-1}\mathsf{P}\mathcal{L}\exp[(t-\tau)\mathsf{P'L}]\mathcal{L}_{12}f_2(\mathbf{y},t_0)f_1(\mathbf{x},\tau),$$
$$M_1(\mathbf{x},\mathbf{y},t_0) = [f_2(\mathbf{y},t_0)]^{-1}\mathsf{PL}\exp[(t-t_0)\mathcal{L}]\mathsf{P'}f_2(\mathbf{y},t_0)f_1(\mathbf{y},t_0).$$

The last term reflects the influence of the initial conditions of the aggregate system on the subsequent movement of subsystem 1. At time $t = t_0$, the integral term reflecting the influence of the process pre-history on $\partial f_1/\partial t$ disappears. It simply means that until that moment the system did not exist. In other words, the system appeared at the moment t_0. But such processes are excluded in classical mechanics; therefore, it is logical to take $t_0 = -\infty$ and suggest

$$\lim_{t_0\to-\infty} M_1(\mathbf{x},\mathbf{y},t_0) = 0.$$

Making a similar change in the lower limit of the memory integral, we get the analog of the quantum *Lindblad equation*

$$\frac{\partial}{\partial t}f_1(\mathbf{x},t) = \mathsf{L}_1 f_1(\mathbf{x},t) + \int\limits_{-\infty}^{t} d\tau \mathsf{Q}_1(t-\tau)f_1(\mathbf{x},\tau), \tag{49}$$

for which we leave the same name.

Despite the absence of explicit expressions for the operators in equation (49), its physical meaning is quite transparent. The change of the state of the observed closed system 1 + 2 in the time interval $(t, t+dt)$ is completely determined by its state at time t. Therefore, to predict its future behavior ($t > t_0$), it is enough to know the state at any moment of time t_0, and the pre-history ($t \in (-\infty < t < t_0)$) does not matter. If only a part of this system such as open system 1 – is observable, then the state of only this system 1 is not enough to predict its motion. This subsystem is (and was earlier) in interaction with subsystem 2, characterized by hidden variables. During some time intervals subsystem 1 transmitted momentum, angular momentum, energy, all of which took part in the evolution of subsystem 2 and after some time returned into subsystem 1. This exchange of dynamic properties is expressed with the hereditary integral. In other words, information about the pre-history of an open system is kept in its surrounding for some time and then returns into it. This process is denoted by the term “memory”, which should not be fully associated with its biological analog, implying the storage of information in the biological object itself.

In classical equilibrium thermodynamics, subsystem 1 is assumed to be large enough so that the influence of this exchange (taking place through the surface and therefore proportional to the surface / volume ratio) can be considered small. At the same time, only a small part of these very small flows

contains information having been taken out before from subsystem 1 and now returned to it, so that in the end we work with a small second-order quantity, which justifies the ordinary neglect of heredity in macroscopic (although not in all) processes. Turning to small (meso- and nano-) sizes, the surface / volume ratio changes and taking into account hereditary effects may turn out to be necessary [55].

Thus, there are two ways to predict the evolution of an open system, which is part of a closed system. The first way is to solve the differential equation of evolution of a closed system with the given initial conditions and extract all the information concerning this subsystem from the obtained solution. The second way is to solve the integro-differential equation for the most open subsystem, without involving its surrounding in the process of solution. The latter does not mean that we ignore the surrounding: the integral term describes the transmission of information from an open system in early times through its surrounding back to it in later times. The information about the initial state of the surrounding is also involved here.

If we are talking about spatially separated subsystems, rather than space-combined ones, such as the electronic and ionic components in a crystal, this dynamic information is transmitted through the surface of subsystem S. The result depends on the surface / volume ratio. For macroscopic samples, this ratio is small and such an exchange can be neglected (a microcanonical ensemble representing the open subsystem S as closed) or limited to the exchange of uncorrelated small portions (canonical and grand canonical ensembles). In both of these cases, the integral disappears, and we get the traditional mechanical basis of thermodynamics. Decreasing the size of the sample, we get into the field of meso- and then nanomechanics. The number of "actors" here is sharply reduced (from 1023 atoms to hundreds of thousands, or even just hundreds of atoms). The role of surface effects in this case becomes crucial, and the integral associated with them in equation (49) turns into an equal partner among the other members of the equation.

11.6. Fractional dynamics of open systems. Returning to the Lindblad equation – the basic equation of the dynamics of open systems (49) and drop, for brevity, the indices 1

$$\frac{\partial f}{\partial t} = \mathsf{L} f + \int\limits_{-\infty}^{t} \mathsf{Q}(t-\tau) f(\tau) d\tau. \tag{50}$$

Expressing the operator function $\mathsf{Q}(t)$ through its Mellin transform

$$\mathsf{Q}(t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} t^{-s} \bar{\mathsf{Q}}(s) ds, \ \sigma = \operatorname{Re} s,$$

we substitute this expression into the integral term of equation (50):

$$\int\limits_{-\infty}^{t} \mathsf{Q}(t-t') f(t') dt' = \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \mathsf{W}(s) \ \mathsf{I}_t^{1-s} f(t) ds.$$

Here

$$\mathsf{W}(s) = \frac{\Gamma(1-s)}{2\pi i} \bar{\mathsf{Q}}(s),$$

and

$$\mathsf{I}_t^{1-s} f(t) = \frac{1}{\Gamma(1-s)} \int\limits_{-\infty}^{t} \frac{f(\tau) d\tau}{(t-\tau)^s}$$

is the integral of complex order $\mu = 1 - s$. As a result, we get the equation

$$\frac{\partial f(t)}{\partial t} = \mathsf{L} f(t) + \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \mathsf{W}(s) \mathsf{I}_t^{1-s} f(t) ds,$$

containing an operator that can be interpreted as an integral operator with an operator-distributed complex order. The transition from the integral to the fractional derivative can be done, for example, by Hadamard regularization (separation of the finite part for $\mu<0$).

On the basis of the above mentioned, we can draw the following conclusion: in contrast to the closed Hamiltonian system controlled by an integer differential equation, its subsystem is controlled by an integro-differential equation of fractional (distributed along the contour in the complex plane) order. It is the spectral operator function that determines the specific kinetics of the open system of this class. Also, in our opinion, the important field of dynamics of open systems lies in the development of mathematical apparatus necessary for evaluating and approximating the spectral function [56].

Conclusion

In conclusion, we present an example of the physical argumentation of the fractional differential model of highly elastic polymer bodies described in the work by G.L. Slonimsky [57], in which the indecomposability of the process under consideration into the sum of the elastic and viscous components is clearly traced. Noting that the most important feature distinguishing highly elastic deformation from deformations of ordinary elastic bodies, "is such a clear-cut complex of relaxation phenomena that the application of Hooke's laws or any other stress-deformation relationship that does not take into account time modes of mechanical interactions, is impossible even in a rough approximation". In many works, deformation delay was associated with the presence of internal friction, and models of elastic bodies with internal friction and viscous bodies with elasticity were constructed. However, for a quantitative or at least semi-quantitative description of the deformation of highly elastic bodies, it would be necessary to construct very complex mechanical models consisting of a large number of different springs and dampers. Taking into account the molecular dynamics of amorphous polymers, according to which the relaxation processes in them are associated with slow processes of rearrangement of long and flexible molecules and tangles (clusters) of such molecules, which are manifested in the lag of strain changes from stress changes. Due to the extremely long length of the chain molecule, the simultaneous motion of all its individual parts (segments) during deformation is impossible, since the energy required to overcome intermolecular interactions far exceeds the energy of chemical bonds. The motion of such a molecule occurs because of the aftermotion of its segments, which are possible due to the flexibility of a highly elastic molecule. Having studied the laws of motion of segments, we can understand the laws of movement of chain molecules, changes in their shape and, ultimately, the laws of deformation of highly elastic bodies. According to G.L. Slonimsky, a new way to determine the laws of deformation of highly elastic bodies is to abandon the notion of high elasticity as a result of summation of elasticity and internal friction. The reason for this failure is seen in the rare difference between high elasticity and the elasticity of ordinary bodies with low values (2–3 orders of magnitude lower than ordinary) of elastic modulus, huge magnitudes (up to 1000% and higher) reversible by deformation, opposite signs of the thermal effects of deformation and temperature elastic modulus. G.L. Slonimsky explains that highly elastic deformation is caused only by a change in the shape of flexible long chain molecules of polymeric substances without changing the energy of their (or their moving parts-segments) interactions. With isothermal high elasticity, the internal energy of the polymer body does not change, and the entire work of deformation turns into heat, so the resistance

of the deformation body is due only to a decrease in entropy. Thus, highly elastic deformation should be considered as an independent type of reversible deformation and one should not try to decompose it into elastic and viscous components. The mathematical symbol of such independence is represented by G.L. Slonimsky as a fractional derivative, "which combines many features of elastic and plastic deformation, but is not a combination of them."

Let us return, in conclusion, to the problem of turbulent diffusion, from which we went off by discussing open systems, the way openness of the system (more precisely, its connection with the surrounding) causes fractional differential "spirits" to appear on stage. To do this, it is enough for us to understand the reason for the delay. Physically, it is associated with a delayed (diffusion type) return from the surrounding of the additive characteristics (kinetic energy, momentum) of the open system transferred there during the initial period of motion. In the case of the description of turbulent diffusion, the splitting of the phase coordinates into molecular and turbulent components can well be interpreted as dividing the system into two subsystems with all ensuing consequences. But this is the subject of another work.

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References

1. Rutman R.S. On physical interpretations of fractional integration and differentiation. *Theoretical and Mathematical Physics*, 1995, vol. 105, no. 3, pp. 1509–1519.
2. Uchaikin V.V. Self-similar anomalous diffusion and Levy-stable laws. *Physics–Uspekhi*, 2003, vol. 46, no. 8, pp. 821.
3. Sibatov R.T., Uchaikin V.V. Fractional differential approach to dispersive transport in semiconductors. *Physics–Uspekhi*, 2009, vol. 52, no. 10, pp. 1019–1043.
4. Uchaikin V.V. Fractional phenomenology of cosmic ray anomalous diffusion. *Physics–Uspekhi*. 2013. vol. 56, no. 11, pp. 1074–1119.
5. Uchaikin V.V. The Method of Fractional Derivatives. Ulyanovsk: «Artishok», 2008 (in Russian).
6. Uchaikin V.V. Fractional Derivatives for Physicists and Engineers. Vol's I–II, Springer Berlin, HEP Beijing, 2013.
7. Boltzmann L. Zur Theorie der Elastishen Nachwirkungen. *Akad. Wiss. Wien, Math. Naturwiss.* 1874, vol.70, no. 2, pp. 275–306.
8. Volterra V. Lectures in the Mathematical Theory of the Struggle for Existence. Gauthier-Villars, Paris, 1931.
9. Volterra V. Theory of Functionals of Integral and Integro-Differential Equations, Dover, New York, 1959.
10. Rabotnov Y.N. Elementy nasledstvennoy mekhaniki tverdykh tel. Moscow, Nauka, 1977 (in Russian).
11. Valanis K.C. and Lee C.F. Endochronic theory of cyclic plasticity with application. *J. Appl. Mech.*, 1984, vol. 51, pp. 367–374.
12. Blatt J.M. An alternative approach to the ergodic problem. *Progress in Theoretical Physics*, 1959, vol. 22, pp. 745–756.
13. Maugin G.A. and Muschik W. Thermodynamics with internal variables. Part I. General concepts. *Journal of Non-Equilibrium Thermodynamics*, 1994, vol. 19, pp. 217–249.
14. Maugin G.A. and Muschik W. Thermodynamics with internal variables. Part II. Applications. *Journal of Non-Equilibrium Thermodynamics*, 1994, vol. 19, pp. 250–289.

15. Maugin G. The Thermomechanics of Nonlinear Irreversible Behaviors (An Introduction). World Scientific, Singapure–New Jersey–London–Hong Kong, 1999.
16. Gemant A. A method of analyzing experimental results obtained from elastoviscous bodies. *Physics*, 1936, vol. 7, pp. 311–317.
17. Gerasimov A.N. A generalization of linear laws of deformation and its application to internal friction problem. *Prikl. Mat. Mekh.*, 1948, vol. 12, no. 3. pp. 251–260 (in Russian).
18. Nigmatullin R.R. Fractional integral and its physical interpretation. *Theoretical and Mathematical Physics*, 1992, vol. 90, no. 3, pp. 242–251.
19. Hilfer R. Classification theory for anequilibrium phase transitions. *Phys Rev E.*, 1993, vol. 48, pp. 2466–2475.
20. Hilfer R. Fractional time evolution, In Applications of Fractional Calculus in Physics. R. Hilfer (ed.), World Scientific, Singapore, 2000, pp. 87–131.
21. Lukashchuk S.Yu. Time-fractional extensions of the Liouville and Zwanzig equation. *Cent. Eur. J. Phys.*, 2013, vol. 11, no. 6, p. 740 .
22. Kwok Sau Fa. A falling body problem through the air in view of the fractional derivative approach. *Physica A*, 2005, vol. 350, pp. 199–206.
23. Narahari Achar B.N., Hanneken J.W., Enck T., Clarke T. Dynamics of the fractional oscillator. *Physica A*, 2001, vol. 297, pp. 361–367.
24. Ryabov Ya.E. and Puzenko A. Damped oscillations in view of the fractional oscillator equation. *Phys. Rev. B*, 2002, vol. 66, 184201.
25. Baleanu D., Golmankhaneh A.K., Nigmatullin R., Golmankhaneh Ali K. Fractional Newtonian mechanics *Cent. Eur. J. Phys.*, 2010, vol. 8, no. 1, pp. 120–125.
26. Slezkin N.A. Dynamics of Viscous Incompressible Fluid, Moscow, Gostekhizdat, 1955 (in Russian).
27. Boussinesq V.J. Sur la resistance quoppose un fluide indefini au repose...*Compt.Rend.de l'Academ. des Sci.* 1885, vol. 100, pp. 935–937.
28. Basset A.B. Treatise on Hydrodynamics 2.,Deighton, Bell and Co., Cambridge. UK, 1888.
29. Zener C.M. Anelasticity of metals. *Suppl. Nuovo Cimento*, 1958, vol. 7, p. 544.
30. Uchaikin V.V., Sibatov R.T. Fractional differential kinetics of dispersive transport as the consequence of its self-similarity. *JETP Letters*, 2007, vol. 86, no. 8, pp. 512–516.
31. Uchaikin V.V., Sibatov R.T. Fractional Kinetics in Solids, World Scientific, 2013.
32. Landau L.D., Lifschitz E.M. Fluid Mechanics. New York: Pergamon Press, 1987.
33. van Hove L. The approach to equilibrium in quantum statistics: A perturbation treatment to general order. *Physica*, 1957, vol. 23, pp. 441–480.
34. Prigogine I., Resibois P. On the kinetics of the approach to equilibrium. *Physica*, 1961, vol. 27, pp. 629–646.
35. Brout R., Prigogine I. Statistical mechanics of irreversible processes, *Physica*, 1956, vol. 22, pp. 35–47, 263–272, 621–636.
36. Resibua P., De Lener M. Classical Kinetic Theory of Liquids and Gases. Moscow, Mir, 1980 (in Russian).
37. Zwanzig R. Nonequilibrium Statistical Mechanics. New York: Oxford University Press, 2001.
38. Montroll E . W. In Fundamental Problems in Statistical Mechanics, ed. E. Cohen, North-Holland, Amsterdam, 1962.
39. Chester G.V. The theory of irreversible processes, *Rep. on Progress in Physics*, 1963, vol. 26, p. 411.

40. Alder B.J., Alley W.E. Generalized Hydrodynamics, *Physics Today*, 1984, vol. 37, pp. 56–83.
41. Richardson L.F. Atmospheric diffusion on a distance-neighbor graph. *Proc. Roy Soc. London, Ser A*, 1926, vol. 110, pp. 709–737.
42. Kolmogorov A.N. Scattering of energy for locally isotropic turbulence. *Dokl. Acad. Sci. USSR*, 1941, vol. 32, pp. 16–18 (in Russian).
43. Obukhov A.M. Energy distribution in the spectrum of a turbulent flow. *Dokl. Acad. Sci. USSR*, 1941, vol. 32, pp. 22–24 (in Russian).
44. Jullien M.C., Paret J., Tabeling P. Richardson pair dispersion in two-dimensional turbulence. *Phys. Rev. Lett.*, 1999, vol. 82, p. 2872.
45. Tchen C.M. Diffusion of Particles in Turbulent Flow. *Advances in Geophysics*, 1959, vol. 9, pp. 165–174.
46. Heisenberg W. Zur Statistischen Theorie der Turbulenz. *Zeitschrift fuer Physik* 1948, vol. 124, pp. 628–657.
47. Monin A.S. Yaglom A.M. Statistical Hydromechanics. Part I. M.: Nauka, 1965; Part II. Moscow: Nauka, 1967 (in Russian).
48. Shlesinger M., Klafter J., West B. Levy walks with applications to turbulence and chaos. *Physica*, 1986, vol. 140A, pp. 212–218.
49. Schönfeld J.C. Integral diffusivity. *Journal of Geophysical Research*, 1962, vol. 67, no. 8, pp. 3187–3199.
50. Tchen C.M. Transport processes as foundations of the Heisenberg and Obukhoff theories of turbulence. *Phys Rev.*, 1954, vol. 93, no. 1, pp. 4–14.
51. Uchaikin V.V. Mechanics. Fundamentals of Continuum Mechanics. SPb, Lan, 2016 (in Russian).
52. Uchaikin V.V. On time-fractional representation of an open system response. *Fractional Calculus and Applied Analysis*, 2016, vol. 19, no. 5, pp. 1306–1315.
53. Klimontovich Yu.L. Introduction to the Physics of Open Systems. M., Janus-K, 2002 (in Russian).
54. Lindenberg K., West B.J. The Nonequilibrium Statistical Mechanics of Open and Closed Systems, Wiley, VCH Publishers, New York, 1990.
55. Di-Ventra M. Electrical Transport in Nanoscale Systems. Cambridge University Press, 2008.
56. Uchaikin V.V. On the fractional-differential Liouville equation as an equation of dynamics of an open system. *Scientific bulletins of the Belgorod University, series: Mathematics. Physics*, 2014, vol. 25(196), no. 37, pp. 58–67 (in Russian).
57. Slonimsky G.L. On the law of deformation of highly elastic polymer bodies. *Dokl. AN SSSR*, 1961, vol. 140, no. 2, pp. 343–346 (in Russian).