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On the typicity of the explosive synchronization phenomenon in oscillator networks with the "ring" and "small world" topologies of links

A. A. Koronovskii, M. K. Kurovskaya, O. I. Moskalenko✉

Saratov State University
Regional Scientific and Educational Mathematical Center "Mathematics of Future Technologies"
E-mail: alexey.koronovskii@gmail.com, mariakurovskaya@gmail.com, ✉o.i.moskalenko@gmail.com
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Abstract. *Purpose* of this study is to investigate the problem of how typical (or, conversely, unique) is the phenomenon of explosive synchronization in networks of nonlinear oscillators with topologies of links such as "ring" and "small world", and, in turn, how the partial frequencies of the interacting oscillators must correlate with each other for the phenomenon of explosive synchronization in these networks can be possible. *Methods.* In this paper, we use an analytical description of the synchronous behavior of networks of nonlinear elements with "ring" and "small world" link topologies. To confirm the obtained results the numerical simulation is used. *Results.* It is shown that in networks of nonlinear oscillators with "ring" and "small world" topologies of links, the phenomenon of explosive synchronization can be observed for the different distributions of partial frequencies of network oscillators. *Conclusion.* The paper considers an analytical description of the behavior of network oscillators with "ring" and "small world" topologies of links and shows that the phenomenon of explosive synchronization in such networks is atypical, but not unique.

Keywords: explosive synchronization phenomenon, Kuramoto oscillators, nonlinear element networks, small-world topology, ring topology, partial frequencies.

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Introduction

When considering the collective dynamics of networks of nonlinear oscillators with a complex topology of links between nodes [1,2] among a wide variety of possible types of synchronous behavior, the so-called explosive synchronization regime occurs [3–6], in which, with an increase in the coupling parameter between the elements, an explosive increase in the number of synchronous oscillators of the network suddenly occurs, and the entire network begins to demonstrate synchronous behavior. And, conversely, when the coupling parameter decreases, the completely synchronous state of the network suddenly collapses, while the number of synchronous oscillators (the size of the synchronous cluster that has arisen) significantly decreases. The phenomenon of explosive synchronization has been described and studied for networks of phase oscillators, and, accordingly, the term "synchronous behavior"of interacting oscillators refers to the phase synchronization regime when there is a coincidence of frequencies of interacting oscillators, and hence the locking of their instantaneous phases [7, 8].

Despite the fact that the phenomenon of explosive synchronization can be observed for complex networks with different link topologies — such as regular networks, where each element of the network is connected to all the others [9], networks with a random link topology [10], scale-free networks [6, 11, 12], small world networks [13] — the explosive synchronization regime undoubtedly has a universal character, but is not typical. It is observed in quite rare cases, has a self-similar character [10] and requires a very specific set of control parameter values, both for individual oscillators and for the entire network as a whole. In this case, the distribution of natural frequencies of interacting oscillators plays an important role $g(\omega)$ (see, for example, section 4.1.1 of the review [3]). In particular, for the case of a regular network of Kuramoto oscillators with a topology of links between elements «each with each», the phenomenon of explosive synchronization is described for the case of uniform equidistant distribution of natural frequencies of oscillators, as well as for special cases of parabolic, triangular and «hat-shaped» equidistant distributions of partial frequencies of interacting systems [9]. Similarly, the phenomenon of explosive synchronization occurs in the case of a uniform equidistant distribution of the natural frequencies of interacting oscillators in networks with a random topology of links [10], and also in small-world networks and regular networks with a topology of connections of the «ring» type [13]. At the same time, in order for the phenomenon of explosive synchronization to take place in scale-free networks, the natural frequencies of the ω_i oscillators must be related to the degree of the corresponding node k_i [14], while the ratio is often used $\omega_i = k_i$ [6, 11]. The literature also considers other rather «exotic» variants of complex networks and the distribution of partial frequencies, such as scale-free networks with delayed connections [15], frequency-weighted networks [16], multilayer networks [17] (including, multilayer networks with partial and weak correlation [18]) and so on.

In this paper, based on the features of the topology of links beween nodes of networks of the «ring» and «small world» type, the question of possible types of distribution of partial frequencies of interacting phase oscillators is considered, in which the «explosive» nature of establishing (or destroying) a fully synchronous state of a network of nonlinear oscillators occurs with a change in strength of connections between oscillators.

1. The considered model of a network of phase oscillators

Although the phenomenon of explosive synchronization is observed for various types of phase oscillators (in particular, for piecewise linear Ressler oscillators [4], generalized Kuramoto oscillators [19]), the Kuramoto oscillator model is used as a reference model of phase oscillators located in network nodes with a complex topology of links [20, 21]

$$\dot{\varphi}_j = \omega_j + \lambda \sum_{k=1}^{N} a_{jk} \sin\left(\varphi_k - \varphi_j\right). \qquad (1)$$

In the ratio (1), each j oscillator is characterized by the instantaneous value of the phase φ_j and the frequency ω_j (which is the control parameter for each oscillator). The topology of the links between the oscillators is given by the elements of the matrix $\mathbf{A} = \{a_{jk}\}$, taking the zero ($a_{jk} = a_{kj} = 0$) or one ($a_{jk} = a_{kj} = 1$) values in the absence or presence of a connection between the j-m and k-m oscillators, respectively. Obviously, the diagonal elements of the matrix $\mathbf{A}$ must be identically equal to zero, $a_{jj} = 0$. The total number of oscillators in the network is set by the control parameter N, the value of which in this paper is assumed to be equal to $N = 10^3$. The intensity of all network links is assumed to be the same and is characterized by the value of the control parameter λ.

The linear transformation $\varphi_j \to \varphi_j + \Omega_0 t$ translates the original ratio (1) into itself. However, after this transformation, the variable φ_j will have the meaning of the phase difference between the original j network oscillator and some reference oscillator (possibly abstract), the oscillation frequency of which is Ω_0. Similarly, the partial frequency ω_j after such a transformation will make sense of the difference between the frequencies of the original oscillator with the number j and the frequency Ω_0. Choosing Ω_0 lying inside the interval of partial frequencies of interacting oscillators, $\Omega_0 \in (\omega_{\min}, \omega_{\max})$, it can be achieved that the natural frequencies of the considered oscillators can be both positive and negative, and if the condition $\Omega_0 = \langle\omega_j\rangle$ is met, the set of partial frequencies will have a zero mean.

Since in this paper we consider a regular network of nonlinear Kuramoto oscillators with a topology of links between nodes of the «ring» type, as well as a network with «small world» topology formed from a regular ring network using the Watts-Strogatz method [22], we introduce into consideration similar to the work [13] a spatial coordinate x directed «along» the ring, the beginning of which relative to the network elements can be chosen arbitrarily. Without loss of generality, we assume that in the coordinate system introduced into consideration, the length of the ring will be $2\mathfrak{L}$ units, and the eigenfrequencies ω_j of the network elements depend on the coordinate x and can be considered as some function of the coordinate, $\omega(x)$. Similarly, the instantaneous values of the quantities φ_j can also be considered as a function of the spatial variable and time, $\varphi(x,t)$, and in the case of a completely synchronous state of all network elements and the choice of the value Ω_0 equal to the frequency of synchronous oscillations — as a function of the spatial variable only, $\varphi(x)$.

Assuming a large number of oscillators $N \gg 1$ and a completely synchronous state of all interacting oscillators for a network of nonlinear oscillators with a topology of links between nodes of the «ring» type, the initial evolution operator (1) can be written as

$$\lambda\Delta \cdot \Phi_s'(x) = \omega(x), \tag{2}$$

where is the value

$$\Phi_s(x) = -\frac{1}{\Delta^2}\int\limits_0^L \eta \sin\left(\varphi'(x)\eta\right)\, d\eta \tag{3}$$

is some function, which we will call potential (see also [13]). In the ratio (2), the variable x runs through the range of values from $-\mathfrak{L}$ to $+\mathfrak{L}$, the parameter $\Delta = 2\mathfrak{L}/(N-1)$ makes sense of the distance (in units of the dimensionless coordinate x, introduced into consideration along the ring network) between two neighboring elements of the network (that is, between j-m and $(j+1)$-m oscillators), and the value of $L = \Delta K$, in turn, characterizes the spatial size of the connection (again in units of the spatial variable x) of each element of the network with $2K$ neighbors with which it is connected.

Indeed, for a fully synchronous network with the «ring» topology of inter-element connections and oscillators with numbers $j \in [K+1, N-K)]$, the relation (1) can be written as

$$0 = \omega(x_j) + \lambda \sum_{k=j-K}^{j+K} \sin\left(\varphi\left(x_j + (k-j)\Delta\right) - \varphi(x_j)\right), \tag{4}$$

where the coordinate x_j corresponds to the position of the jth oscillator. Due to the closeness of the oscillator network into a ring, the ratio (4) (with appropriate reinterpretation of the oscillator numbers) will be valid for oscillators with numbers less than $(K+1)$ and more $(N-K)$.

Using reinterpretations $k \to k-j$, $x \to x_j$

$$-\lambda \sum_{k=-K}^{+K} \sin\left(\varphi\left(x + k\Delta\right) - \varphi(x)\right) = \omega(x), \tag{5}$$

and a sequential chain of transformations

$$-\lambda \sum_{k=0}^{K} \sin\left(\varphi\left(x - k\Delta\right) - \varphi(x)\right) - \lambda \sum_{k=0}^{K} \sin\left(\varphi\left(x + k\Delta\right) - \varphi(x)\right) = \omega(x), \tag{6}$$

$$-\lambda \sum_{k=0}^{K} \left\{\sin\left(\varphi\left(x + k\Delta\right) - \varphi(x)\right) - \sin\left(\varphi(x) - \varphi\left(x - k\Delta\right)\right)\right\} = \omega(x), \tag{7}$$

the relation (4) can be reduced to the form

$$-\lambda \sum_{k=0}^{K} \left\{U_k(x + k\Delta) - U_k(x)\right\} = \omega(x), \tag{8}$$

where

$$U_k(x) = \sin\left(\varphi(x) - \varphi(x - k\Delta)\right). \tag{9}$$

Assuming a large number of ensemble oscillators, $N \gg 1$, and, accordingly, the distance between the oscillators tending to zero, $\Delta \ll 1$, the ratio (8) can be written as

$$-\lambda\Delta \sum_{k=0}^{K} kU'_k(x) = \omega(x). \tag{10}$$

Introducing the function

$$\tilde{\Phi}_s(x) = -\sum_{k=1}^{K} kU_k(x) = -\sum_{k=1}^{K} k\sin(\varphi(x) - \varphi(x - k\Delta)) = -\sum_{k=1}^{K} k\sin(\varphi_j - \varphi_{j-k}), \tag{11}$$

and using the linearity property of the sum operator, the relation (10) can be written as

$$\lambda\Delta \cdot \tilde{\Phi}'_s(x) = \omega(x). \tag{12}$$

Now, to move from the resulting ratio (12) to the form (2), it remains only to take into account

$$U_k(x) = \sin\left(\varphi(x) - \varphi(x - k\Delta)\right) \approx \sin\left(\varphi'(x)k\Delta\right) \tag{13}$$

and go from discrete summation to (11) to continuous integration

$$\sum_{k=1}^{K} kU_k(x) = \frac{1}{\Delta^2}\sum_{k=1}^{K}(k\Delta)U_k(x)\times\Delta \approx \frac{1}{\Delta^2}\int\limits_0^L \eta\sin\left(\varphi'(x)\eta\right)\,d\eta, \tag{14}$$

which leads to the relations (12) and (11) to a form completely identical to the expressions (2) and (3), respectively.

It is clear that when constructing a network of elements with a topology of links «small world» using the Watts–Strogatz [22] method, as a result of re-linking ring connections, «long» connections appear that are not taken into account by the relations (2) and (3), and, accordingly, the behavior of small-world networks will be distorted relative to the analytical description. At the same time, it can be expected that, since the properties of the «small world» in networks of nonlinear elements manifest themselves at small values of the probability of p reconnection of connections in the Watts-Strogatz method, the distortions that appear will be insignificant and will not greatly affect the main analytical results obtained during the review.

2. Fully synchronous regime and its destruction

As already discussed in the Introduction, the explosive synchronization regime consists in a sharp, seemingly sudden increase in the number of synchronized oscillators in the network and the transition of the network to a fully synchronous state with an increase in the strength of the couplings between the network elements. Similarly, with a decrease in the strength of the coupling, there is a sudden destruction of the fully synchronous state of the network and a sharp decrease in the number of synchronized oscillators. It is important to note that the establishment and destruction of a fully synchronous network state is usually accompanied by hysteresis [3], in other words, the critical values of the coupling parameter with increasing and decreasing the strength of the connection may be different.

Analytical relations (2) and (3) in the section 1 were obtained for the case of a fully synchronous state of the Kuramoto oscillator network. At the same time, it is obvious that in the case when the regularity $\omega(x)$ is known, according to which the partial frequencies of interacting oscillators are given (let's make an assumption about the integrability of this regularity $\omega(x)$), for the relation (2) there is always the solution is in the form of

$$\Phi_s(x) = \frac{1}{\lambda\Delta}\int \omega(x)\,dx \tag{15}$$

over the entire review interval $x \in [-\mathfrak{L};\mathfrak{L}]$, regardless of the magnitude of the coupling strength λ (of course, provided $\lambda \neq 0$) and the degree of synchronicity of the dynamic regime of the network oscillators. Taking into account the above, it seems obvious that the potential function $\Phi_s(x)$, defined by the relation (15), should have (again, over the entire consideration interval $x \in [-\mathfrak{L};\mathfrak{L}]$) a certain property in the regime of a fully synchronous state of the network, and, conversely, not to have this property if it is impossible for a fully synchronous state of the network to exist with small values of the coupling parameter. Moreover, the loss of this property by the potential function $\Pi_s(x)$ must correspond to the critical value of the control parameter λ, corresponding to the moment of destruction of the fully synchronous state of the network of coupled oscillators.

Since from the review conducted in the section 1, we know some information about the potential function $\Phi_s(x)$ (see the relation (3)), and considering the results of [13], we can assume that this property is the property of the limitation of the values of the potential function $\Phi_s(x)$

over the entire interval of consideration $x \in [-\mathfrak{L}; \mathfrak{L}]$ in a certain range of values

$$-S \leqslant \Phi_s(x) \leqslant S, \qquad S > 0. \tag{16}$$

The value of S can be estimated from the ratio (3): despite the fact that the explicit form of the dependence of the phases of the oscillators on the coordinate $\varphi(x)$ (the derivative of which appears in the relation (3)) we do not know, obviously, that the maximum and minimum possible values of the integrand function are $\sin(\varphi'(x)\eta)$ are ± 1, and accordingly, the maximum and minimum possible values of $\Phi_s(x)$ can be estimated (from above) as

$$\pm\frac{1}{\Delta^2}\int\limits_0^L \eta\, d\eta = \pm\frac{L^2}{2\Delta^2} = \pm\frac{K^2}{2}, \tag{17}$$

and, accordingly,

$$0 < S \leqslant \frac{K^2}{2}. \tag{18}$$

Thus, based on the results of the above consideration, the following intermediate conclusions can be formulated: in networks of phase oscillators with «ring» and «small world» topologies of links between nodes, specifying an explicit type of distribution of partial frequencies of interacting oscillators ω_j (and, respectively, $\omega(x)$) and by tracking the fulfillment of the condition (16) for the potential function $\Phi_s(x)$ (the type of which is known from the relation (15)), it is possible to «construct» situations, in which, with an increase/decrease in the strength of the coupling between the interacting oscillators, an explosive transition (*explosive synchronization*) will be realized between a fully synchronous state of the network and a state in which there are synchronous clusters of oscillators whose sizes are significantly smaller than the total number of network oscillators N. Obviously, there can be quite a lot of such dependencies $\omega(x)$, and, accordingly, network configurations in which explosive synchronization can be implemented, which, in turn, suggests that in networks with «ring» and «small world» link topologies the phenomenon of explosive synchronization is at least not unique.

3. Example: harmonic dependence of frequency on oscillator number

In order to illustrate the above, let's consider networks with topologies of links between nodes of the type «ring» and «small world», consisting of $N = 10^3$ Kuramoto oscillators (1), the partial frequencies of which depend on the number of the oscillator according to the harmonic law

$$\omega_j = \Omega \sin\left(\frac{2m\pi}{N} j\right), \qquad j = 1, 2, \ldots, N, \tag{19}$$

where $\Omega > 0$ is the maximum possible value of the partial frequency of the oscillators, m is the number of periods depending on the partial frequencies of the oscillator numbers. Let's choose the value of the parameter $K = 8$, which will mean that each element of the network is connected to 16 nearest oscillators. Since the theoretical consideration in the sections 1 and 2 was carried out in the range of coordinate values $x \in [-\mathfrak{L}; \mathfrak{L}]$, for a continuous model (2), the dependency (19) will correspond to the dependency

$$\omega(x) = -\Omega \sin\left(\frac{m\pi}{\mathfrak{L}} x\right), \qquad x \in [-\mathfrak{L}; \mathfrak{L}]. \tag{20}$$

When considering this example, we will choose for convenience $\mathfrak{L} = 0.5$, then, as it is easy to see, the value of the control parameter Δ will be equal to 10^{-3}. It is not difficult to see that for the dependency under consideration (20), the potential function (15) will have the form

$$\Phi_s(x) = \frac{\mathfrak{L}\Omega}{m\pi\lambda\Delta}\cos\left(\frac{m\pi x}{\mathfrak{L}}\right) + C, \tag{21}$$

where C is the integration constant, which, due to the symmetry of the relations (3), (16), (17), (20), we will assume identically equal to zero.

Obviously, the condition (16) will be met when

$$\frac{\mathfrak{L}\Omega}{m\pi\lambda\Delta} \lesssim \frac{K^2}{2}, \tag{22}$$

what makes it possible to estimate the critical value of the coupling parameter λ_c, at which (in case of a decrease in the strength of the connection) the destruction of the fully synchronous state of the Kuramoto oscillator network should occur

$$\lambda_c \gtrsim \lambda_* = \frac{2\mathfrak{L}\Omega}{m\pi\Delta K^2}. \tag{23}$$

It is easy to see that with the selected values of the control parameters, $\Omega = 0.5$ and $m = 1$ the value of λ_c must be equal to or greater than $\lambda_* = 2.4868$. In other words, it can be expected that for the specified values of the control parameters, with the strength of the coupling between the elements of the network being below the value of λ_*, the network of Kuramoto oscillators (1) with the «ring» topology of links between nodes cannot be in completely synchronous regime, but above the critical value phenomenon of explosive synchronization is possible from this point of view.

In Fig. 1 the dependences of the number of oscillators in the maximum synchronous cluster of the network on the value of the coupling parameter λ for different values of the control parameters characterizing the Kuramoto oscillator network obtained by direct numerical

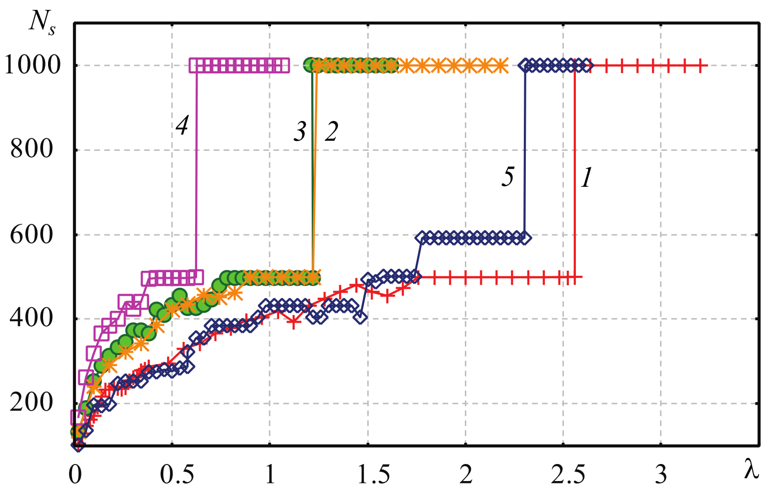

Fig. 1. The dependence of the number of the synchronous oscillators into the maximal synchronous cluster, N_s, on the coupling strength λ for the Kuramoto oscillator network (1) with topology of links belonging to the types "ring" (curves *1*–*4*) and "small world" (curve *5*). The control parameter values are the following: $\mathfrak{L} = 0.5$, $N = 10^3$, $K = 8$; $\Omega = 0.5$, $m = 1$ (curve *1*); $\Omega = 0.5$, $m = 2$ (curve *2*); $\Omega = 0.25$, $m = 1$ (curve *3*); $\Omega = 0.25$, $m = 2$ (curve *4*); $\Omega = 0.5$, $m = 1$, $p = 0.01$ (curve *5*). All curves have been obtained for the decreasing coupling parameter λ (color online)

simulation of the model system are shown. The figure clearly shows that for all the cases under consideration in the oscillator network, there is a sharp transition between the fully synchronous state of the oscillators of the network and the state when there is no such fully synchronous state. The curve *1* corresponding to $\Omega = 0.5$ and $m = 1$, undergoes a sharp transition at $\lambda_c = 2.5224$, which correlates very well with the received value $\lambda_* = 2.4868$. Above the critical value, the Kuramoto oscillator network turns out to be fully synchronized, whereas below λ_c there are two equivalent synchronous oscillator clusters in the network, just as is the case in of a Kuramoto oscillator network with a uniform equidistant partial frequency distribution (see [13]).

It is also interesting to note that an increase in the number of oscillation periods of the $\omega(x)$ dependence on the considered interval $[-\mathfrak{L}; \mathfrak{L}]$ leads to a corresponding decrease in the value of λ_* and, accordingly, the critical value of λ_c (see the relation (23), and also compare the curves *1* and *2*, *3* and *4* in Fig. 1). In the case when the frequency of Ω changes simultaneously with the number of oscillation periods m so that the ratio Ω/m remains unchanged, the corresponding dependencies $N(\lambda)$ will be almost identical to each other (see (23) and cf. the curves *2* and *3* in Fig. 1).

The figure also shows that in the case of a small-world network, an explosive transition occurs at a lower (although close) value of the control parameter λ compared to a similar network of Kuramoto oscillators of the «ring» type (cf. curves *1* and *5*, Fig. 1). Such a difference can be explained by the fact that a small number of «long» links formed due to the reconnection of connections between oscillators when constructing a small-world network from a ring using the Watts-Strogatz [22] method introduces some distortions in the behavior of the small-world network compared to the theoretical relations given in this paper.

The obtained theoretical relation (21) for the potential function $\Phi_s(x)$ can be compared with the results of direct numerical simulation of the Kuramoto oscillator network (1), while the numerical dependence of the potential function on the coordinate x can be calculated using the relation (11), since for the number of elements of the network N tending to infinity, the discrete analogue of the potential function $\tilde{\Phi}_s(x)$ tends to the potential function $\Phi_s(x)$:

$$\lim_{N \to +\infty} \tilde{\Phi}_s(x) = \Phi_s(x). \tag{24}$$

Since the numerical simulation of the behavior of the Kuramoto oscillator network uses the element number j as the spatial coordinate, for matching (11) c (21) it makes sense to move to a dimensionless coordinate $x = (j - N/2)/N$.

In Fig. 2 it is shown the comparisons of profiles of theoretical $\Phi_s(x)$ and numerical $\tilde{\Phi}_s(x)$ representations of a potential function for a network of Kuramoto oscillators (1), which is in a fully synchronous state. Fig. 2, *a* corresponds to the case of a network of oscillators with a «ring» topology of links, Fig. 2, *b* corresponds to the «small world» topology of links. From Fig. 2, *a* it can be shown an excellent correspondence between the result of numerical modeling and the theoretical dependence (21), which suggests the adequacy of the proposed theoretical approach. In the case of the «small world» «long» network, the connections that were not taken into account when constructing the theoretical description somewhat distort the profile of the potential function. «Long» connections resulting from the application of the Watts-Strogatz [22] method can be considered as some random effect on a potential function that distorts its profile. Distortions of the smooth profile of the potential function lead to the fact that for a small-world network at some points x_e, the potential function $\tilde{\Phi}_s(x)$ reaches restrictive values of S at lower values of the control parameter λ compared to a similar network (characterized by the same set values of control parameters) with the «ring» topology of inter-element connections, which leads to a shift in the area of a sharp transition towards smaller values of the coupling parameter λ (see Fig. 1, curves *1* and *5*). Nevertheless, these differences are mainly of an insignificant quantitative

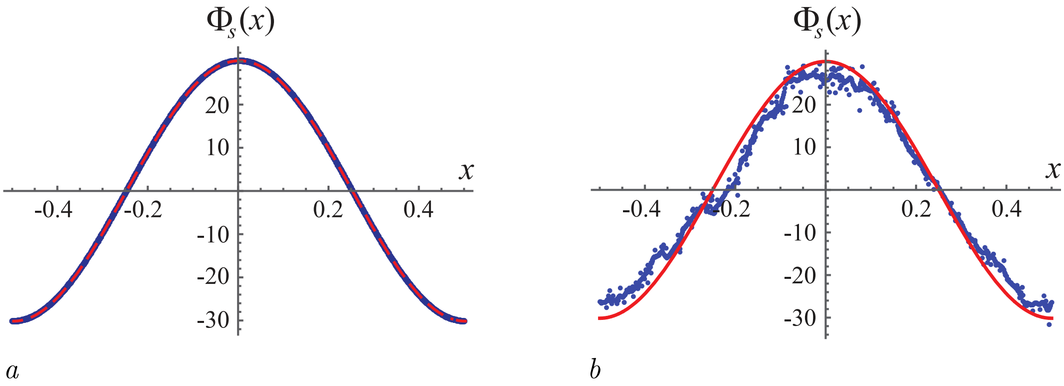

Fig. 2. The dependence of the discrete and continuous representations of the potential function on coordinate x for the Kuramoto oscillator network with "ring" (a) and "small world" (b) topologies, $\lambda = 2.64$, $\Omega = 0.5$, $m = 1$, $p = 0.01$ (color online)

nature. The general patterns in the behavior of a potential function for a network of synchronous oscillators turn out to be the same for the «ring» and «small world» topologies of interelement connections, therefore all the conclusions obtained for a network with the «ring» topology remain valid for small world networks.

Conclusion

Thus, based on the theoretical results set out in sections 1 and 2, confirmed by the results of numerical modeling (section 3) of the dynamics of Kuramoto oscillator networks with «small world» and «ring» topologies, it can be concluded that the explosive nature of the transition to a fully synchronous state in such networks can be observed for a sufficiently large number of possible dependencies of the partial frequencies of ω_j oscillators on the oscillator number j, consequently, such an explosive transition in such networks is a phenomenon, although atypical, but not unique. Moreover, the approach proposed in this paper, based on specifying a sequence of partial frequencies ω_j, allows us to purposefully «design» networks with «ring» and «small world» topologies, which will demonstrate the phenomenon of explosive synchronization.

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