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Nonlinear waves of the sine-Gordon equation in the model with three attracting impurities

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Abstract. *Purpose* of this work is to use analytical and numerical methods to consider the problem of the structure and dynamics of coupled localized nonlinear waves in the sine-Gordon model with "impurities" (or spatial inhomogeneity of the periodic potential). *Methods.* Using the analytical method of collective coordinates for the case of the arbitrary number the same point impurities on the same distance each other, differential equation system was got for localized waves amplitudes as the functions on time. We used the finite difference method with explicit scheme for the numerical solution of the modified sine-Gordon equation. We used a discrete Fourier transform to perform a frequency analysis of the oscillations of localized waves calculate numerically. *Results.* We found of the differential equation system for three harmonic oscillators with the elastic connection for describe related oscillations of nonlinear waves localized on the three same impurity. The solutions obtained from this system of equations for the frequencies of related oscillation well approximate the results of direct numerical modeling of a nonlinear system. *Conclusion.* In the article shows that the related oscillation of nonlinear waves localized on three identical impurities located at the same distance from each other represent the sum of three harmonic oscillations: in-phase, in-phase-antiphase and antiphase type. The analysis of the influence of system parameters and initial conditions on the frequency and type of associated oscillations is carried out.

Keywords: sine-Gordon equation, kink, soliton, breather, the method of collective coordinates, impurity.

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Introduction

The study of nonlinear wave processes allowed us to make a number of fundamental discoveries [1–3]. Solutions of nonlinear differential equations have been found that describe solitary waves that retain their shape and velocity over time, including when interacting with each other — solitons. One of the intensively studied nonlinear differential equations is the sine-Gordon equation [2–5]. This equation describes wave processes in a wide variety of fields of natural science: geology, molecular biology, physics, cosmology. For example, in condensed matter physics, it is applicable in describing the dynamics of magnetization waves in ferromagnetic crystals, the movement of dislocations in crystals, processes in Josephson superconducting contacts, the propagation of charge density waves in one-dimensional organic conductors, the propagation of electromagnetic waves in a graphene-based superlattice, the dynamics of an ensemble of interacting dislocations in a linear defect of the electroconvective structure of a liquid crystal [4–8]. The sine-Gordon equation is a nonlinear partial differential equation and is fully integrable.

Various exact solutions of the sine-Gordon equation of the kink, soliton, breather type and some other solutions of the more complex multisoliton type [2–4, 9, 10] are found. Finding new solutions to the sine-Gordon equation and studying their properties and interactions is an urgent task of the theory of nonlinear waves. Usually, a modification of the sine-Gordon equation is required for use in real physical models. For example, by adding additional terms. These terms can describe the external force, dissipation, heterogeneity of the parameters of the medium, etc. The resulting modified sine-Gordon equation no longer has exact analytical solutions. However, a number of analytical methods have been developed and widely used (for example, perturbation theory for solitons or the method of collective coordinates [2, 5, 11]). Using these methods, a wide range of different tasks has been investigated. For example, the problem of dynamics of kinks, solitons and breathers under the action of an external force of various types (depending on time and spatial variables) [12, 13] is investigated.

Many works are devoted to the study of the influence of spatial modulation of the periodic potential (or impurity) on the dynamics of solitons of the sine-Gordon equation [5, 14–30]. The sine-Gordon model with impurities is applicable to describe the case of a multilayer ferromagnet [31–34]. Spatial modulation of the periodic potential is often modeled as a delta function or in other special forms. The excitation of an impurity-localized wave (impurity mode) as a result of kink scattering leads to a significant change in its dynamics [5, 19, 20, 24–28]. The structure and properties of localized nonlinear waves excited on one and two impurities were analyzed in [19, 21, 25, 28, 29]. It was shown that the attracting admixture can be used to excite the multisolitons of the sine-Gordon equation. for example, when localized impurity waves (four-kink multisolitons) are excited on two impurities. It is analytically shown that their oscillations can be described by a system of two harmonic oscillators with elastic coupling. This model qualitatively describes the results of numerical modeling (for both point and extended impurities). The case of two impurities gives a wide variety of new multisoliton solutions and dynamic effects compared to the case of a single impurity. We can expect an even greater variety of solutions and effects in the presence of three or more impurities in the system. In this paper, we study the coupled dynamics of nonlinear localized waves on three identical point impurities.

1. Basic equations and results of analytical calculations

Consider the dimensionless sine-Gordon equation in a model with N identical point impurities located at a distance of d from each other, of the following form:

$$u_{tt} - u_{xx} + \sin u - \sum_{k=1}^{N} \varepsilon_k \delta\left(x - x_k\right) \sin u = 0, \qquad (1)$$

where $\varepsilon_k = \varepsilon$, $x_k = kd$ for all $k = 1..N$. In the equation (1) $u = u(x,t)$, the spatial modulation of the periodic potential of the sine-Gordon equation is taken into account by adding point impurities of the form $\varepsilon\delta(x)$, where ε — constant, $\delta(x)$ — Dirac delta function. This equation, for example, can describe the dynamics of magnetization waves in a multilayer uniaxial ferromagnet [5, 32, 33, 35] with a magnetic anisotropy constant inhomogeneous in coordinate. Then the function $u = u(x,t)$ determines the double angle between the magnetization vector at a given point at a given time and the direction of the magnetization vector in the domain, the coordinate x will be normalized to δ_0, where δ_0 is the width of the static Bloch domain boundary, and the time t is normalized to δ_0/c, where c is the maximum Walker speed of stationary motion [33, 35]. The presence of such magnetic anisotropy inhomogeneities has a significant impact on the dynamics of domain boundaries and can lead to the formation of various kinds of localized magnetic inhomogeneities [31–35].

The Lagrange function corresponding to the equation (1) has the form

$$L = \int_{-\infty}^{\infty} \left[\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \cos u - 1 + \sum_{k=1}^{N} \delta(x - x_k)(\varepsilon_k - \varepsilon_k \cos u) \right] dx. \tag{2}$$

First, let's consider the analytical solution of the equation (1) describing the amplitude fluctuations of waves localized on impurities using the method of collective variables [2, 5]. The method is variational, based on the allocation of collective coordinates and the construction of the averaged Lagrangian [2, 5]. When using this method, a transition is made from a continuous field $u = u(x, t)$ to a finite set of functions $f(t)$ that depend only on time. In this case, u is approximated by the paragraph — by the sum of solutions from localized waves containing time functions characterizing the state of the waves, called collective variables. We will take the ansatz as the sum of N impurity modes:

$$u_a = \sum_{n=1}^{N} u_n = \sum_{n=1}^{N} a_n(t) e^{\frac{-\varepsilon_n |x - x_n|}{2}}, \tag{3}$$

where $a_n(t)$ — the amplitudes of the impurity modes at time t on the nth impurity. Within the framework of the considered approximation, the amplitudes of the impurity modes and the parameter ε will be considered sufficiently small, so that $u_a \ll 1$. The ansatz (3) is then substituted into the Lagrangian (2). Within the framework of our approximation, the nonlinear Lagrangian term (2) in the form of $\cos u$ can be decomposed into a Taylor series up to second-order terms. After taking integrals, similarly to the previously investigated cases with $N = 1$ and $N = 2$ [2, 5, 28, 34], we obtain a new effective Lagrange function, which is a function of the collective variables $a_n(t)$. By substituting the effective Lagrange function into the Lagrange equations−Euler, after taking the derivatives, one can obtain N differential equations for N collective variables $a_n(t)$ of the following type:

$$\sum_{n=1}^{N} \left\{ [\ddot{a}_n + a_n] E_{ln+} + a_n \frac{\varepsilon}{2} \left[\frac{\varepsilon}{2} E_{ln-} - \sum_{k=1}^{N} e_{nk} e_{lk} \right] \right\} = 0, \tag{4}$$

where

$$e_{nl} = e^{\frac{-\varepsilon |l-n| d}{2}} = e_d^{|l-n|}, \quad E_{ln\pm} = E_{nl\pm} = \left(\frac{1}{\varepsilon} \pm \frac{|l-n| d}{2} \right) e_d^{|l-n|} = E_{|l-n|\pm}, \quad l = 1..N. \tag{5}$$

Next, we consider possible solutions of the equation (4) for the case of $N = 3$. From the equation (4), taking $l = 1$, $l = 2$ and $l = 3$, we can obtain three differential equations for three collective variables $a_{1,2,3}(t)$ of the following type

$$\begin{cases} \dfrac{(\ddot{a}_1 + a_1)}{\varepsilon} + (\ddot{a}_2 + a_2)E_{1+} + (\ddot{a}_3 + a_3)E_{2+} + \varepsilon/2[-a_1(1/2 + e_d^2 + e_d^4) + \\ \qquad + a_2(\varepsilon E_{1-}/2 - 2e_d - e_d^3) + a_3(\varepsilon E_{2-}/2 - 3e_d^2)] = 0, \\ (\ddot{a}_1 + a_1)E_{1+} + (\ddot{a}_2 + a_2)/\varepsilon + (\ddot{a}_3 + a_3)E_{1+} + \varepsilon/2[a_1(\varepsilon E_{1-}/2 - 2e_d - e_d^3) - \\ \qquad - a_2(1/2 + 2e_d^2) + a_3(\varepsilon E_{1-}/2 - e_d^3 - 2e_d)] = 0, \\ (\ddot{a}_1 + a_1)E_{2+} + (\ddot{a}_2 + a_2)E_{1+} + (\ddot{a}_3 + a_3)/\varepsilon + \varepsilon/2[a_1(\varepsilon E_{2-}/2 - 3e_d^2) + \\ \qquad + a_2(\varepsilon E_{1-}/2 - e_d^3 - 2e_d) - a_3(1/2 + e_d^4 + e_d^2)] = 0, \end{cases} \tag{6}$$

where

$$e_d = e^{\frac{-\varepsilon d}{2}}, \quad E_{k\pm} = (1/\varepsilon \pm kd/2)e_d^k. \tag{7}$$

We will reduce them to a more convenient form for solving, leaving only one acceleration in each of the equations $\ddot{a}(t)$. To do this, we subtract from the first equation multiplied by $[1-\varepsilon^2 E_{1+}^2]$ (6) multiplied by $\varepsilon E_{1+}[1-\varepsilon E_{2+}]$ the second equation (6). Then, in the resulting equation, we take out and insert the expression $[\ddot{a_3} + a_3]$ obtained from the third equation (6). Next, we will take out and get rid of the expression $[\ddot{a_2} + a_2]$, and also regroup the terms. Repeating similar algebraic transformations for the other two equations (6), we obtain:

$$\begin{cases} \ddot{a}_1 + a_1\omega_1^2 + a_2k_{12} + a_3k_{13} = 0, \\ \ddot{a}_2 + a_2\omega_2^2 + (a_1+a_3)k_{21} = 0, \\ \ddot{a}_3 + a_3\omega_1^2 + a_1k_{13} + a_2k_{12} = 0, \end{cases} \tag{8}$$

where

$$\begin{aligned}
\omega_1^2 &= 1 - \frac{\varepsilon^2}{4} + \frac{\varepsilon^3 d e_d^2/4}{1-(1+\varepsilon d+\varepsilon^2 d^2/2)e_d^2}\left[1 + e_d^2 + \frac{\varepsilon d(e_d^2-1)e_d^2/2}{1-(1+\varepsilon d)e_d^2}\right], \\
\omega_2^2 &= 1 - \frac{\varepsilon^2}{4} + \frac{\varepsilon^3 d(1-e_d^2)e_d^2/2}{1-(1+\varepsilon d+\varepsilon^2 d^2/2)e_d^2}, \\
k_{12} &= \frac{\varepsilon^2[(1+\varepsilon d)e_d^2 - 1]e_d/2}{1-(1+\varepsilon d+\varepsilon^2 d^2/2)e_d^2}, \\
k_{13} &= \frac{\varepsilon^2 e_d^2/2}{1-(1+\varepsilon d+\varepsilon^2 d^2/2)e_d^2}\left[\varepsilon d(1+e_d^2)/2 + e_d^2 - 1 + \frac{\varepsilon^2 d^2(1-e_d^2)e_d^2/4}{1-(1+\varepsilon d)e_d^2}\right], \\
k_{21} &= \frac{\varepsilon^2(\varepsilon d e_d^2/2 - 1)(1-e_d^2)e_d/2}{1-(1+\varepsilon d+\varepsilon^2 d^2/2)e_d^2}, \\
e_d &= e^{\frac{-\varepsilon d}{2}}.
\end{aligned} \tag{9}$$

The case of non-interacting impurity modes corresponding to the previously studied case of a single impurity [5] is obtained by finding the limits of expressions (9) for $d \to \infty$:

$$\lim_{d\to\infty} e_d = 0, \quad \lim_{d\to\infty} \omega_1^2 = \lim_{d\to\infty} \omega_2^2 = 1 - \frac{\varepsilon^2}{4}, \quad \lim_{d\to\infty} k_{12} = \lim_{d\to\infty} k_{13} = \lim_{d\to\infty} k_{21} = 0. \tag{10}$$

Note that in the previously investigated case of two impurities [21, 28], it was possible to obtain approximate linear dynamic equations for the amplitudes of impurity modes, which are the equations of an oscillatory system with two degrees of freedom (two coupled oscillators). Similar linear dynamic equations can be obtained for the case of an arbitrary number of impurities. Approximate linear dynamic equations for the amplitudes of impurity modes in our case are the equations of an oscillatory system with three degrees of freedom (or three coupled oscillators). Their solution is the sum of three harmonic oscillations of the form [36]:

$$\begin{aligned}
a_1(t) &= a_{01}\cos(\Omega_1 t + \theta_1) + \eta_{12}a_{02}\cos(\Omega_2 t + \theta_2) - a_{03}\cos(\Omega_3 t + \theta_3), \\
a_2(t) &= \eta_{21}a_{01}\cos(\Omega_1 t + \theta_1) + a_{02}\cos(\Omega_2 t + \theta_2), \\
a_3(t) &= a_{01}\cos(\Omega_1 t + \theta_1) + \eta_{12}a_{02}\cos(\Omega_2 t + \theta_2) + a_{03}\cos(\Omega_3 t + \theta_3),
\end{aligned} \tag{11}$$

where a_{01}, a_{02}, a_{03} are constants determined from the initial conditions, and

$$\begin{aligned}
\Omega^2_{1,2} &= \frac{\omega_1^2+\omega_2^2+k_{13} \mp \sqrt{(\omega_1^2-\omega_2^2+k_{13})^2+8k_{12}k_{21}}}{2},\\
\eta_{21} &= \frac{\omega_2^2-\omega_1^2-k_{13} - \sqrt{(\omega_1^2-\omega_2^2+k_{13})^2+8k_{12}k_{21}}}{2k_{12}},\\
\eta_{12} &= \frac{\omega_1^2-\omega_2^2+k_{13} + \sqrt{(\omega_1^2-\omega_2^2+k_{13})^2+8k_{12}k_{21}}}{4k_{12}},\\
\Omega_3^2 &= \omega_1^2 - k_{13}.
\end{aligned} \tag{12}$$

For $d \to \infty$, using (9), we have:

$$\Omega^2_{1,2,3} = 1 - \frac{\varepsilon^2}{4}, \quad \eta_{21} = \sqrt{2}, \quad \eta_{12} = -\frac{1}{\sqrt{2}}. \tag{13}$$

The first of the formulas (13) describes the frequency of the impurity mode for a single impurity [5]. In the case under consideration, by introducing the substitution $y_1 = a_1 + a_3, y_2 = a_1 - a_3$, the equations (8) are reduced to a system of two coupled oscillators with respect to a_2 and y_1 (previously considered [28,34]) and an unrelated oscillator described by the function y_2. Therefore, the solutions of the resulting system are a combination of the previously considered solutions of equations for a system with a single local inhomogeneity and a system with two local inhomogeneities. You can also switch to the normal or main coordinates of [1,36], each of which oscillates with the same frequency:

$$\begin{cases}
\dfrac{a_3(t)+a_1(t)-2a_2(t)\eta_{12}}{2(1-\eta_{12}\eta_{21})} = a_{01}\cos(\Omega_1 t+\theta_1) = \phi_1,\\
\dfrac{2a_2(t)-(a_3(t)+a_1(t))\eta_{21}}{2(1-\eta_{12}\eta_{21})} = a_{02}\cos(\Omega_2 t+\theta_2) = \phi_2,\\
\dfrac{a_3(t)-a_1(t)}{2} = a_{03}\cos(\Omega_3 t+\theta_3) = \phi_3.
\end{cases} \tag{14}$$

In Fig. 1 the dependencies of $\Omega_{1,2,3}$ on d are presented, constructed according to the formulas (12) for the values of ε equal to 0.5 and 0.3333. At large distances between impurities, all frequencies tend to the limit value (13). As the distance between the impurities decreases, the value of Ω_1 decreases. The larger ε is, the faster it happens and the smaller Ω_1 becomes.

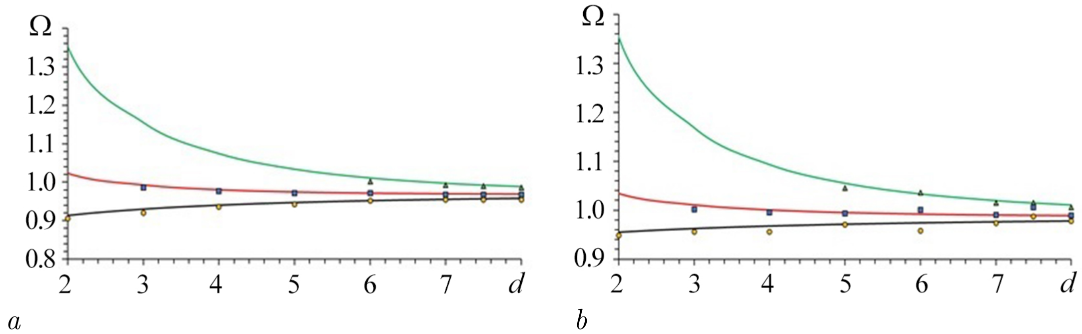


Fig. 1. Зависимости частот $\Omega_{1,2,3}$ от d (нижняя, верхняя, средняя линии). Аналитическое решение (12) — сплошные линии, численное решение (1) — точки. $\varepsilon = 0.5$ (a); $\varepsilon = 0.3333$ (b)

Fig. 1. Dependences of the $\Omega_{1,2,3}$ frequency on the d value (lower, upper, middle lines). The analytical solution (12) — solid lines, the numerical solution (1) — points. $\varepsilon = 0.5$ (a); $\varepsilon = 0.3333$ (b)

The frequencies of $\Omega_{2,3}$ increase with decreasing d, tending to infinity. Moreover, Ω_2 increases faster, and the difference in its behavior at different ε is not very noticeable. Ω_3 increases more slowly and its graph is located between the graphs Ω_1 and Ω_2, and the dependence on ε is more pronounced than at Ω_2.

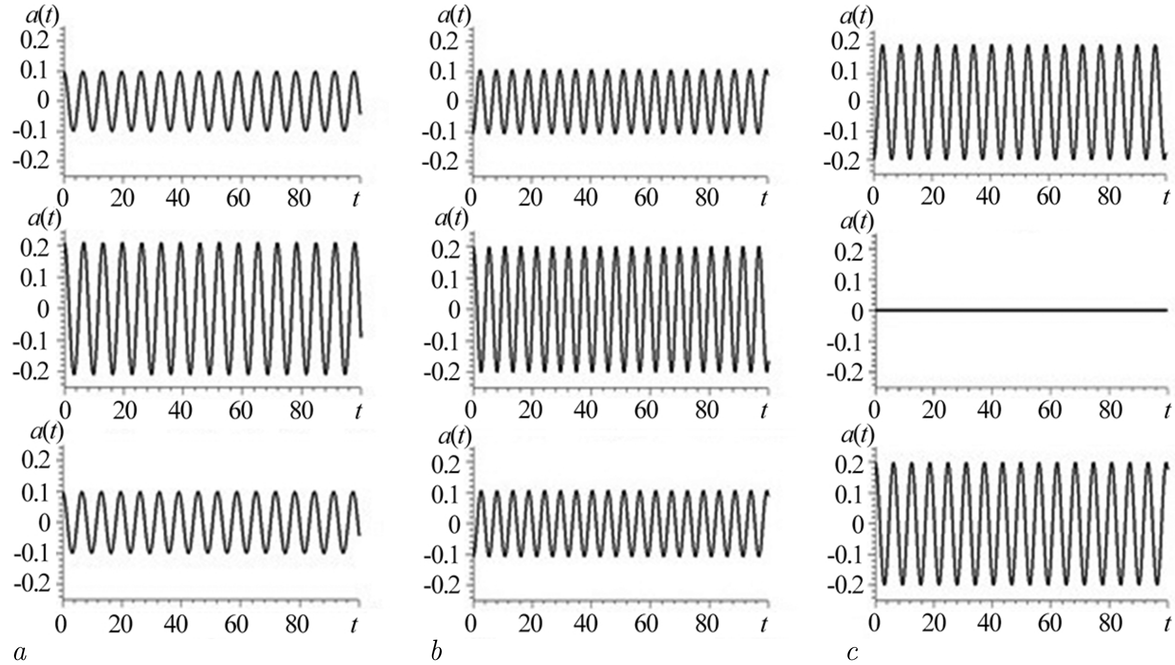


Fig. 2. Зависимости амплитуд $a_{1,2,3}(t)$ (верхний, средний, нижний графики) от времени при $\varepsilon = 0.3333$, $d = 3$. $a_1(0) = 0.1$, $a_2(0) = 0.2098$, $a_3(0) = 0.1$ (a); $a_1(0) = -0.1075$, $a_2(0) = 0.2$, $a_3(0) = -0.1075$ (b); $a_1(0) = -0.2$, $a_2(0) = 0$, $a_3(0) = 0.2$ (c)

Fig. 2. Dependences of the $a_{1,2,3}$ amplitudes (upper, middle, lower figures) on time at $\varepsilon = 0.3333$, $d = 3$. $a_1(0) = 0.1$, $a_2(0) = 0.2098$, $a_3(0) = 0.1$ (a); $a_1(0) = -0.1075$, $a_2(0) = 0.2$, $a_3(0) = -0.1075$ (b); $a_1(0) = -0.2$, $a_2(0) = 0$, $a_3(0) = 0.2$ (c)

The initial phases of the oscillations will then be considered zero for simplicity $\theta_1 = \theta_2 = \theta_3 = 0$. By setting different ratios of the initial amplitudes of localized waves, it is possible to obtain different types of oscillations with a given set of parameters ε and d. Consider for example the case of $\varepsilon = 0.3333$, $d = 3$. The dependences of the amplitudes $a_{1,2,3}(t)$ on time when only one harmonic oscillation is excited are shown in Fig. 2. In this case, all impurity modes oscillate with the same frequency. The first type of oscillation is in-phase — all impurity modes oscillate in the same phase (Fig. 2, a). It is characteristic that the frequency of Ω_1 of this type of oscillation decreases with a decrease in the parameter d (Fig. 1). The second type of oscillations — in-phase-antiphase — the first and third impurity modes oscillate in the same phase, and the second between them — in the opposite to them (Fig. 2, b). Its frequency Ω_2 increases as the distance d decreases. The third type of oscillations is antiphase — the first and third impurity modes oscillate in opposite phases, and the second one is not excited between them (Fig. 2, c). Its frequency Ω_3 does not increase as fast as Ω_2 when d decreases. In this case, the oscillation amplitudes of the first and third impurity modes are the same in all cases.

Varying the initial conditions, we consider possible more complex cases of coupled oscillations of impurity modes. If $a_{02} = 0$, then the expressions (11) can be written as

$$
\begin{aligned}
a_1(t) &= 2a_{01}\cos\left(\frac{(\Omega_1+\Omega_3)t+\theta_1+\theta_3}{2}\right)\cos\left(\frac{(\Omega_1-\Omega_3)t+\theta_1-\theta_3}{2}\right)-(a_{03}+a_{01})\cos(\Omega_3 t+\theta_3),\\
a_2(t) &= \eta_{21}a_{01}\cos(\Omega_1 t+\theta_1), \qquad (15)\\
a_3(t) &= 2a_{01}\cos\left(\frac{(\Omega_1+\Omega_3)t+\theta_1+\theta_3}{2}\right)\cos\left(\frac{(\Omega_1-\Omega_3)t+\theta_1-\theta_3}{2}\right)+(a_{03}-a_{01})\cos(\Omega_3 t+\theta_3),
\end{aligned}
$$

that is, the oscillations on the first and third impurities have the form of beats with a beat frequency equal to $|\Omega_1 - \Omega_3|$, and amplitudes varying from $|a_{01} - a_{03}|$ to $|a_{01} + a_{03}|$, by the second impurity vibrations are harmonic. Assuming the initial phases are still zero $\theta_1 = \theta_2 = \theta_3 = 0$, let's take the initial conditions under which the amplitude of the beats decreases to zero. For example, if $a_{02} = 0$, take $a_{03} = a_{01}$, then the initial conditions will be $a_1(0) = a_{01} - a_{03} = 0$, $a_2(0) = \eta_{21}a_{01} \approx 2.098a_{01}$, $a_3(0) = a_{01}+a_{03} = 2a_{01}$. The amplitudes will vary from $|a_{01}-a_{03}| = 0$ to $|a_{01} + a_{03}| = 2|a_{01}|$ on the first and third impurities, and $|\eta_{21}a_{01}| \approx 2.098|a_{01}|$ on the second impurity (fig. 3, *a*).

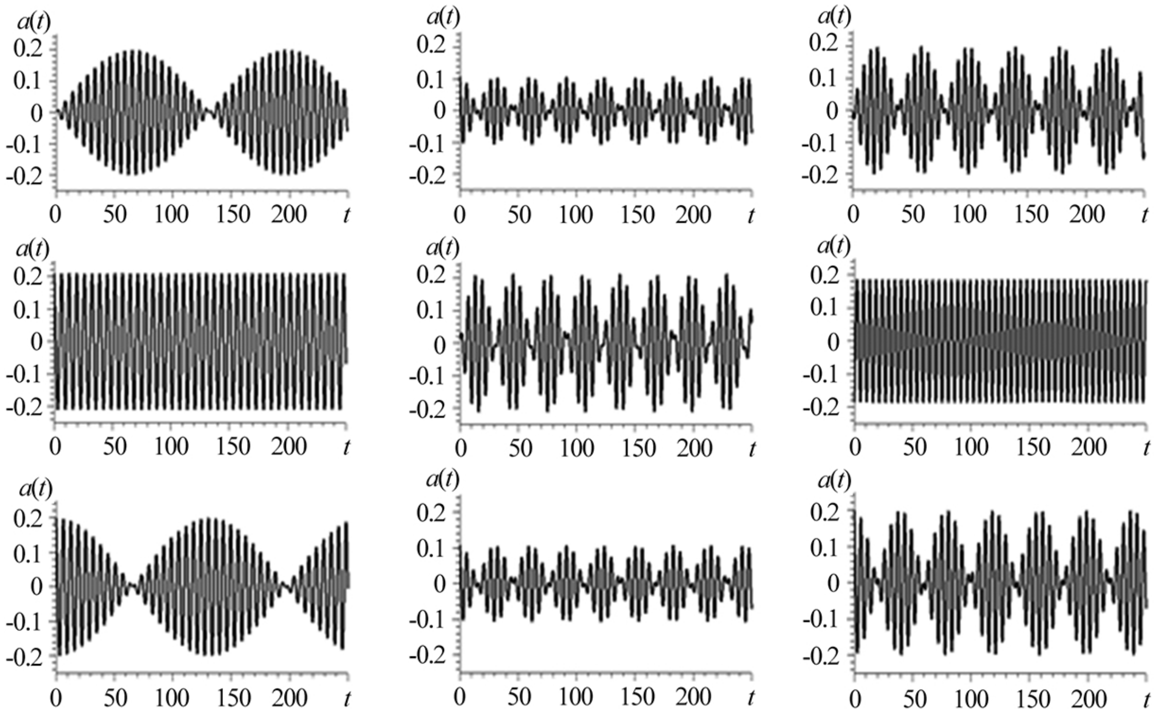


Fig. 3. Зависимости амплитуд $a_{1,2,3}(t)$ (верхний, средний, нижний графики) от времени при $\varepsilon = 0.3333$, $d = 3$. Значения амплитуд при $t = 0$: $a_1(0) = 0$, $a_2(0) = 0.2098$, $a_3(0) = 0.2$ (*a*); $a_1(0) = 0.1075$, $a_2(0) = 0.01277$, $a_3(0) = 0.1075$ (*b*); $a_1(0) = 0$, $a_2(0) = -0.186$, $a_3(0) = 0.2$ (*c*)

Fig. 3. Dependences of the $a_{1,2,3}$ amplitudes (upper, middle, lower figures) on time at $\varepsilon = 0.3333$, $d = 3$. Amplitude values at $t = 0$: $a_1(0) = 0$, $a_2(0) = 0.2098$, $a_3(0) = 0.2$ (*a*); $a_1(0) = 0.1075$, $a_2(0) = 0.01277$, $a_3(0) = 0.1075$ (*b*); $a_1(0) = 0$, $a_2(0) = -0.186$, $a_3(0) = 0.2$ (*c*)

If $a_{03} = 0$, then the expressions (11) can be written as:

$$
\begin{aligned}
a_1(t) = a_3(t) = 2a_{01}\cos\left(\frac{(\Omega_1+\Omega_2)t+\theta_1+\theta_2}{2}\right)\cos\left(\frac{(\Omega_1-\Omega_2)t+\theta_1-\theta_2}{2}\right)+\\
+(\eta_{12}a_{02}-a_{01})\cos(\Omega_2 t+\theta_2),\\
a_2(t) = 2a_{02}\cos\left(\frac{(\Omega_1+\Omega_2)t+\theta_1+\theta_2}{2}\right)\cos\left(\frac{(\Omega_1-\Omega_2)t+\theta_1-\theta_2}{2}\right)+\\
+(\eta_{21}a_{01}-a_{02})\cos(\Omega_1 t+\theta_1),
\end{aligned} \tag{16}
$$

that is, the oscillations have the form of beats with a beat frequency equal to $|\Omega_1 - \Omega_2|$, and amplitudes varying from $|a_{01} - \eta_{12}a_{02}|$ to $|a_{01} + \eta_{12}a_{02}|$ on the first and third impurities and from $|a_{02} - \eta_{21}a_{01}|$ to $|a_{02} + \eta_{21}a_{01}|$ on the second impurity. For example, if $a_{03} = 0$, take $a_{01} = \eta_{12}a_{02}$, then the initial conditions will be $a_1(0) = a_3(0) = a_{01} + \eta_{12}a_{02} = 2\eta_{12}a_{02} \approx -1.075a_{02}$, $a_2(0) = a_{02} + \eta_{21}a_{01} = (1+\eta_{21}\eta_{12})a_{02} \approx -0.1277a_{02}$. The amplitudes will vary from $|a_{01} - \eta_{12}a_{02}| = 0$ to $|a_{01} + \eta_{12}a_{02}| = 2|\eta_{12}a_{02}| \approx 1.075|a_{02}|$ on the first and third impurities and from $|a_{02} - \eta_{21}a_{01}| = |(1 - \eta_{21}\eta_{12})a_{02}| \approx 2.128|a_{02}|$ to $|a_{02} + \eta_{21}a_{01}| = |(1 + \eta_{21}\eta_{12})a_{02}| \approx 0.1277|a_{02}|$ on the second impurity (рис. 3, *b*).

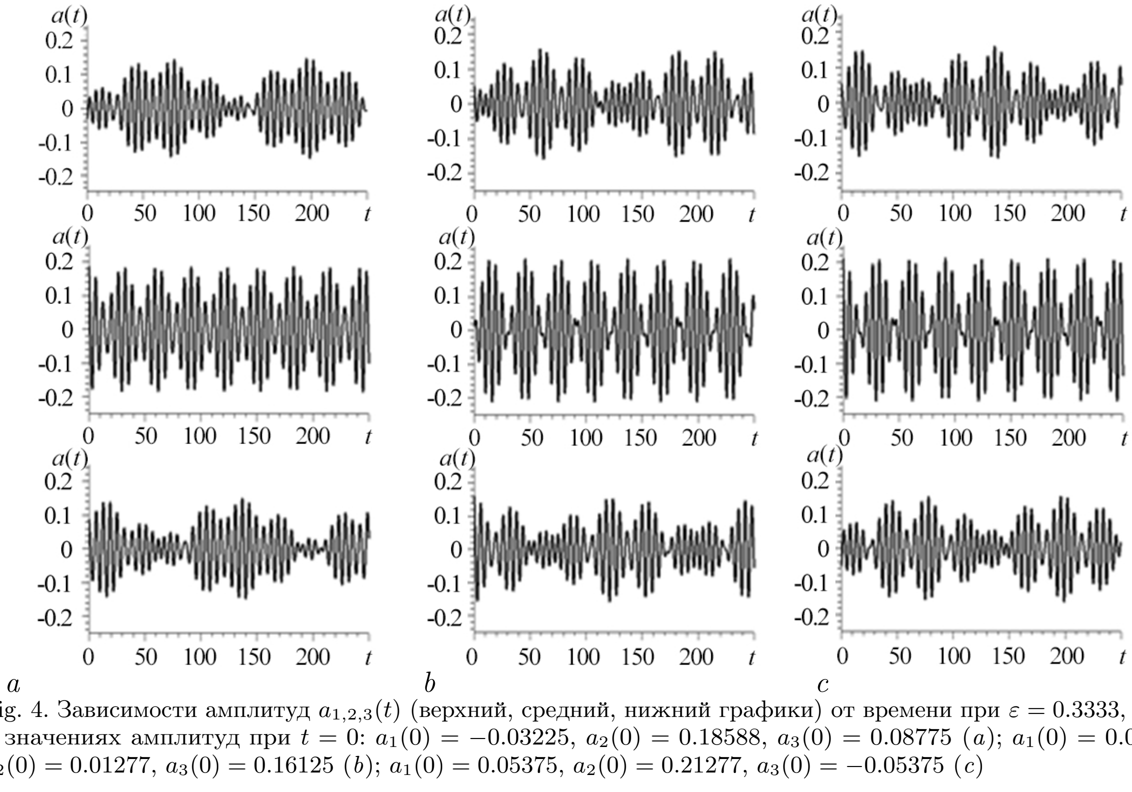


Fig. 4. Зависимости амплитуд $a_{1,2,3}(t)$ (верхний, средний, нижний графики) от времени при $\varepsilon = 0.3333$, $d = 3$ и значениях амплитуд при $t = 0$: $a_1(0) = -0.03225$, $a_2(0) = 0.18588$, $a_3(0) = 0.08775$ (*a*); $a_1(0) = 0.05375$, $a_2(0) = 0.01277$, $a_3(0) = 0.16125$ (*b*); $a_1(0) = 0.05375$, $a_2(0) = 0.21277$, $a_3(0) = -0.05375$ (*c*)

Fig. 4. Dependences of the $a_{1,2,3}$ amplitudes (upper, middle, lower figures) on time at $\varepsilon = 0.3333$, $d = 3$. Amplitude values at $t = 0$: $a_1(0) = -0.03225$, $a_2(0) = 0.18588$, $a_3(0) = 0.08775$ (*a*); $a_1(0) = 0.05375$, $a_2(0) = 0.01277$, $a_3(0) = 0.16125$ (*b*); $a_1(0) = 0.05375$, $a_2(0) = 0.21277$, $a_3(0) = -0.05375$ (*c*)

If $a_{01} = 0$, then the expression (11) can be written as

$$
\begin{aligned}
a_1(t) &= (a_{03} + \eta_{12}a_{02})\cos(\Omega_2 t + \theta_2) - \\
&\quad - 2a_{03}\cos\left(\frac{(\Omega_2+\Omega_3)t+\theta_2+\theta_3}{2}\right)\cos\left(\frac{(\Omega_2-\Omega_3)t+\theta_2-\theta_3}{2}\right),\\
a_2(t) &= a_{02}\cos(\Omega_2 t + \theta_2),\\
a_3(t) &= (\eta_{12}a_{02} - a_{03})\cos(\Omega_2 t + \theta_2) + \\
&\quad + 2a_{03}\cos\left(\frac{(\Omega_2+\Omega_3)t+\theta_2+\theta_3}{2}\right)\cos\left(\frac{(\Omega_2-\Omega_3)t+\theta_2-\theta_3}{2}\right),
\end{aligned}
\tag{17}
$$

that is, the oscillations on the first and third impurities have the form of beats with a frequency of $|\Omega_3 - \Omega_2|$ and amplitudes varying from $|a_{03} - \eta_{12}a_{02}|$ to $|a_{03} + \eta_{12}a_{02}|$, on the second impurity the oscillations are harmonic. For example, if $a_{01} = 0$, take $a_{02} = a_{03}/\eta_{12}$, then the initial conditions will be $a_1(0) = \eta_{12}a_{02} - a_{03} = 0$, $a_2(0) = a_{02} = a_{03}/\eta_{12} \approx -1.860a_{03}$, $a_3(0) = \eta_{12}a_{02} + a_{03} = 2a_{03}$. The amplitudes will vary from $|a_{03} - \eta_{12}a_{02}| = 0$ to $|a_{03} + \eta_{12}a_{02}| = 2|a_{03}|$ on the first and third impurities, and $|a_{02}| = |a_{03}/\eta_{12}| \approx .860|a_{03}|$ on the second impurity (Fig. 3, *c*). The oscillations in these cases are similar to the oscillations on two identical impurities considered earlier [28].

Let us further consider the case of coupled oscillations of impurity modes in the presence of all three harmonics (Fig. 4). Fluctuations of the second impurity mode for the considered case $\varepsilon = 0.3333$, $d = 3$ retain their character as in Fig. 3, *b*, since they involve only two harmonics. The oscillation form of the first and third impurity modes becomes more complicated due to the addition of the third harmonic.

2. Results of numerical calculations

In order to analyze to what extent the analytical solution obtained using perturbation theory is applicable to describe solutions of a nonlinear differential equation (1), it is necessary to solve it using numerical methods. Currently, a large number of methods have been developed for the numerical solution of such equations [3,4,11,25,27]. Let's use the finite difference method. Let's choose a three-layer explicit solution scheme, with approximation of derivatives on a five-point pattern of the "cross"type, which was used earlier for simpler modifications of the sine-Gordon equation (see, for example, [19,21]). This second-order numerical scheme approximates Δx and τ, where Δx is a coordinate step, τ is a time step. It has conditional stability $(\tau/\Delta x) \leqslant 1$. In our case, the scheme is a "one-step" [21,27,37], uses a relatively small number of memory accesses and has the potential to optimize the computational algorithm.

Frequency analysis of localized wave oscillations, which are calculated numerically, is performed using a discrete Fourier transform. The fast Fourier transform algorithm is used to calculate it. This algorithm has good performance. However, the most optimized implementations of the fast Fourier transform algorithm impose certain restrictions on the original series. To prepare the data, the original discrete dependence is interpolated by a cubic spline with natural boundary conditions, from which a new discrete dependence is constructed on a uniform grid with an increased number of approximation points. The frequency spectrum is calculated from the new discrete dependence using the fast Fourier transform algorithm. To increase the accuracy of frequency determination, the points of the maxima of the frequency spectrum are refined using the interpolation by the Akim spline.

The numerical experiment is performed as follows. At the initial moment of time, at some distance from the impurities, there is a kink moving at a constant speed. When the kink passes through the region of point impurities, localized breeze-type waves are excited on them. The amplitude and type of localized waves depend on the initial velocity of the kink, the parameters ε and d. Since localized impurity waves are excited as a result of the passage of a kink, its initial velocity determines their initial phase difference, as for the case of an analytical solution. As a result, it is not possible to excite the entire spectrum of possible associated oscillations of localized waves.

In Fig. 5 and fig. 6 the dependences of the amplitude of localized waves on time at the impurity location at $\varepsilon = 0.3333$ and $\varepsilon = 0.5$ are given for three different cases corresponding to different values of the parameter d. By the nature of the frequency spectra $A(\omega)$ they can be attributed to different modes of oscillation. It can be seen from the figures that at small distances between the impurities, the connection between the waves is very strong. Under any initial conditions, they begin to oscillate in phase at a single frequency after a certain period of time. Similar behavior is typical for the case of two point impurities [28].

From fig. 5 and fig. 6 for the case of a small and large distance between impurities, it can be seen that there are associated oscillations of localized breeze-type waves with characteristic strong beats. Let's compare the harmonics obtained using Fourier decomposition with the frequencies

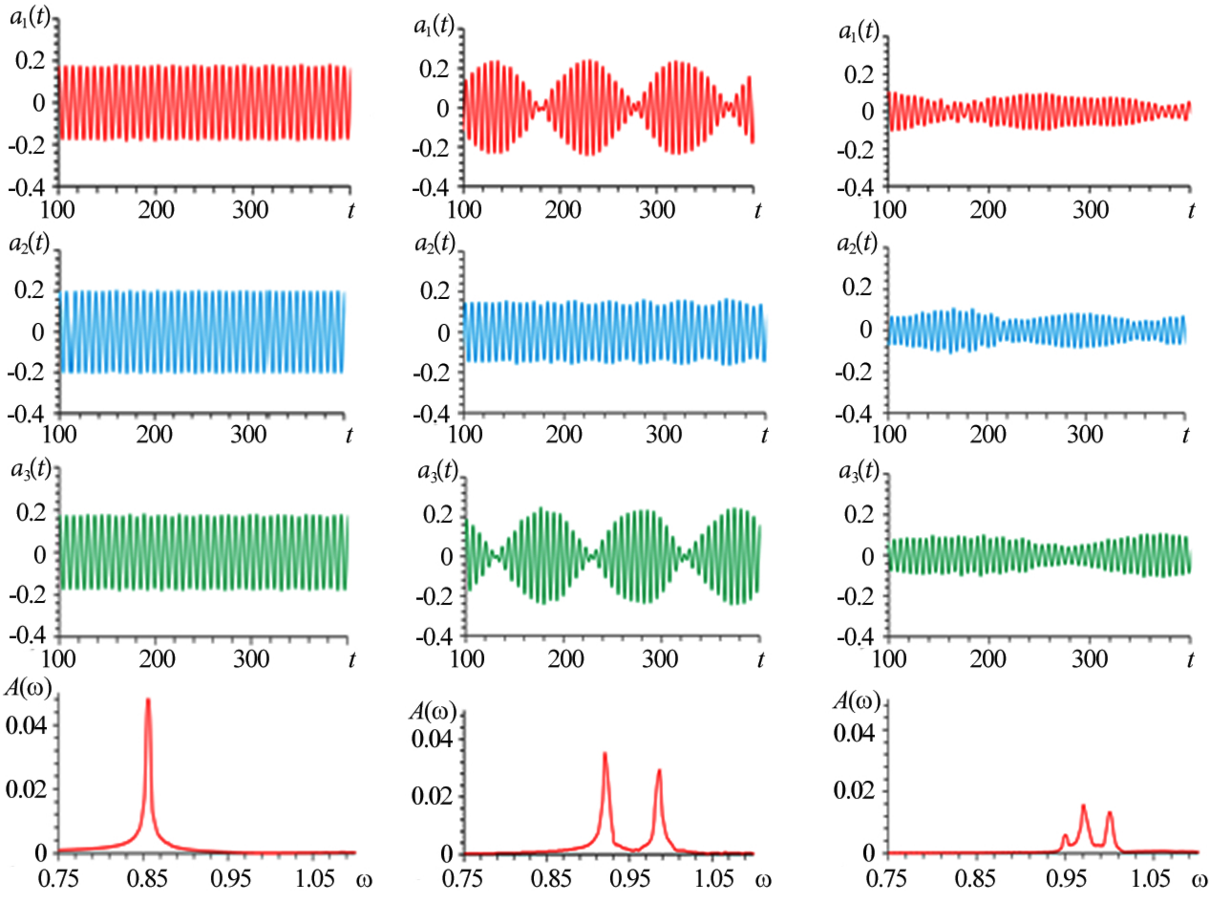
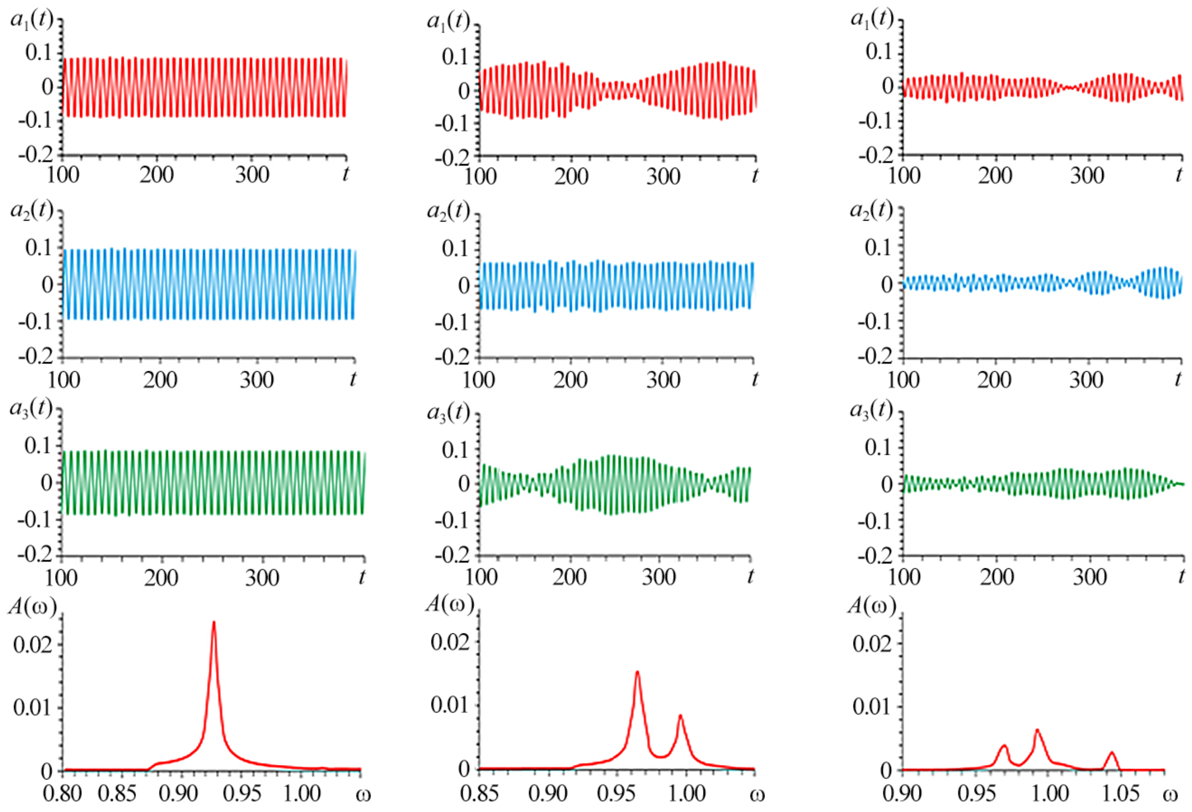


Fig. 5. Зависимости амплитуд $a_{1,2,3}(t)$ от времени для $\varepsilon = 0.5$ при различных значениях параметра d, рассчитанные численно из уравнения (1), и соответствующие дискретные фурье-разложения $A(\omega)$. $d = 1$ (a); $d = 3$ (b); $d = 6$ (c)

Fig. 5. Dependences of the $a_{1,2,3}$ amplitudes on time for $\varepsilon = 0.5$ at different the d values calculated numerically from the equation (1) and corresponding discrete Fourier expansion $A(\omega)$. $d = 1$ (a); $d = 3$ (b); $d = 6$ (c)

$\Omega_{1,2,3}$ obtained analytically earlier. On fig. 1 solid lines are the analytically calculated frequencies Ω_1, Ω_2 and Ω_3, and the points are the numerically obtained frequencies. From fig. 1 it can be seen that there is a good coincidence of numerical and analytical results. The numerically obtained harmonic values for the considered cases with an accuracy of 1–2% coincide with the corresponding values of $\Omega_{1,2,3}$. For example, for Ω_1 and Ω_3 with $\varepsilon = 0.3333$ and $d = 4$, the values obtained analytically are 0.968 and 1.000, and the values obtained numerically are 0.956 and 0.996, respectively. The value of Ω_2 for $\varepsilon = 0.3333$ and $d = 6$ for the analytical solution is 1.034, for the numerical — 1.036.

From the comparison of numerical and analytical results, it follows that the analytical results obtained using the equations for collective coordinates remain relatively reliable and are close to the results of direct numerical calculation for d greater than or equal to one, ε less than one, and the amplitudes of impurity modes of the order of 0.3 or less. Localized waves containing frequencies Ω_1 and Ω_3, are excited numerically starting from small distances between impurities. Localized waves containing the frequency Ω_2 are excited numerically only for large distances between impurities (in this case starting from $d = 6$), when the "binding force"between them is greatly reduced. Moreover, for the case of localized waves with one frequency, Ω_1 is excited, for the case of oscillations with two frequencies — Ω_1 и and Ω_3.



a *b* *c*

Fig. 6. Зависимости амплитуд $a_{1,2,3}(t)$ от времени для $\varepsilon = 0.3333$ при различных значениях параметра d, рассчитанные численно из уравнения (1), и соответствующие дискретные фурье-разложения $A(\omega)$. $d = 1$ (a); $d = 3$ (b); $d = 6$ (c)

Fig. 6. Dependences of the $a_{1,2,3}$ amplitudes on time for $\varepsilon = 0.3333$ at different the d values calculated numerically from the equation (1) and corresponding discrete Fourier expansion $A(\omega)$. $d = 1$ (a); $d = 3$ (b); $d = 6$ (c)

Conclusion

In the article, for the sine-Gordon model with an arbitrary number of identical point impurities located at the same distance from each other, using the method of collective variables, a system of equations describing the oscillations of waves localized on impurities is obtained. The obtained differential equations for the case of three impurities are the equations of an oscillatory system with three degrees of freedom or three coupled harmonic oscillators. The oscillations of the system are the sum of three types of harmonic oscillations: in-phase, in-phase-antiphase and antiphase. Approximate analytical solutions for frequencies are obtained that well approximate the results of direct numerical simulation of a nonlinear system. It is shown that when the distance between the impurities decreases, the frequency of common-mode oscillations decreases, the frequency of common-mode-antiphase oscillations increases, the frequency of antiphase oscillations does not increase as fast as common-mode-antiphase. When two frequencies are excited, beats occur, vibrations are similar to vibrations in the case of two identical impurities. When three frequencies are excited, the oscillation form becomes more complicated.

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