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Local dynamics of laser chain model with optoelectronic delayed unidirectional coupling

E. V. Grigorieva[1], *S. A. Kashchenko*[2]✉

[1]Belarus State Economic University, Minsk, Republic of Belarus
[2]Regional Scientific and Educational Mathematical Center
of the Yaroslavl State University, Yaroslavl, Russia
E-mail: grigorieva@tut.by, ✉kasch@uniyar.ac.ru
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Abstract. *Purpose.* The local dynamics of the laser chain model with optoelectronic delayed unidirectional coupling is investigated. A system of equations is considered that describes the dynamics of a closed chain of a large number of lasers with optoelectronic delayed coupling between elements. An equivalent distributed integro-differential model with a small parameter inversely proportional to the number of lasers in the chain is proposed. For a distributed model with periodic edge conditions, the critical value of the coupling coefficient is obtained, at which the stationary state in the chain becomes unstable. It is shown that in a certain neighborhood of the bifurcation point, the number of roots of the characteristic equation with a real part close to zero increases indefinitely when the small parameter decreases. In this case, a two-dimensional complex Ginzburg–Landau equation with convection is constructed as a normal form. Its nonlocal dynamics determines the behavior of the solutions of the original boundary value problem. *Research methods.* Methods for studying local dynamics based on the construction of normal forms on central manifolds are used as applied to critical cases of (asymptotically) infinite dimension. An algorithm for reducing the original boundary value problem to the equation for slowly varying amplitudes is proposed. *Results.* The simplest homogeneous periodic solutions of Ginzburg–Landau equation and corresponding to them inhomogeneous solutions in the form of traveling waves in a distributed model are obtained. Such solutions can be interpreted as phase locking regimes in the chain of coupled lasers. The frequencies and amplitudes of oscillations of the radiation intensity of each laser and the phase difference between adjacent oscillators are determined.

Keywords: bifurcation analysis, wave structures, delay, laser dynamics.

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Introduction

Cooperative effects in natural communities and in technical devices are the subject of constant active research. A description of various manifestations of synchronization in ensembles of coupled oscillators, basic models and classification of coupling functions are given, for example, in [1, 2]. It was noted that in the processes of functioning of many networks, an unavoidable factor may be a delay due to the finiteness of the speed of propagation and conversion of signals between elements. The features of network dynamics caused by the presence of time delays in communication lines were discussed in the review [3]. Many theoretical results were obtained on the basis of local analysis of the model of coupled phase oscillators [4] or the method of reduction to maps with a pulse coupling function. In particular, the possibility of delay-induced multistability of synchronous modes [5], the formation of traveling waves in a closed ring of oscillators [6, 7], cluster states [8], extinction of oscillations due to the presence of delayed connections [9] taking into account the amplitude-phase interaction.

The systems of coupled lasers attract special attention in connection with the promising applications of such networks in photonics and information technologies [10]. The dynamics of a small number of coupled lasers was studied in the most detail. Optical communication is carried out by mutual (or unidirectional) injection of the generated radiation into the active medium of lasers. In this case, due to the symmetric coupling, it was reported that the generation of two semiconductor lasers was completely synchronized when their optical frequencies were detuned [12]. In the work [11], localized synchronization (with different amplitudes) of fluctuations in the radiation intensity of two non-identical lasers with an asymmetric mutual coupling force is demonstrated. The problems of reliable radiation stabilization of a large number of optically coupled laser diodes were considered in [13].

Optoelectronic communication is usually carried out by modulating the pumping current of one laser in accordance with changes in the intensity of the generated radiation of another (other) laser (lasers). In the work [14] it is shown that in the model of two lasers connected through pumping, oscillation synchronization, resonance effects and dynamic chaos were observed even in the absence of lag. The paper [15] numerically shows the possibility of synchronization of chaos in two lasers on microchips Nd:YVO_4 or in a two-mode laser [16]. The role of delay in the coupling circuit of two laser diodes was experimentally studied in [17]. In the works [18, 19], a model with two mutually connected identical lasers was considered under the condition of different delay times in the communication circuits. A quasi-periodic scenario of transition to chaos with a change in delay was observed, and the possibility of suppressing oscillations in the case of its own delayed feedback for each laser was shown. The expansion of the region of stable synchronization of oscillations in small laser networks in the presence of their own feedback was also noted in [20]. Models of dynamics of a large number of coupled lasers were also considered. A numerical study of collective effects in a chain containing up to 50 coupled semiconductor lasers with a saturating absorber operating in the mode of short pulsations under the action of additive noise is presented in [21]. Stable temporal synchronization of pulse moments and synchronization by pulse amplitude is demonstrated.

In this paper, we study the local dynamics of a large number of coupled lasers by constructing a normalized boundary value problem. The critical values of the parameters at which stationary generation (equilibrium state) becomes unstable are determined. The correspondence between the solutions of the quasi-normal form and the solutions of the initial system for coupled lasers in the supercritical region of parameters is established.

The material of the article is presented as follows.

Part 1 discusses a closed (ring-shaped) chain model based on velocity equations for a single laser. The connection is assumed to be unidirectional, carried out through the pumping current. The choice of optoelectronic communication avoids the complexity associated with modeling multiple reflections and the dynamics of the phase of the electric field, which plays a crucial role when considering coherent optical communication. The number of lasers in the chain is assumed to be large.

A distributed integro-differential model containing a small parameter inversely proportional to the number of lasers in the chain is proposed. Part 2 presents the results of the stability analysis of the equilibrium state of the distributed model. It is shown that when a small parameter tends to zero, the number of roots of a characteristic equation with a real part close to zero tends to infinity. The critical values of the coupling coefficient, frequencies and wave numbers of the emerging traveling waves in the linearized system are obtained. In the part 3, solutions of a nonlinear system are constructed in the form of series by degrees of a small parameter. For the amplitude of the first term of the expansion, a boundary value problem with periodic boundary conditions is obtained — the two-dimensional complex Ginzburg −Landau equation with convection. Based on the solution of this quasinormal form, all the other terms of the series are found sequentially, that is, the solution of the original nonlinear system is constructed with a given accuracy. An example of the simplest periodic solutions of the quasi-normal form and the corresponding traveling waves in the initial integro-differential model is given.

About the research methodology. A distinctive feature of the problems considered here is the fact that the critical cases in the problem of stability of the equilibrium state have infinite dimension. Standard methods of study based on the application of the theory of integral invariant manifolds and the theory of normal forms are not directly applicable. In [22–26], a method for constructing quasi -normal forms for infinite-dimensional critical cases was developed. In this paper, this method is effectively used in the study of distributed laser systems.

1. A model of a chain of lasers with unidirectional connections

The standard system of velocity equations for the photon density $u(t)$ and the population inversion of the active medium $y(t)$ in a two-level approximation is chosen as the basic model of the dynamics of a separate element of the chain — laser [27],

$$\begin{aligned} \dot{u} &= vu(y-1), \\ \dot{y} &= q - y(1+u), \end{aligned} \tag{1}$$

where the dot is the derivative of the function in time t, which is normalized for the relaxation time of the inversion of populations; v — the ratio of the photon attenuation rate in the resonator to the relaxation rate of populations; resonator losses are normalized to one, q — the normalized pumping rate.

Note that at $q < 1$ (low pumping rate), all solutions of the (1) system tend to a single stable equilibrium state with zero radiation density, $u = 0$, which corresponds to the mode of no radiation generation. With an increase in the pumping rate to the values of $q > 1$, the zero state becomes unstable and a stable non-zero equilibrium state corresponding to the stationary generation mode appears. When the initial conditions deviate from the stationary, weakly damped oscillations with a frequency of $\omega_R = \sqrt{v(q-1)} + O(v^{-1})$ can be observed, which is typical for many solid-state and semiconductor lasers with a sufficiently large value of v [27].

Consider a chain of lasers, the connection between which is realized by optoelectronic means so that the pumping speed of each laser depends on the radiation density of one neighboring laser. The model of a chain closed in a ring with unidirectional communication takes the form:

$$\begin{aligned} \dot{u}_j &= vu_j(y_j-1), \\ \dot{y}_j &= q - y_j(1+u_j) - \gamma u_{j+1}(t-T), \end{aligned} \tag{2}$$

where $j = 0, \pm1, \pm2, \dots, \pm N$ — the number of the laser in the chain and the conditions $(2N+1)aremet$-periodicity by index j: $u_{j\pm(2N+1)} = u_j$; the term $\gamma u_{j+1}(t-T)$ describes the effect on the pumping of the jth laser at the current time t of the radiation intensity of the neighboring laser at the moment $(t - T)$; γ — the coupling coefficient between the lasers, which can take both positive and negative small values $|\gamma| < 1$; $T > 0$ — the signal delay time in the optoelectronic communication circuit, which it can vary from negligibly small to large values; the arguments of other functions u_j, y_j, if not specified, correspond to the current value of t.

Model (2) chains with unidirectional connections can be considered as a special case of a more general model of an ensemble of lasers, each of which is connected to all through a pump current,

$$\begin{aligned}\dot{u}_j &= vu_j(y_j - 1),\\ \dot{y}_j &= q - y_j(1 + u_j) - \gamma \sum_{l=-N}^{N} (2N+1)^{-1} g_{lj} u_{j+l}(t - T),\end{aligned} \tag{3}$$

where g_{lj} are the coefficients determining the strength of the bonds, and $\sum\limits_{l=-N}^{N} (2N+1)^{-1} g_{lj} = 1$. We assume that g_{lj} depends on the distance between the elements, so that the greater the distance, then is the modulus of the index difference $|l - j|$, the smaller the bond strength. Then, for a closed chain with unidirectional coupling, all g_{lj} are close to zero, except for the coefficients g_{1j}, and the conditions $(2N+1) aremet$-periodicity according to the index j.

In this paper, we consider a chain with a sufficiently large number of elements, $N \gg 1$, and then we introduce the parameter $\varepsilon = 2\pi(2N+1)^{-1}$ for which the condition is met

$$0 < \varepsilon \ll 1. \tag{4}$$

The parameter ε can be interpreted as the distance between the lasers in a chain of length 2π. Then the value of the functions $u_j(t)$, $y_j(t)$ can be associated with the value of the functions of two variables $u(t,x)$, $y(t,x)$, respectively, at the points of some circle with phase coordinates $x_j = \varepsilon j$. The condition (4) gives a reason to move from the system (3) to a continuous spatially distributed system for $u(t,x)$, $y(t,x)$,

$$\begin{aligned}\frac{\partial u}{\partial t} &= vu(y - 1),\\ \frac{\partial y}{\partial t} &= q - y(1 + u) - \gamma \int\limits_{-\infty}^{\infty} F(s) u(t - T, x + s) ds\end{aligned} \tag{5}$$

with periodic boundary conditions

$$u(t, x + 2\pi) \equiv u(t,x), \quad y(t, x + 2\pi) \equiv y(t,x), \tag{6}$$

where $F(s)$ is the Gaussian function,

$$F(s) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(s - \varepsilon)^2}{2\sigma^2}\right), \quad \int\limits_{-\infty}^{\infty} F(s) ds = 1. \tag{7}$$

Integral term in the system (5) generalizes the corresponding discrete expressions describing the relationship between the elements in the equations (3). For a chain model with unidirectional connection with a neighboring element, it is convenient to choose $F(s)$ as a Gaussian function for several reasons.

First, the function (7) describes a situation in which an element of the chain at point x is most strongly connected to an element at point $x+\varepsilon$, and the strength of the connection between the other elements of the chain decreases exponentially with increasing distance between them. Indeed, for every fixed function $W(x)$ with $\sigma \to 0$, there is an asymptotic equality

$$\int\limits_{-\infty}^{\infty} F(s)W(x+s)ds = W(x+\varepsilon) + o(1). \tag{8}$$

The parameter σ characterizes the width of the spatial area of effective interaction of elements. It is natural to choose this area in a continuous model less than the distance ε between the lasers in a discrete chain model. Therefore, instead of σ, we will consider the parameter $\varepsilon^2\sigma \ll 1$. Secondly, using an integral over an infinite interval for a 2π-periodic function is more convenient than an integral over a period of 2π for more complex functions. Thirdly, the connection under consideration is convenient for analytical research, since the corresponding integrals of exponential functions are accurately calculated:

$$\int\limits_{-\infty}^{\infty} F(s)\exp(iks)ds = \exp(ik\varepsilon)\exp(-\sigma^2\varepsilon^4k^2/2). \tag{9}$$

Thus, next we will investigate the solutions of the integro differential system (5), (6) with infinite dimensional phase space $C_{[0,2\pi]}(R^2) \times C_{[-T,0]}(R^2)$.

2. Stability analysis of the equilibrium state

The system (5), (6) for $q > 1$ has a homogeneous nonzero equilibrium state $u(x,t) \equiv u_s$, $y(x,t) \equiv y_s$, where

$$u_s = \frac{q-1}{1+\gamma}, \quad y_s = 1. \tag{10}$$

To study the stability of the equilibrium state, (10) is convenient in the system (5), (6) go to small deviations from the equilibrium values $\Delta u = u - u_s$ и $\Delta y = y - y_s$. As a result, omitting the prefix «Δ», we proceed to the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= vy(u_s+u), \\ \frac{\partial y}{\partial t} &= -y(1+u_s+u) - u - \gamma\int\limits_{-\infty}^{\infty} F(s)u(t-T,x+s)ds \end{aligned} \tag{11}$$

with periodic boundary conditions

$$u(t,x+2\pi) \equiv u(t,x), \quad y(t,x+2\pi) \equiv y(t,x). \tag{12}$$

Let us further consider the problem of the local dynamics of the system (11), (12) at $t \to \infty$ and at small ε with initial conditions from some sufficiently small and independent of

ε neighborhood of zero. The behavior of solutions to this boundary value problem is largely determined by the behavior of solutions of a linearized system

$$\begin{aligned}\frac{\partial u}{\partial t} &= vu_s y,\\ \frac{\partial y}{\partial t} &= -y(1+u_s) - u - \gamma \int\limits_{-\infty}^{\infty} F(s)u(t-T, x+s)ds.\end{aligned} \tag{13}$$

Taking into account the boundary conditions (12), we are looking for solutions of the linear system (13) in the form

$$u = u_k \exp\left(ikx + \lambda_k t\right), \quad y = y_k \exp\left(ikx + \lambda_k t\right), \quad k = 0, \pm1, \pm2, \ldots.$$

Then, to find the values of $\lambda = \lambda_k$, we obtain the characteristic equation

$$\det \begin{pmatrix} -\lambda & vu_s \\ -1 - \gamma g(z)\exp\left(-\lambda T\right) & -1 - u_s - \lambda \end{pmatrix} = 0$$

or

$$\lambda^2 + (1+u_s)\lambda + vu_s\left[1 + \gamma g(z)\exp\left(-\lambda T\right)\right] = 0, \tag{14}$$

где $g(z) = \exp\left(iz - \sigma^2\varepsilon^2 z^2/2\right)$, $z = \varepsilon k$, $k = 0, \pm1, \pm2, \ldots$.

The following general statements are true.

Statement 1. *Let for all $z \in (-\infty, \infty)$ all the roots of the equation (14) have negative real parts. Then all solutions of the boundary value problem (12), (13) with initial conditions from some sufficiently small and independent of ε neighborhood of the zero equilibrium state tend to zero at $t \to \infty$.*

Under this condition, the problem of the local dynamics of a nonlinear system (11), (12) is trivial.

Statement 2. *Let there be such a z_0 that for $z = z_0$ the equation (14) has a root with positive real part. Then for all sufficiently small ε, the null solution in (12), (13) is unstable and there are no attractors of this boundary value problem in its small neighborhood.*

In the condition of this Statement, the problem of the dynamics of the system (11), (12) becomes non-local.

2.1. Critical case when $\sigma = 0$. Let us consider a critical case in the problem of stability of a periodic equilibrium state. Let the characteristic equation (14) for all $z \in (-\infty, \infty)$ have no roots with a positive real part and let there be such a z_0 that for $z = z_0$ this equation has a root with a zero real part.

Putting first in (14) formally $\sigma = 0$, we come to the equation

$$\lambda^2 + (1+u_s)\lambda + vu_s = vu_s\gamma\exp\left(iz - \lambda T\right). \tag{15}$$

Let's highlight for (15) the critical case when there is a root $\lambda = i\omega$ and there are no roots with a positive real part. We introduce a polynomial

$$P(\omega) = -\omega^2 + i\omega(1+u_s) + vu_s,$$

and let

$$\min_{\omega} |P(\omega)|^2 = |P(\omega_0)|^2.$$

When finding the minimum $|P(\omega)|^2$ we come to an algebraic equation of the fourth degree with respect to γ,

$$(q+\gamma)^2[4v(q-1)(1+\gamma)-(q+\gamma)^2]-4[v(q-1)\gamma(1+\gamma)]^2=0. \tag{16}$$

The real root γ_0 of the equation (16) determines the coupling coefficient at which the stationary solution loses stability. Calculations show that there are two such roots $\gamma_0^{\pm}$, positive and negative, satisfying the physical requirement for the coupling coefficient $|\gamma_0^{\pm}| < 1$. Next, we will specify the upper indices $^{\pm}$ of critical parameters, if necessary, and omit them in the general case.

The value γ_0 determines the oscillation frequency $\omega_0 = \omega(\gamma_0)$, where

$$\omega_0^2 = vu_0 - \frac{(1+u_0)^2}{2}, \quad u_0 = \frac{q-1}{1+\gamma_0}. \tag{17}$$

Denote the phase $P(\omega_0)$ at the extremum point

$$\phi_0 = \arctan\frac{\omega_0(1+u_0)}{vu_0-\omega_0^2} \tag{18}$$

and note that from the equation (15) follows $\phi_0^+ + \pi(2n+1) = z - \omega_0^+ T$ at $\gamma_0^+ > 0$ или $\phi_0^- + 2\pi n = z - \omega_0^- T$ at $\gamma_0^- < 0$, $n = 0, \pm1, \pm2....$

The following statement defines the conditions for the occurrence of critical cases for the characteristic equation (14).

Lemma 1. *The characteristic equation (14) has a solution* $\lambda = i\omega_0^+$ *with* $\gamma = \gamma_0^+$ *and* $z = z_n^+ = \phi_0^+ + \pi(2n+1) + \omega_0^+ T$, $n = 0, \pm1, \pm2...$, *as well as the solution* $\lambda = i\omega_0^-$ *when* $\gamma = \gamma_0^-$ *и* $z = z_n^- = \phi_0^- + 2\pi n + \omega_0^- T$, $n = 0, \pm1, \pm2....$

Since the values of the parameter $z = \varepsilon k$ are related to the wave numbers of k solutions of the form $\exp(ikx + i\omega t)$ of the linearized system, the critical value of z determines the «central» wave number $k \sim z_n\varepsilon^{-1}$ those modes that are excited when the stability of the stationary state is lost. The wave numbers k must be integers due to periodic boundary conditions, so we introduce the correction $\theta = \theta(\varepsilon) \in [0,1)$, which complements the value of $z_0\varepsilon^{-1}$ to an integer. Then, in the supercritical domain, the linear system has solutions of the form $\exp(ik_{mn}x + i\omega_0 t)$ with wave numbers k_{mn} from the set of integers K_ε, which can be represented as:

$$K_\varepsilon = \{z_0\varepsilon^{-1} + \theta + 2\pi n\varepsilon^{-1} + m, \quad n, m = 0, \pm1, \pm2, ...\}, \tag{19}$$

where $2\pi n\varepsilon^{-1} = (2N+1)n$ — integer, $z_0 = \phi_0^+ + \pi + \omega_0^+ T$ when $\gamma = \gamma_0^+ > 0$ or $z_0 = \phi_0^- + \omega_0^- T$ when $\gamma = \gamma_0^- < 0$.

The introduction of the θ correction, which complements the wave number to an integer, entails a deviation of the parameter z from the critical z_0 and the frequency ω from ω_0 in the critical case. The relevant amendments are given in the following paragraph.

2.2. Critical case when $\sigma \neq 0$. Here we obtain an asymptotic representation of the roots of the characteristic equation (14), which correspond to the eigenfunctions $\exp(ik_{mn}x + \lambda t)$ of a linearized system with wave numbers from the set of integers K_ε defined in (19).

We also take into account $\sigma \neq 0$ and the small deviation of the coupling parameter from the critical value of γ_0 by an amount of the order of ε^2, that is, we put in the equation (14)

$$\gamma = \gamma_0 + \varepsilon^2\gamma_1, \tag{20}$$

at the same time, in (14), the value of u_s is represented as $u_s = u_0 - \varepsilon^2\gamma_1 u_0(1+\gamma_0)^{-1} + O(\varepsilon^4)$.

For those roots $\lambda = \lambda_{mn}(\varepsilon)$, $n, m = 0, \pm1, \pm2, \ldots$ whose real parts tend to zero at $\varepsilon \to 0$, the asymptotic equality is fulfilled:

$$\begin{aligned} \lambda_{mn} &= i\omega_0 + \varepsilon\lambda_{mn1} + \varepsilon^2\lambda_{mn2} + \ldots, \\ \lambda_{mn1} &= i\delta(\theta + m), \\ \lambda_{mn2} &= -\delta_2(\theta+m)^2 - \delta\sigma^2(z_0 + 2\pi n)^2/2 - \gamma_1\delta_3, \end{aligned} \tag{21}$$

where

$$\begin{aligned} \delta &= \frac{1+u_0}{2+T(1+u_0)}, \quad \delta > 0, \\ \delta_2 &= \delta\big[(1-\delta T)^2/2 + \delta^2(vu_0\gamma_0)^{-1}e^{-i\phi_0}\big], \\ \delta_3 &= \delta\big[(1+i\omega_0 v^{-1})e^{-i\phi_0} - 1\big](\gamma_0+\gamma_0^2)^{-1}. \end{aligned}$$

For the subsequent interpretation of the results, it should be noted that $\varepsilon\lambda_{mn1}$ is a purely imaginary quantity, which is a correction to the fundamental frequency ω_0 for the wavenumber k_{mn}.

It follows from the expansion of (21) that for $\gamma = \gamma_0 + \varepsilon^2\gamma_1$ there are infinitely many roots of the characteristic equation (14), the real parts of which tend to zero for $\varepsilon \to 0$. Thus, the specificity of the problem under consideration is that the critical cases have infinite dimension.

2.3. Solutions of a linearized system (13). To each characteristic root $\lambda_{mn}(\varepsilon)$ answers a particular solution of a linearized system (13)

$$\begin{pmatrix} u_{mn} \\ y_{mn} \end{pmatrix} = [a_0 + \varepsilon(m+\theta)a_1]\xi_{mn}\exp\big(ik_{mn}x + \lambda_{mn}t\big), \tag{22}$$

where ξ_{mn} is the amplitude of the mode determined by the initial conditions, and

$$a_0 = \begin{pmatrix} 1 \\ r \end{pmatrix}, \quad a_1 = \begin{pmatrix} 1 \\ r\left(1+\delta\omega_0^{-1}\right) \end{pmatrix}, \quad r = \frac{i\omega_0}{vu_0}.$$

The general solution of the linear boundary value problem (12), (13) is a set of partial solutions found,

$$\begin{pmatrix} u(t,x,\varepsilon) \\ y(t,x,\varepsilon) \end{pmatrix} = \sum_{m,n=-\infty}^{\infty} [a_0 + \varepsilon(m+\theta)a_1]\xi_{mn}\exp\big(ik_{mn}x + \lambda_{mn}t\big). \tag{23}$$

This solution can be written in the following form, given the expansions (21) for characteristic roots $\lambda_{mn} = i\omega_0 - \varepsilon i\delta(\theta+m) + O(\varepsilon^2)$ and expressions for wave numbers k_{mn},

$$\begin{pmatrix} u(t,x,\varepsilon) \\ y(t,x,\varepsilon) \end{pmatrix} = e^{iR} \sum_{m,n=-\infty}^{\infty} [(a_0 + \varepsilon\theta a_1) + \varepsilon m a_1]\, \xi_{mn} e^{\left(imX+inY+O(\varepsilon^2)t\right)}, \tag{24}$$

where notation is used: $R = \kappa x + \Omega t$ — running variable, $\kappa = z_0\varepsilon^{-1} + \theta$ — main wave number, $\Omega = \omega_0 + \varepsilon\delta\theta$ — the main frequency, and variables are introduced $X = x + \varepsilon\delta t$ и $Y = 2\pi\varepsilon^{-1}x$.

Note that the expression

$$\sum_{m,n=-\infty}^{\infty} \xi_{mn} e^{\left(imX+inY\right)}$$

it can be considered as a Fourier decomposition for the function $\xi(X, Y)$, which is 2π-periodic over both ‘quasi"spatial variables, $\xi(X, Y) = \xi(X + 2\pi, Y)$, $\xi(X, Y) = \xi(X, Y + 2\pi)$. Considering that

$$\frac{\partial\xi}{\partial X} = \sum_{m,n=-\infty}^{\infty} im\xi_{mn} e^{\left(imX+inY\right)},$$

the solution (24) of a linearized system can be written as

$$\begin{pmatrix} u \\ y \end{pmatrix} = \left[(a_0 + \varepsilon a_1\theta)\xi - \varepsilon i a_1 \frac{\partial\xi}{\partial X}\right] e^{iR} e^{O(\varepsilon^2)t} + c.c. \tag{25}$$

3. Asymptotics of solutions of a nonlinear system (11)

The solution of the nonlinear boundary value problem (11), (12) for $\gamma = \gamma_0 + \varepsilon^2\gamma_1$ we will search using the solution (25) of a linearized system and assuming that the amplitudes ξ_{mn} change slowly over time.

We introduce the function $\xi(\tau, X, Y)$, which depends on the slow time variable $\tau = \varepsilon^2 t$ and which is 2π-periodic over the spatial variables X, Y. Then the solution of $u(t, x, \tau, X, Y)$ and $y(t, x, \tau, X, Y)$ of the nonlinear boundary value problem (11), (12) is represented as a series of degrees ε

$$\begin{pmatrix} u \\ y \end{pmatrix} = \varepsilon\left[(a_0 + \varepsilon a_1\theta)\xi - \varepsilon i a_1 \frac{\partial\xi}{\partial X}\right] e^{iR} + \varepsilon^2 \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} u_3 \\ y_3 \end{pmatrix} + \ldots + c.c. \tag{26}$$

Here are the functions

$$\varepsilon^2 u_2 = \varepsilon^2(u_{20} + u_{22}e^{i2R} + c.c.),\ \varepsilon^2 y_2 = \varepsilon^2(y_{20} + y_{22}e^{i2R} + c.c.),$$
$$\varepsilon^3 u_3 = \varepsilon^3(u_{31}e^{iR} + u_{32}e^{i2R} + u_{33}e^{i3R} + c.c.),\ \ldots$$

denote the magnitudes of the second, third and t d. order in degrees of ε and the amplitudes of $u_{2j}(\tau, X, Y)$, $y_{2j}(\tau, X, Y), \ldots$ harmonics of the fundamental frequency in the same way as $\xi(\tau, X, Y)$, slowly depend on time and are 2π-periodic in spatial variables X, Y.

Substitute (20) and the rows (26) into the nonlinear system (11). Auxiliary series for derivatives and integral terms are given in the appendix. We collect coefficients at the same degrees of ε and harmonics of the fundamental frequency. In this case , slow amplitudes are consistently found:

$$u_{20} = 0, \quad y_{20} = 0, \tag{27}$$

$$u_{22} = C\frac{r}{u_0}\xi^2, \quad y_{22} = (2rC - 1)\frac{r}{u_0}\xi^2, \tag{28}$$

where

$$C = \frac{1 + 2i\omega_0}{4i\omega_0 r + 2r(1 + u_0) + 1 + \gamma_0 e^{2i\phi_0}}.$$

For the solvability of the corresponding system with respect to u_{31}, y_{31}, the existence condition must be met. The latter leads to the following complex equation for the function $\xi(\tau, X, Y)$:

$$\frac{\partial\xi}{\partial\tau} = \alpha\xi + L\xi|\xi|^2 + \upsilon_X\frac{\partial\xi}{\partial X} + \upsilon_Y\frac{\partial\xi}{\partial Y} + D_X\frac{\partial^2\xi}{\partial X^2} + D_Y\frac{\partial^2\xi}{\partial Y^2}, \tag{29}$$

with periodic boundary conditions

$$\xi(\tau, X, Y) = \xi(\tau, X + 2\pi, Y), \quad \xi(\tau, X, Y) = \xi(\tau, X, Y + 2\pi), \tag{30}$$

where

$$\alpha = \gamma_1\delta_3, \quad L = \frac{i\omega_0(i\omega_0 + 1)(rC - 1)}{u_0^2(2i\omega_0 + 1 + u_0 - T\gamma_0 v u_0 e^{i\phi_0})}$$

$$\upsilon_X = -2i\delta_2\theta, \quad \upsilon_Y = -2i\delta\pi\sigma^2, \quad D_X = \delta_2, \quad D_Y = \delta\sigma^2 2\pi^2$$

and the expressions for δ, δ_2, δ_3 are given in (21). Note that the coefficient α for the linear term of the equation (29) is determined by the value of the supercritical parameter γ_1, and $\operatorname{Re}\alpha > 0$ if $\gamma_0^+ > 0$ and $\gamma_1 > 0$ ($\gamma_0^- < 0$ and $\gamma_1 < 0$). The transfer coefficients υ_X and diffusion coefficients D_X for spatial derivatives in the equation (29) are complex and are determined only by critical parameter values. The coefficients υ_Y and D_Y are determined by the parameter σ, which describes the width of the effective coupling area between the elements of the laser ring. Using the expressions for δ and δ_2, it can be shown that the diffusion coefficients have a positive real part:

$$\operatorname{Re} D_X = \delta\frac{8\omega_0^2}{(2 + T(1 + u_0))^2(\omega_0^2 + (1 + u_0)^2/2)} > 0, \quad \operatorname{Re} D_Y = 2\pi^2\sigma^2\delta > 0,$$

therefore, the equation (29) describes a dissipative system.

The boundary value problem (29), (30) plays the role of a normal form: its nonlocal dynamics determines, for sufficiently small ε, the behavior of all solutions of the nonlinear equation (11) with initial conditions from some sufficiently small and independent of ε neighborhood of the zero equilibrium state. By the very construction of the boundary value problem (29), (30), there follows a connection between its solutions and solutions of the equation (11), which is established by the following theorem.

Theorem 1. *Let the condition* $\gamma = \gamma_0 + \varepsilon^2\gamma_1$ *be satisfied and let the boundary value problem* (29), (30) *have a solution* $\xi_0(\tau, X, Y)$ *bounded at* $\tau \to \infty$ *and* $X \in (0, 2\pi]$, $Y \in (0, 2\pi]$. *Then the functions*

$$u(t, x) = \varepsilon\xi_0(\varepsilon^2 t, x + \varepsilon\delta t, 2\pi\varepsilon^{-1}x)e^{i(z_0\varepsilon^{-1} + \theta)x + i\omega_0 t} + c.c.,$$

$$y(t, x) = \varepsilon r\xi_0(\varepsilon^2 t, x + \varepsilon\delta t, 2\pi\varepsilon^{-1}x)e^{i(z_0\varepsilon^{-1} + \theta)x + i\omega_0 t} + c.c. \tag{31}$$

satisfy the equation (11) *up to* $O(\varepsilon^2)$.

3.1. Periodic solutions of the boundary value problem (29)**,** (30)**.** The boundary value problem (29), (30) is a two-dimensional complex Ginzburg –Landau equation with convection [28].

Here we will consider the simplest solutions for typical class lasers in parameter values.

In the quasi-normal form (29), (30) there are terms with coefficients depending on the parameter σ. The latter characterizes the size of the area of effective spatial interaction between the elements of the distributed ring model (5). Discrete model (3) the chains of lasers with unidirectional coupling will most naturally correspond to the quasi-normal form (29), (30) with the parameter $\sigma \to 0$, then the coefficients $\upsilon_Y, D_Y \to 0$ and we come to the quasi-normal form for the function $\xi(\tau, X)$, depending on a single spatial variable,

$$\frac{\partial \xi}{\partial \tau} = \alpha\xi + L\xi|\xi|^2 + \upsilon_X \frac{\partial \xi}{\partial X} + D_X \frac{\partial^2 \xi}{\partial X^2},$$
$$\xi(\tau, X) = \xi(\tau, X + 2\pi). \tag{32}$$

The simplest solution of the quasinormal form (32) is a solution homogeneous in space and periodic in time

$$\xi_0(\tau, X) = \rho e^{i\omega_2\tau + i\psi}, \tag{33}$$

where

$$\rho^2 = -\frac{\operatorname{Re}\alpha}{\operatorname{Re}L}, \quad \omega_2 = \operatorname{Im}\alpha + \rho^2 \operatorname{Im}L$$

and ψ — the initial phase of the oscillations determined by the initial conditions. The coefficient for the nonlinear term in the equation (32) is a Lyapunov quantity that determines the direction of bifurcation. It is known that at $\operatorname{Re}L < 0$ a supercritical bifurcation takes place, as a result of which a stable cycle of small amplitude occurs in the supercritical region of $\operatorname{Re}\alpha > 0$. At $\operatorname{Re}L > 0$, a subcritical bifurcation takes place, as a result of which an unstable cycle is formed in the subcritical region $\operatorname{Re}\alpha < 0$. Then, in the initial system, it is possible to observe the bistability of the modes: with small deviations of the initial conditions from the stationary, the system remains stable or, with sufficiently large deviations, goes to another attractor.

The solution of the (33) equation (32) homogeneous in spatial variable corresponds to the inhomogeneous solution of the equation (13) in the form of a stable traveling wave with a wave number $\kappa = z_0\varepsilon^{-1} + \theta$:

$$\begin{aligned} u(t,x) &= 2\varepsilon\rho\cos\left(\kappa x + (\omega_0 + \varepsilon\delta\theta + O(\varepsilon^2))t\right) + O(\varepsilon^2), \\ y(t,x) &= 2\varepsilon\frac{\omega_0}{vu_0}\rho\sin\left(\kappa x + (\omega_0 + \varepsilon\delta\theta + O(\varepsilon^2))t\right) + O(\varepsilon^2), \end{aligned} \tag{34}$$

moreover, by virtue of the integer value κ, the conditions $u(t,x) = u(t, x+2\pi)$ and$y(t,x) = y(t, x+2\pi)$.

In turn, the inhomogeneous solution (34) corresponds to a stable phase synchronization mode of a chain of coupled lasers. Each element of the chain (laser) experiences fluctuations in the intensity of radiation with the same amplitude and frequency for all elements with a phase shift of $z_0 + O(\varepsilon)$ relative to the neighboring element, $u_j(t) = u(t, \varepsilon j)$.

Here is an example of calculating solutions in the form of a traveling wave with the following parameter values: pumping rate $q = 1.5$, photon attenuation rate in the resonator $v = 100$, delay in the optoelectronic communication circuit $T = 0.2$ and let 63 lasers be connected in the chain, then $\varepsilon = 2\pi/63 = 0.09973$. Note that the value of ε must be calculated with high accuracy so that periodic boundary conditions are satisfied for a large wave number. Considering that the parameter v takes rather large values in the case of Class B lasers, you can use estimation formulas for the coefficients of the quasi-normal form (32). From the equation (16) we find the critical values of the coupling coefficient between the lasers:

$$\gamma_0^{\pm} = \pm\frac{q}{\sqrt{v(q-1)}} + O(v^{-1}). \tag{35}$$

These values of the coupling coefficients correspond to the frequencies of the emerging traveling waves, comparable to the frequency of damped oscillations $\omega_R = \sqrt{v(q-1)}$ for a solitary laser,

$$\omega_0^{\pm} = \omega_R \pm \frac{q}{2} + O(v^{-1}) \tag{36}$$

in the vicinity of the (unstable) stationary state

$$u_0^{\pm} = q - 1 \pm \frac{q\sqrt{q-1}}{\sqrt{v}} + O(v^{-1}). \tag{37}$$

The complex diffusion coefficient is represented as

$$D_X^{\pm} = \frac{2(1+u_0^{\pm}) - 4i(1+u_0^{\pm})^2/\omega_0^{\pm}}{[2+T(1+u_0^{\pm})]^3} + O(v^{-1}), \tag{38}$$

it follows from this that the diffusion coefficient always has a positive real part and decreases modulo with increasing delay in the coupling chain T. For the Lyapunov magnitude determining the direction of bifurcation, we obtain

$$L^{\pm} = \frac{-5q \pm i(\sqrt{v}q \pm 6\sqrt{v(q-1)})}{18(u_0^{\pm})^2[2+T(1+u_0^{\pm})]^3} + O(v^{-1/2}), \tag{39}$$

whence it follows that $\operatorname{Re} L^{\pm} < 0$ in the case of a large v, which is true for class B lasers.

We obtain for a positive value of the critical level of the coupling coefficient $\gamma_0^+ = 0.219$, frequency $\omega_0^+ = 6.327$, parameter $\theta^+ = 0.332$, central wave number $\kappa^+ = 59$, diffusion coefficient $D_X^+ = 0.234 + 0.026i$, Lyapunov magnitude $L^+ = -1.059 - 6.274i$, supercritical parameter $\alpha^+ = 2.719 - 2.286i$ at $\gamma_1 = 1$, the amplitude of the cycle $\rho^+ = 0.764$, the frequency of the resulting oscillations, adjusted for an integer wavenumber $\omega^+ = 6.347$. On fig. 1 the instantaneous distribution of the radiation intensity is shown u_j, $j = 0, \pm 1, \ldots 31$ lasers in the chain relative to the equilibrium level u_s, where $u_j = u(\varepsilon j, 0)$ according to the formula (34). The phase difference between adjacent chain elements is $z_0^+ + O(\varepsilon) \approx 5.87$. Over time, a traveling wave circulates through the ring.

For a negative value of the critical level of the coupling coefficient $\gamma_0^- = -0.204$, the frequency of the resulting oscillations is $\omega_0^- = 7.843$, phase difference between adjacent chain elements $z_0^- =$

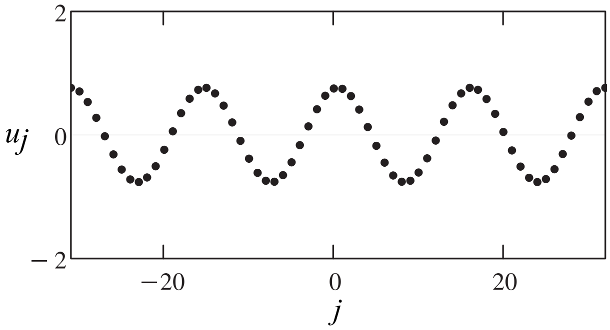

Fig. 1. Мгновенное распределение интенсивности излучения u_j, $j = 0, \pm 1, \ldots 31$ лазеров в цепочке относительно равновесного уровня u_s при $\gamma = \gamma_0^+ - \varepsilon^2 = 0.229$. Разность фаз между колебаниями соседних элементов цепочки $z_0^+ = 5.87$

Fig. 1. Instantaneous distribution of radiation intensity u_j, $j = 0, \pm 1, \ldots 31$ of lasers in the chain with respect to the equilibrium level u_s at $\gamma = \gamma_0^+ - \varepsilon^2 = 0.229$. Phase difference between oscillations of adjacent elements of the chain $z_0^+ = 5.87$

3.04, parameter $\theta^{-} = 0.64$, central wave number $\kappa^{+} = 31$, diffusion coefficient $D_X^{-} = 0.256 + 0.027i$, Lyapunov magnitude $L^{-} = -0.484 - 2.436i$, supercritical parameter $\alpha^{-} = -3.526 - 4249i$ at $\gamma_1 = 1$, cycle amplitude $\rho^{-} = 1.202$, oscillation frequency adjusted for the whole wave the number $\omega^{-} = 7.89$. In Fig. 2 the instantaneous distribution of the radiation intensity u_j, j=$0, \pm 1, ...31$ of lasers in the chain relative to the equilibrium level u_s is shown. Since the phase difference of the intensity fluctuations of neighboring lasers is close to π, it is possible to observe striped structures that live for some time.

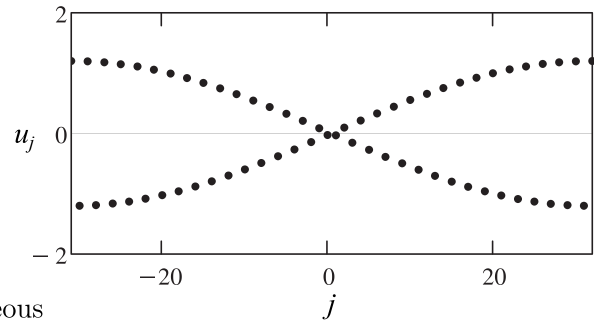

Fig. 2. Мгновенное распределение интенсивности излучения u_j, $j = 0, \pm 1, ...31$ лазеров в цепочке относительно равновесного уровня u_s при $\gamma = \gamma_0^{-} + \varepsilon^2 = -0.214$. Разность фаз между колебаниями соседних элементов цепочки $z_0^{-} = 3.04$

Fig. 2. Instantaneous distribution of radiation intensity u_j, $j = 0, \pm 1, ...31$ of lasers in the chain with respect to the equilibrium level u_s at $\gamma = \gamma_0^{-} - \varepsilon^2 = -0.214$. Phase difference between oscillations of adjacent elements of the chain $z_0^{-} = 3.04$

Note that the complex equation Ginzburg −Landau can have other nonstationary inhomogeneous solutions, including diffusion chaos, which were considered earlier in many papers, for example in [28]. Then the solutions of the distributed chain model can be significantly more complicated. Additional studies are required, taking into account the convection present in the normalized equation, the stability of possible regimes, which seems to be the subject of a separate study.

Conclusion

In this paper, a distributed integro-differential model with a lagging argument is proposed to describe the dynamics of the laser chain if the number of elements is sufficiently large.

As a result of linear analysis of the distributed model, taking into account a small parameter inversely proportional to the number of lasers in the chain, critical values of the coupling coefficient are obtained at which the stationary state in the chain becomes unstable. It is shown that in a certain neighborhood of the bifurcation point, the number of roots of a characteristic equation with a real part close to zero increases indefinitely with a decrease in a small parameter. An algorithm is proposed for reducing the initial boundary value problem to an equation for slowly varying amplitudes in relation to critical cases of (asymptotically) infinite dimension. A two-dimensional complex Ginzburg equation−Landau with convection and periodic boundary conditions as a normal form is obtained. Its nonlocal dynamics determines the behavior of solutions to the original boundary value problem.

The simplest homogeneous periodic solutions of the Ginzburg equation−Landau and their corresponding inhomogeneous solutions in the form of traveling waves in a distributed model are obtained. Such solutions can be interpreted as phase synchronization modes in a chain of coupled lasers. The frequencies and amplitudes of fluctuations in the radiation intensity of each laser and the phase difference between neighboring oscillators are determined.

Auxiliary series for constructing a normal form

We give auxiliary expressions for the terms in the equation (11). Since $\tau = \varepsilon^2 t$ and $X = x - \varepsilon\delta t$, then $\xi'_t = \varepsilon^2\xi'_\tau - \varepsilon\delta\xi'_X$. Further, taking into account the expressions for a_0 and a_1, we obtain series expansions by degrees of the small parameter ε for time derivatives t:

$$\frac{\partial u}{\partial t} = \varepsilon e^{iR} i\omega_0\xi + \varepsilon^2\Big[e^{iR}(\omega_0+\delta)(\xi'_X + i\theta\xi) + e^{i2R}(i2\omega_0)u_{22}\Big] + \\ +\varepsilon^3 e^{iR}\Big[\big(\xi'_\tau + i\delta\xi''_{XX} + 2\delta\theta\xi'_X + i\delta\theta^2\xi\big) + i\omega_0 u_{31}\Big] + \\ +\varepsilon^3 i\delta(u_{20})'_X + \varepsilon^3 e^{i2R} i\delta\big[(2\theta u_{22} + (u_{22})'_X)\big] + \varepsilon^4 ... + c.c. \tag{40}$$

and

$$\frac{\partial y}{\partial t} = \varepsilon e^{iR} r i\omega_0\xi + \varepsilon^2\Big[e^{iR} r(i\theta(\omega_0 + 2\delta)\xi\omega_0\xi'_X) + e^{i2R}(2i\omega_0)y_{22}\Big] + \\ +\varepsilon^3 e^{iR}\Big[-i\delta(1+\delta\omega^{-1})\big(i\omega_0\xi'_\tau + \xi''_{XX} - \theta^2\xi\big) + i\omega_0 y_{31}\Big] + \\ +\varepsilon^3 i\delta(y_{20})'_X + \varepsilon^3 e^{i2R} i\delta\big[(2\theta y_{22} + (y_{22})'_X)\big] + \varepsilon^4 ... + c.c. \tag{41}$$

Nonlinear terms in the system are represented as a series

$$\begin{aligned} uv &= \varepsilon^2 e^{2iR} r\xi^2 + \varepsilon^3\delta(|\xi|^2)'_X + \\ &+ \varepsilon^3 e^{iR}\big[\xi(y_{20} + ru_{20}) + \bar{\xi}(y_{22} - ru_{22})\big] + \\ &+ \varepsilon^3 e^{2iR}(\delta + 2\omega_0)(\xi\xi'_X + i\theta\xi^2) + \varepsilon^3 e^{3iR} ... + c.c. \end{aligned} \tag{42}$$

To decompose the integral term in the (11) system into a series, we use the equality (9) and decomposition of the function

$$u(t-T, x+s) = \sum_{m,n} \varepsilon(1+\varepsilon(m+\theta))\xi_{mn} e^{ik_{mn}(x+s) + i(\omega_0 + \varepsilon\delta(m+\theta))(t-T)} + \\ +\varepsilon^2(u_{20} + e^{i2R}u_{22}) + \varepsilon^3(e^{iR}u_{31} + ...) + c.c.$$

Note that the values of the amplitudes of harmonics depending on the slow time $\tau = \varepsilon^2 t$ are taken at the moment $\varepsilon^2(t - T)$, then

$$\xi_{mn}(\tau - \varepsilon^2 T) = \xi_{mn} - \varepsilon^2 T(\xi_{mn})'_\tau + ...$$

We get the decomposition

$$\int_{-\infty}^{\infty} F(s)u(t-T, x+s)ds = \varepsilon e^{iR} r\xi + \\ +\varepsilon^2 e^{iR} e^{i\phi_0}\Big(r_1\xi + r_2\xi'_X\Big) + \varepsilon^2 u_{20} + \varepsilon^2 e^{2iR} e^{2i\phi_0} u_{22}(1 + \varepsilon^2 ...) + \\ +\varepsilon^3 e^{iR} e^{i\phi_0}\Big(-T\xi'_\tau + r_3\xi'_X + r_4\xi'_Y + r_5\xi + r_6\xi''_{XX} + r_7\xi''_{YY}\Big) + \\ +\varepsilon^3 e^{iR} e^{i\phi_0} u_{31} + ... + c.c., \tag{43}$$

where $\phi_0 = z_0 - \omega_0 T$ и

$$r_1 = i\theta(1 + T\delta - i), \quad r_2 = (1 + T\delta - i),$$
$$r_3 = i\theta(1 + T\delta)(1 + T\delta - 2i), \quad r_4 = 2i\pi\sigma^2 z_0,$$
$$r_5 = -[z_0^2\sigma^2 + \theta^2(1 + T\delta)(1 + T\delta - 2i)]/2,$$
$$r_6 = (1 + T\delta)(1 + T\delta - 2i)/2, \quad r_7 = 2\pi^2\sigma^2.$$

Let's put (20) and the series (40)–(43) into a nonlinear system (11) and collect coefficients at the same degrees of ε and harmonics of the fundamental frequency.

The coefficients for εe^{iR} in the first and second equations of the system (11) give correct equalities for critical parameter values,

$$\xi\Big(\omega^2 - (1 + u_0)i\omega_0 - vu_0\xi - \gamma_0 vu_0 \exp(i\phi_0)\Big) = 0.$$

In the second step, by collecting the coefficients for $\varepsilon^2 e^{iR}$ in the first equation, we also get the correct equalities for critical parameters, which we will not give. Collecting coefficients at $\varepsilon^2 e^0$, we come to the system,

$$0 = vu_0 y_{20}, \quad 0 = -(1 + u_0)y_{20} - u_{20} - \gamma_0 u_{20},$$

from where we have $y_{20} = 0,\ u_{20} = 0$.

Collecting coefficients at $\varepsilon^2 e^{2iR}$, we come to a system from which we find $u_{22}{=}C(r/u_0)\xi^2$, $y_{22} = (2rC - 1)(r/u_0)\xi^2$, where

$$C = \frac{1 + 2i\omega_0}{4i\omega_0 r + 2r(1 + u_0) + 1 + \gamma_0 e^{2i\phi_0}}.$$

At the third step, collecting coefficients at $\varepsilon^3 e^0$, we come to a system from which the amplitudes u_{31}, y_{31} can be excluded by multiplying the first equation by $(i\omega_0 + 1 + u_0)$ and the second by vu_0, then adding up these equations. For the solvability of the system, it is necessary that the function ξ satisfies the equation (29) for the function $\xi(\tau, X, Y)$.

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