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Variational approach to the construction of discrete mathematical model of the pendulum motion with vibrating suspension with friction

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Abstract. *The main purpose* of this work is, first, a construction of the indirect Hamilton's variational principle for the problem of motion of a pendulum with a vibration suspension with friction, oscillating along a straight line making a small angle with the vertical line. Second, the construction on its basis of the difference scheme. Third, to carry out its investigation by methods of numerical analysis. *Methods.* The problem of motion of the indicated pendulum is considering as a particular case of the given boundary problem for a nonlinear second order differential equations. For the solution of problem of its variational formulation there is used the criterion of potentiality of operators — the symmetry of the Gâteaux derivative of nonlinear operator of the given problem. This criterion is also used for the construction of variational multiplier and the corresponding Hamilton's variational principle. On its basis there is constructed and investigated a discrete analog of the given boundary problem and a problem of motion of the pendulum. *Results.* It is proved that the operator of the given boundary problem is not potential with respect to the classical bilinear form. There is found a variational multiplier and constructed the corresponding indirect Hamilton's variational principle. On its basis there is obtained a discrete analog of the given boundary problem and its solution is found. As particular cases one can deduce from that the corresponding results for the problem of motion of the pendulum. There are performed numerical experiments, establishing the dependence of solutions of the problem of motion of the pendulum on the change of parameters. *Conclusion.* There is worked out a variational approach to the construction of two difference schemes for the problem of a pendulum with a suspension with friction, oscillating along a straight line making a small angle with the vertical line. There are presented results of numerical simulation under different parameters of the problem. Numerical results show that under sufficiently small amplitude and sufficiently big frequency of the oscillations of the point of suspension the pendulum realizes a periodical motion.

Keywords: inverted pendulum, indirect variational formulation, Hamilton's equations, difference scheme.

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Introduction

This work goes back to the papers [1–3] and aims to construct and study a discrete model of the motion of a pendulum with a vibration suspension with friction based on the indirect Hamilton variational principle. A mathematical model with a continuous time of motion of such a pendulum after the works of P. L. Kapitsa, N. N. Bogolyubov became the basic one in nonlinear mechanics. Its interconnection with some problems of physics is revealed (see the literature review in [4, 5]).

At the same time, not all properties of the basic nonlinear model have been studied. Additional opportunities are opened with the use of finite difference methods [6]. In the transition from a dynamic system with continuous time to a difference scheme, it is important that both models retain the qualitative characteristics of the same phenomenon.

As noted in [7], it is advisable to use variational principles for the mathematical description of a number of physical phenomena and processes.

The authors of this article are not aware of works in which a variational approach to the construction of two different difference schemes for the problem of the motion of a pendulum with a suspension point making small oscillations along a straight line, which makes a small angle with a vertical, would be developed.

1. Construction of an indirect variational principle for one boundary value problem for a second-order differential equation

Consider the following boundary value problem:

$$N(u) \equiv \ddot{u} + k(t)\dot{u} + \varphi(t)\sin u + \psi(t)\cos u = 0, \quad t \in (0, l), \tag{1}$$

$$D(N) = \left\{u \in U = C^2[0, l] : u|_{t=0} = a_0, u|_{t=l} = a_1\right\}. \tag{2}$$

Here $u(t)$ is an unknown function; $k \in C^2[0, l]$, φ, $\psi \in C[0, l]$ are given functions; a_0, a_1 are given numbers; $\dot{u} = \dfrac{du(t)}{dt}$, $\ddot{u} = \dfrac{d^2u(t)}{dt^2}$.

The equations studied in the papers [1–3], are special cases (1).

Denote $V = C[0, l]$. Let us set the bilinear form $\langle \cdot, \cdot \rangle : V \times U \to \mathbb{R}$ of the kind

$$\langle v, g \rangle = \int_0^l v(t)\, g(t)\, dt. \tag{3}$$

We shall say that problem (1), (2) admits a direct variational formulation with respect to (3), if there exists a Gateaux differentiable functional $F_N : D(N) \to \mathbb{R}$ such that its differential

$$\delta F_N[u, h] = \langle N(u), h \rangle, \quad \forall u \in D(N), \forall h \in D\left(N'_u\right).$$

Here $D(N'_u)$ is the domain of definition of the Gateaux derivative N'_u of the operator N at the point $u \in D(N)$. At the same time, it is also said that operator N is potential on $D(N)$ with respect to bilinear form (3).

The criterion of potentiality of N on a convex set $D(N)$ is a condition of symmetry of the form [8, c. 18], [9, c. 23]

$$\left\langle N'_u h, g \right\rangle = \left\langle N'_u g, h \right\rangle, \quad \forall u \in D(N), \forall h, g \in D\left(N'_u\right). \tag{4}$$

When it is satisfied, the functional F_N — Hamilton action — can be found by the formula

$$F_N[u] = \int_0^1 \left\langle N\Big(\widetilde{u} + \lambda(u - \widetilde{u})\Big), u - \widetilde{u}\right\rangle d\lambda + \text{const}, \tag{5}$$

where $\widetilde{u}$ is an arbitrary fixed element of $D(N)$.

Theorem 1. *At $k(t) \neq 0$ problem (1), (2) does not allow a direct variational formulation with respect to bilinear form (3).*

Proof. Let us make sure that the given operator of form (1) does not satisfy condition (4). We have

$$N'_u h = \ddot{h} + k(t)\dot{h} + \varphi(t) h \cos u - \psi(t) h \sin u,$$

$$\left\langle N'_u h, g\right\rangle = \int_0^l \Big[\ddot{h} + k(t)\dot{h} + \varphi(t) h \cos u - \psi(t) h \sin u\Big] g\, dt, \tag{6}$$

$$\forall u \in D(N), \forall g, h \in D\left(N'_u\right).$$

Integrating by parts and taking into account that

$$h|_{t=0} = g|_{t=0} = h|_{t=l} = g|_{t=l} = 0, \quad \forall h, g \in \left(N'_u\right), \tag{7}$$

from (6) we get

$$\left\langle N'_u h, g\right\rangle = \int_0^l [\ddot{g} - k(t)\dot{g} + \varphi(t) g \cos u - \psi(t) g \sin u]\, h\, dt \not\equiv$$

$$\not\equiv \left\langle N'_u g, h\right\rangle = \int_0^l [\ddot{g} + k(t)\dot{g} + \varphi(t) g \cos u - \psi(t) g \sin u]\, h\, dt$$

with $k(t) \neq 0$.

The theorem is proved. □

Let us denote $\widetilde{N}(u) = M(t) N(u)$, $u \in D(N)$, where $M(t) \neq 0$ on $[0, l]$ — the required variational multiplier determined from the condition that the operator $\widetilde{N}$ is potential on $D\big(\widetilde{N}\big) = D(N)$ with respect to bilinear form (3).

Theorem 2. *For problem (1), (2) there is a variational multiplier of the form $M(t) = e^{\int k(t)dt}$.*

Proof. Let us denote

$$Q(u, h, g) = \left\langle \widetilde{N}'_u h, g\right\rangle - \left\langle \widetilde{N}'_u g, h\right\rangle, \quad \forall u \in D(N), \forall g, h \in D\left(N'_u\right).$$

We have

$$\widetilde{N}'_u h = M(t) N'_u h,$$

$$\left\langle \widetilde{N}'_u h, g\right\rangle = \int_0^l M(t) N'_u h \cdot g\, dt = \int_0^l M(t)\Big[\ddot{h}g + k(t)\dot{h}g + \varphi(t) hg \cos u - \psi(t) hg \sin u\Big] dt,$$

$$\left\langle \widetilde{N}'_u g, h\right\rangle = \int_0^l M(t)\,[\ddot{g}h + k(t)\dot{g}h + \varphi(t) hg \cos u - \psi(t) hg \sin u]\, dt.$$

With this in mind, we get

$$Q(u, h, g) = \int_0^l \Big[M(t)\ddot{h}g + M(t)k(t)\dot{h}g - M(t)\ddot{g}h - M(t)k(t)\dot{g}h\Big] dt.$$

Integrating by parts and taking into account conditions (7), from here we find

$$Q\left(u,h,g\right)=\int_0^l\left\{2\left[\frac{dM}{dt}-Mk\right]\frac{dg}{dt}+\left[\frac{d^2M}{dt^2}-\frac{d\left(Mk\right)}{dt}\right]g\right\}h\,dt, \tag{8}$$
$$\forall u\in D\left(N\right),\forall h,g\in D\left(N'_u\right).$$

To fulfill the condition

$$Q\left(u,h,g\right)=0,\quad \forall u\in D\left(N\right),\forall g,h\in D\left(N'_u\right)$$

it is necessary and sufficient that

$$\frac{dM}{dt}-Mk=0,\quad \forall t\in\left[0,l\right], \tag{9}$$

$$\frac{d^2M}{dt^2}-\frac{d\left(Mk\right)}{dt}=0,\quad \forall t\in\left[0,l\right]. \tag{10}$$

Condition (10) is a consequence of (9). Thus, the variational multiplier $M\left(t\right)$ is a solution of equation (9) and has the form

$$M\left(t\right)=e^{\int k(t)dt}. \tag{11}$$

The theorem is proved. □

Theorem 3. *The equation*

$$\widetilde{N}\left(u\right)\equiv e^{\int k(t)dt}N\left(u\right)=0,\quad u\in D\left(N\right), \tag{12}$$

where N has form (1), can be represented in the form of Hamilton's equations

$$\begin{aligned}\dot{u}&=-e^{-\int k(t)dt}p,\\ \dot{p}&=e^{\int k(t)dt}\left(\varphi\sin u+\psi\cos u\right).\end{aligned} \tag{13}$$

Proof. Using formula (5), we find the Hamilton's action for (12) in the form

$$F_{\widetilde{N}}\left[u\right]=\int_0^l M\left(-\frac{1}{2}\dot{u}^2-\varphi\cos u+\psi\sin u+\varphi\right)dt. \tag{14}$$

Thus, Lagrangian of the equation (12) is equal to

$$\mathcal{L}=e^{\int k(t)dt}\left(-\frac{1}{2}\dot{u}^2-\varphi\cos u+\psi\sin u+\varphi\right). \tag{15}$$

By introducing a generalized impulse

$$p=\frac{\partial\mathcal{L}}{\partial\dot{u}}=-M\dot{u},$$

we obtain that the Lagrangian (15) corresponds to the Hamiltonian

$$H\left(t,p,u\right)=-\frac{p^2}{2M}+M\varphi\cos u-M\psi\sin u-M\varphi.$$

From here we find

$$\frac{\partial H}{\partial p} = -\frac{p}{M},$$
$$\frac{\partial H}{\partial u} = -M\varphi \sin u - M\psi \cos u$$

and, therefore, we obtain Hamilton's equations (13).

The theorem is proved. □

Equations (13) can be derived from Hamilton's variational principle with the action

$$J\left[p,u\right] = \int_0^l \left[p\dot{u} - H\left(t,p,u\right)\right] dt. \tag{16}$$

2. Construction and investigation of a discrete analog of problem (1), (2) based on functional (14)

We divide the segment $[0,l]$ into m equal parts by nodes $t_i = i\tau$ $(i = \overline{0,m})$, where $\tau = m^{-1}l$. Let us introduce the narrowing operator

$$T_r u\left(t\right) = \overline{u}_r = \left(u\left(t_0\right), u\left(t_1\right), u\left(t_2\right), \ldots, u\left(t_{m-1}\right), u\left(t_m\right)\right)^T$$

(column height $r = m+1$). Such columns form a linear space, which we will denote $\overline{U}_r$. For convenience, we shall write $u_i = u\left(t_i\right)$.

Denote $\overline{N}_F$ the operator of the discrete analog of problem (1), (2) based on functional (14).

Suppose $D(\overline{N}_F) = \left\{\overline{u}_r \in \overline{U}_r : u_0 = a_0, u_m = a_1\right\}$ и $D(\overline{N}'_F) = \left\{\overline{h}_r \in \overline{U}_r : h_0 = h_m = 0\right\}$.

Let us write (14) as

$$F_{\widetilde{N}}\left[u\right] = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} M\left(-\frac{1}{2}\dot{u}^2 - \varphi\cos u + \psi\sin u + \varphi\right) dt.$$

Next , we approximate the integrals

$$\int_{t_i}^{t_{i+1}} M\left(-\frac{1}{2}\dot{u}^2 - \varphi\cos u + \psi\sin u + \varphi\right) dt \approx$$
$$\approx \frac{l}{m} M^i \left[-\frac{1}{2}\left(\frac{u_{i+1}-u_i}{\tau}\right)^2 - \varphi^i\cos u_i + \psi^i \sin u_i + \varphi^i\right],$$

where $M^i = M\left(t_i\right)$, $\varphi^i = \varphi\left(t_i\right)$ и $\psi^i = \psi\left(t_i\right)$.

Functional (14) is replaced by the Hamilton's difference action

$$\overline{F}\left[\overline{u}_r\right] = \frac{l}{m}\sum_{i=0}^{m-1} M^i \left[-\frac{1}{2}\left(\frac{u_{i+1}-u_i}{\tau}\right)^2 - \varphi^i\cos u_i + \psi^i \sin u_i + \varphi^i\right].$$

Equating partial derivatives to zero

$$\frac{\partial \overline{F}\left[\overline{u}_r\right]}{\partial u_i} = \frac{l}{m}\left(M^i\frac{u_{i+1}-u_i}{\tau^2} - M^{i-1}\frac{u_i-u_{i-1}}{\tau^2} + M^i\varphi^i\sin u_i + M^i\psi^i\cos u_i\right),$$
$$i=\overline{1,m-1},$$

we obtain the system of difference equations

$$\overline{N}_F^i\left(\overline{u}_r\right) \equiv M^i\frac{u_{i+1}-u_i}{\tau^2} - M^{i-1}\frac{u_i-u_{i-1}}{\tau^2} + \\ + M^i\varphi^i\sin u_i + M^i\psi^i\cos u_i = 0, \quad i=\overline{1,m-1}. \tag{17}$$

From here we find the solution to this system

$$u_{i+1} = u_i + \frac{M^{i-1}}{M^i}\left(u_i-u_{i-1}\right) - \tau^2\varphi^i\sin u_i - \tau^2\psi^i\cos u_i, \quad i=\overline{1,m-1},$$
$$u_0=a_0, \quad u_m=a_1.$$

Let us pass to the next special case of equation (1). Consider the equation of motion of a pendulum, the suspension point of which oscillates according to a sinusoidal law along a straight line inclined to the vertical axis OY at an angle α [10]

$$N_1(u) \equiv \ddot{u} + \sigma\dot{u} + \frac{g - A\omega^2\sin\left(\omega t\right)\cos\alpha}{d}\sin u - \frac{A\omega^2\sin\left(\omega t\right)\sin\alpha}{d}\cos u = 0, \quad t\in\left(0,l\right), \tag{18}$$

$$u_0=a_0, \quad u_m=a_1, \tag{19}$$

where u is the angle of deviation of the pendulum from the lower vertical equilibrium position, σ an attenuation coefficient, d is the length of the pendulum, g is an acceleration of gravity, ω is a frequency of point oscillations suspension, A is the amplitude of the suspension point oscillations.

By virtue of theorems 1, 2 for $\sigma \neq 0$ operator N_1 (18) is non-potential with respect to bilinear form (3) and for problem (18),(19) there is a variational multiplier of the form $e^{\sigma t}$. Denote

$$\widetilde{N}_1\left(u\right) \equiv e^{\sigma t}N_1\left(u\right) = 0. \tag{20}$$

According to formula (14), we have the Hamilton's action for (20):

$$F_{\widetilde{N}_1}\left[u\right] = \int_0^l e^{\sigma t}\left(-\frac{1}{2}\dot{u}^2 - \frac{g - A\omega^2\sin\left(\omega t\right)\cos\alpha}{d}\cos u - \frac{A\omega^2\sin\left(\omega t\right)\sin\alpha}{d}\sin u + \\ + \frac{g - A\omega^2\sin\left(\omega t\right)\cos\alpha}{d}\right)dt, \tag{21}$$

and the corresponding finite-difference functional has the form

$$\overline{F}_{\widetilde{N}_1}\left[\overline{u}_r\right] = \frac{l}{m}\sum_{i=0}^{m-1} e^{\sigma t_i}\left[-\frac{1}{2}\left(\frac{u_{i+1}-u_i}{\tau}\right)^2 - \frac{g - A\omega^2\sin\left(\omega t_i\right)\cos\alpha}{d}\cos u_i + \\ + \left(-\frac{A\omega^2\sin\left(\omega t_i\right)\sin\alpha}{d}\right)\sin u_i + \frac{g - A\omega^2\sin\left(\omega t_i\right)\cos\alpha}{d}\right].$$

Using (17), we write a discrete analog of problem (18),(19) based on functional (21)

$$\overline{N}^{i}_{1,F}(\overline{u}_r) \equiv e^{\sigma t_i}\frac{u_{i+1}-u_i}{\tau^2} - e^{\sigma t_{i-1}}\frac{u_i-u_{i-1}}{\tau^2} + e^{\sigma t_i}\frac{g - A\omega^2\sin(\omega t_i)\cos\alpha}{d}\sin u_i + \\ + e^{\sigma t_i}\left(-\frac{A\omega^2\sin(\omega t_i)\sin\alpha}{d}\right)\cos u_i = 0, \quad i=\overline{1,m-1},$$

$$u(0)=a_0, \quad u(l)=a_1.$$

The solution of this system is given by the formulas

$$u_{i+1} = u_i + e^{-\sigma\tau}(u_i-u_{i-1}) - \tau^2\frac{g - A\omega^2\sin(\omega t_i)\cos\alpha}{d}\sin u_i + \\ + \tau^2\frac{A\omega^2\sin(\omega t_i)\sin\alpha}{d}\cos u_i, \quad i=\overline{1,m-1}, \tag{22}$$

$$u_0=a_0, \quad u_m=a_1.$$

To conduct numerical experiments , we assume:

- attenuation coefficient $\sigma = -0.01$ c^{-1},
- gravity acceleration $g = 9.8$ м/с^2,

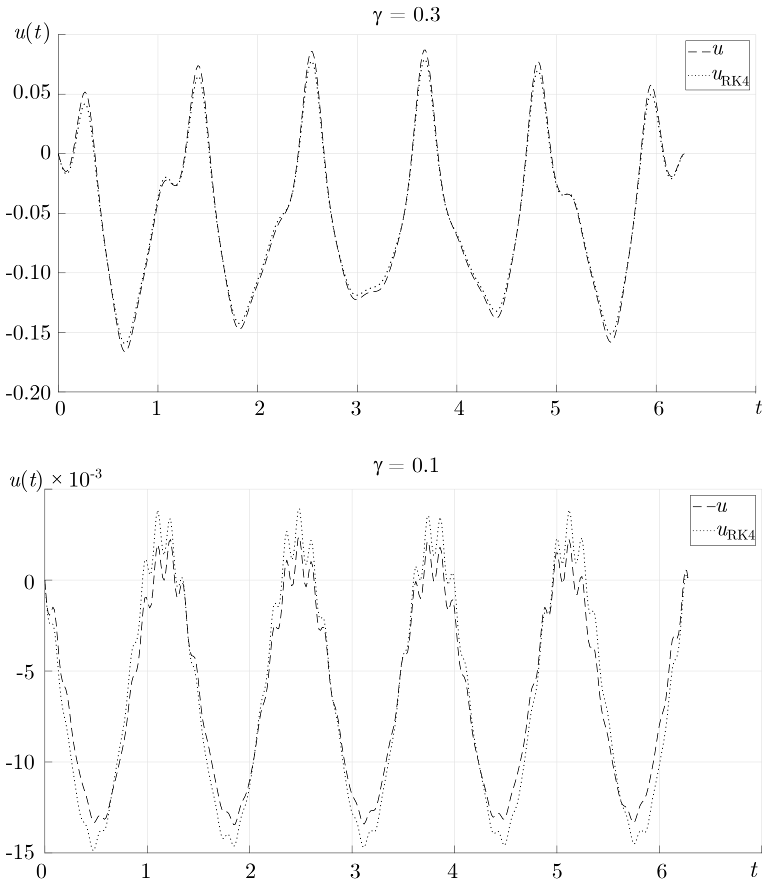

Fig. 1. u is a solution (22), u_{RK4} is a solution by the Runge–Kutta method

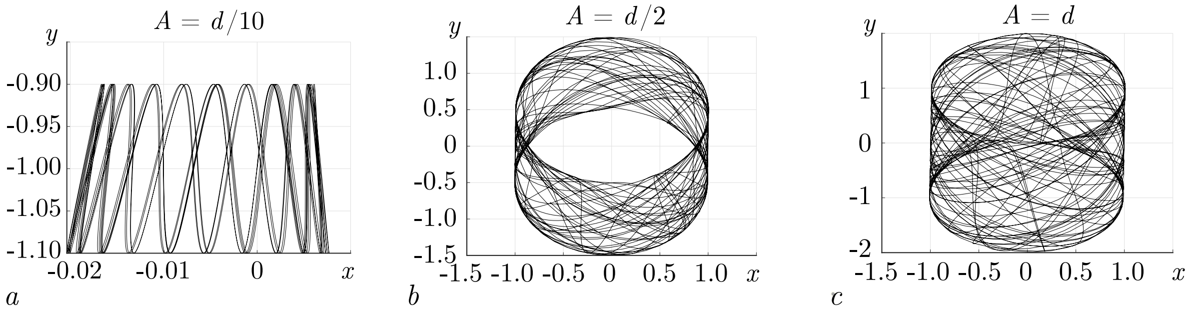

Fig. 2. Portrait of the system on the plane OXY for a pendulum with large amplitude of forcing oscillations. The parameters of the pendulum are given in the text

- length $d = 1$ м,
- small parameter γ ($\gamma = 0.3,\ 0.1$),
- oscillation amplitude of the suspension point $A = \gamma A_0$, $A_0 = 1$ м,
- angle $\alpha = \gamma^2 \alpha_0$, $\alpha_0 = \pi/6$,
- vibration frequency of the suspension point $\omega = \omega_0/\gamma$, $\omega_0 = 5$ Гц,
- $l = 5T$, $T = 2\pi/\omega_0$ with the number of nodes $m = 400$, the function values at the endpoints $u_0 = u_m = 0$.

In Fig. 1 the solutions of (22) are shown at various specified values γ.

It should be noted that since (1), (2) is a boundary value problem, the well known shooting method is additionally used to construct the graphs given in the work. It allows us to take into account the value of the solution at the right end of the segment. In this regard, the Runge–Kutta method was chosen because of its simplicity and effectiveness for verifying the correctness of the found solution (22).

It is easy to see that when $A_0\omega_0 > \sqrt{2gd}$ и γ are sufficiently small ($\gamma = 0.1$), the graphs of solution (22) have a T-periodic form. This property in the case of continuous time is noted in the articles [10, 11].

If we increase the value of the oscillation amplitude of the pendulum suspension point $A = d/2$, then we come to the solution shown in Fig. 2, *b* (with the parameters of the pendulum: $\gamma = 0.1$ and $m = 4000$). With a further increase in the amplitude $A = d$, the trajectory of the pendulum motion fills all the empty space inside. This is clearly seen in Fig. 2, *c*. When the amplitude increases, the patterns do not change.

3. Construction and investigation of the difference scheme of equation (13) based on functional (16)

Similarly to section 2, we write (16) as a sum

$$J\left[p, u\right] = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left[p\dot{u} - H\left(t, p, u\right)\right] dt.$$

Approximating, we get

$$\int_{t_i}^{t_{i+1}} \left[p\dot{u} - H\left(t, p, u\right)\right] dt \approx \frac{l}{m}\left[p_i\left(\frac{u_{i+1} - u_i}{\tau}\right) - H^i\right],$$

where $p_i = p\left(t_i\right)$ and $H^i = H\left(t_i, p_i, u_i\right)$.

Thus, we have the difference Hamilton's action

$$\overline{J}\left[\overline{p}_r, \overline{u}_r\right] = \frac{l}{m}\sum_{i=0}^{m-1}\left[p_i\left(\frac{u_{i+1}-u_i}{\tau}\right) - H^i\right].$$

We find partial derivatives

$$\frac{\partial \overline{J}\left[\overline{p}_r, \overline{u}_r\right]}{\partial p_i} = \frac{l}{m}\left(\frac{u_{i+1}-u_i}{\tau} - \frac{\partial H^i}{\partial p_i}\right), \quad i = \overline{0, m-1},$$

$$\frac{\partial \overline{J}\left[\overline{p}_r, \overline{u}_r\right]}{\partial u_i} = \frac{l}{m}\left(-\frac{p_i - p_{i-1}}{\tau} - \frac{\partial H^i}{\partial u_i}\right), \quad i = \overline{1, m}.$$

Equating them to zero, we obtain the system of difference equations

$$\overline{N}_J^{1,i} \equiv \frac{u_{i+1}-u_i}{\tau} - \frac{\partial H^i}{\partial p_i} = \frac{u_{i+1}-u_i}{\tau} - \frac{p_i}{M^i} = 0, \quad i = \overline{0, m-1},$$

$$\overline{N}_J^{2,i} \equiv -\frac{p_{i+1}-p_i}{\tau} - \frac{\partial H^{i+1}}{\partial u_{i+1}} =$$

$$= -\frac{p_{i+1}-p_i}{\tau} - M^{i+1}\varphi^{i+1}\sin u_{i+1} - M^{i+1}\psi^{i+1}\cos u_{i+1} = 0, \quad i = \overline{0, m-1}.$$

From here we find the solution of this system by the formulas

$$u_{i+1} = u_i + \tau\frac{p_i}{M^i}, \quad i = \overline{0, m-1},$$

$$p_{i+1} = p_i - \tau M^{i+1}\varphi^{i+1}\sin u_{i+1} - \tau M^{i+1}\psi^{i+1}\cos u_{i+1}, \quad i = \overline{0, m-1},$$

$$u_0 = a_0, \quad u_m = a_1.$$

Let us pass to problem (18),(19). By virtue of theorem 3, equation (20) can be represented in the form of Hamilton's equations

$$\dot{u} = -e^{-\sigma t}p,$$

$$\dot{p} = e^{\sigma t}\left(\frac{g - A\omega^2\sin(\omega t)\cos\alpha}{d}\sin u - \frac{A\omega^2\sin(\omega t)\sin\alpha}{d}\cos u\right).$$

The Hamilton's action has the form

$$J_{\tilde{N}_1}[t, p, u] = \int_0^l \left[p\dot{u} - H_1(t, p, u)\right] dt,$$

where

$$H_1(t, p, u) = -\frac{p^2}{2e^{\sigma t}} + e^{\sigma t}\frac{g - A\omega^2\sin(\omega t)\cos\alpha}{d}\cos u + e^{\sigma t}\frac{A\omega^2\sin(\omega t)\sin\alpha}{d}\sin u -$$

$$- e^{\sigma t}\frac{g - A\omega^2\sin(\omega t)\cos\alpha}{d}.$$

The corresponding difference functional is equal to

$$\overline{J}_{\tilde{N}_1}\left[\overline{p}_r, \overline{u}_r\right] = \frac{l}{m}\sum_{i=0}^{m-1}\left[p_i\left(\frac{u_{i+1}-u_i}{\tau}\right) - H_1^i\right],$$

where $H_1^i = H_1(t_i, p_i, u_i)$. Based on it, we obtain the following system of difference equations:

$$\overline{N}_{1,J}^{1,i} \equiv \frac{u_{i+1} - u_i}{\tau} - \frac{p_i}{e^{\sigma t_i}} = 0, \quad i = \overline{0, m-1},$$
$$\overline{N}_{1,J}^{2,i} \equiv -\frac{p_{i+1} - p_i}{\tau} - e^{\sigma t_{i+1}} \frac{g - A\omega^2 \sin(\omega t_{i+1}) \cos\alpha}{d} \sin u_{i+1} +$$
$$+ e^{\sigma t_{i+1}} \frac{A\omega^2 \sin(\omega t_{i+1}) \sin\alpha}{d} \cos u_{i+1} = 0, \quad i = \overline{0, m-1}.$$

From here we find

$$\begin{aligned}
&u_{i+1} = u_i + \tau \frac{p_i}{e^{\sigma t_i}}, \quad i = \overline{0, m-1}, \\
&p_{i+1} = p_i - \tau e^{\sigma t_{i+1}} \frac{g - A\omega^2 \sin(\omega t_{i+1}) \cos\alpha}{d} \sin u_{i+1} - \\
&\qquad - \tau e^{\sigma t_{i+1}} \frac{A\omega^2 \sin(\omega t_{i+1}) \sin\alpha}{d} \cos u_{i+1}, \quad i = \overline{0, m-1}, \\
&u_0 = a_0, \quad u_m = a_1.
\end{aligned} \tag{23}$$

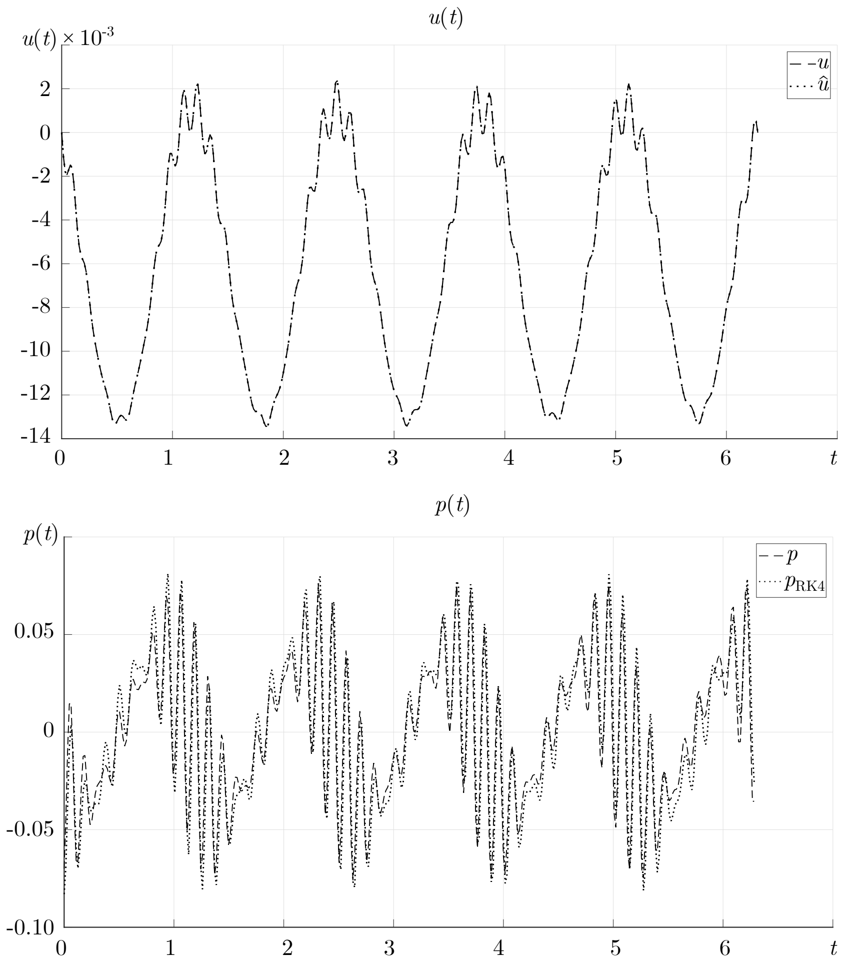

Fig. 3. $\widehat{u}$ is a solution (22), (u, p) is a solution (23), p_{RK4} is a solution by the Runge–Kutta method

To conduct numerical experiments, we assume:

- small parameter $\gamma = 0.1$,

- oscillation amplitude of the suspension point $A = \gamma A_0$, $A_0 = 1$ м,
- other parameters σ, g, d, α_0, ω_0, l, T, m, u_0 и u_m do not change as in section 2.

Conclusion

A variational approach to the construction of two difference schemes for the problem of the motion of a pendulum, the suspension point of which oscillates along a straight line, making a small angle with the vertical, is presented. The results of numerical simulation for various parameters of the problem are presented. Numerical solutions show that with a sufficiently small amplitude of oscillations and a sufficiently high frequency of oscillations of the suspension point, the pendulum performs periodic motion.

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