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Mutual synchronization of oscillations in a system of coupled evolutionary games

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Abstract. The *purpose* of this study is to investigate collective dynamics of coupled communities that evolve according to the population game «Battle of the Sexes». A separate community includes two interacting populations of players of opposite sex, where each player has one of two possible competing behavior strategies. It is necessary to determine the possibility of mutual synchronization of oscillations in the number of players adhering to a particular strategy, build a synchronization region, and also evaluate the dependence of the properties of oscillations on the coupling strength. *Methods.* In this paper, we study the system of evolutionary games «Battle of the Sexes» interacting through migration. To simulate the evolutionary game dynamics we make use of the stochastic Moran process, as well as the Monte Carlo method to sample game trajectories. Mutual synchronization is defined by the appropriately generalized criteria of frequency and phase locking. *Results.* It is shown that the system of coupled evolutionary games «Battle of the Sexes» demonstrates mutual synchronization of oscillations under sufficiently strong coupling. In particular, oscillation frequencies of two communities get adjusted to each other and begin to coincide at some interaction parameter, while the oscillations themselves become almost identical. A similar result was also observed for an ensemble of more than two communities. *Conclusion.* The dependence of the average frequencies of community oscillations on the coupling strength was determined, the adjustment of oscillations with an increase in the coupling strength was demonstrated, thereby showing the possibility of mutual synchronization in the model of coupled evolutionary games «Battle of the Sexes». The region of frequency synchronization was numerically found.

Keywords: evolutionary game theory, game trajectories, stochastic oscillations, mutual synchronization.

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Introduction

Synchronization of oscillations is a well-known phenomenon that has been extensively studied for regular, noisy and chaotic oscillations [1]. It is also ubiquitous in biological systems and populations, to name synchronization of noisy electrosensitive paddlefish cells [2], between activity in remote human brain areas [4], between the human cardiovascular and respiratory systems [3], in the neuronal population model [5], in predator–prey populations [6, 7], and population synchronization in epidemic models [8].

Here we focus on synchronization of evolutionary game oscillations, with the emphasis on a specific and yet not well-understood case when the population dynamics is (i) essentially discrete and oscillations emerge due to discreteness (i.e. the population size is finite, and the mean-field approximation is invalid), and (ii) metastable (i.e. the game fixation occurs on a finite time-scale, so that oscillations are only transient). We consider a system of coupled communities that evolve according to the population game «Battle of the Sexes», where individuals may follow one of several behavioral strategies [9]. In other words, the game models the competition of two strategies for choosing a partner and raising offspring in two populations of individuals of the same species, but of the opposite sex (males and females) [10].

More precisely, individuals from populations of the opposite sex play against each other. Each player, depending on his strategy and the strategy of its counterpart, receives some payoff reflecting his total costs and benefits. The evolutionary dynamics consists in repeated rounds of play between randomly selected males and females. The interaction of players and the update of the populations composition are described using a stochastic frequency-dependent Moran process [11, 12]. According to this process, the total size of populations is finite and constant, as a result of which the birth of new individuals and the death of existing ones occur at the same moment in time.

When the game «Battle of the Sexes» is played by finite populations, asymptotic equilibrium states are absorbing states (complete dominance of one of the behavioral strategies). Since mutations are not included in the model, once populations enter the state of absorption, they can no longer get out of it, and the evolutionary dynamics is completed. Earlier we showed that the game has a non-trivial transitional dynamics. In particular, it was found that before absorption, stochastic cyclical fluctuations in the number of players adhering to one or another strategy are observed in the model.

Here we describe and study a system of two or more coupled communities (evolutionary games «Battle of the Sexes»). If isolated, such communities generate their own transient oscillations with a frequency determined by the properties of the subsystem. We introduce migration between communities, and in the case of more than two connected games, consider two topologies (chain and ring structures).

1. Methods

1.1. Model of the game «Battle of the Sexes». «Battle of the Sexes» model [10] describes the process of population reproduction by means of a game-theoretic approach. Denote the male population as A and the female population as B . Assume that each population consists of N individuals, and this number is constant in time. In each population, there are players (agents) with one of two behavioral strategies that differ in reproduction and raising offspring. Male categories are defined as «faithful» and «philanderer», and female – «coy» and «fast».

The evolutionary dynamics consists in repeated game rounds between the chosen particular male and female. Interacting with each other, the players receive the following payoffs

$$\left[\begin{array}{cc} a_{11} = 1; b_{11} = -1 & a_{12} = -1; b_{12} = 1 \\ a_{21} = -1; b_{21} = 1 & a_{22} = 1; b_{22} = -1 \end{array} \right], \quad (1)$$

where $a_{ss'}$ is the payoff for a male with $s \in \{1, 2\}$ strategy interacting with a female with $s' \in \{1, 2\}$ strategy. Similarly, the values $b_{ss'}$ determine payments for females. A negative payoff means that the cost to the player exceeds the benefit of the interaction.

At each round of the game, the dynamics is described using the stochastic frequency-dependent Moran process [11, 12], which determines the rules for choosing players and further updating the composition of populations. The Moran process consists of three steps.

1. In each population, a player is selected randomly with a probability proportional to the fitness of the strategies.
2. The chosen pair of players gives birth to two offspring (male and female) that inherit the strategy of the parent of the same gender.
3. Each offspring replaces an randomly dead individual in the corresponding population.

According to the described process, the population size N remains constant throughout the game, so the state of each population after some game round m can be described by the number of players with the first strategy: i males and j females, where $0 \leq i, j \leq N$. Then the state of the game consisting of two populations is given by a pair of values (i, j) .

As mentioned above, the fitness (reproducibility) of strategies affects the probability of a player choice. Fitness is determined in terms of average payoffs. The average payoffs of males with the s strategy and females with the s' strategy are (2) and (3), respectively

$$\pi_s^A(j) = a_{s1} \frac{j}{N} + a_{s2} \frac{N-j}{N}, \quad (2)$$

$$\pi_{s'}^B(i) = b_{1s'} \frac{i}{N} + b_{2s'} \frac{N-i}{N}. \quad (3)$$

The average payoffs of the entire population of males or females are given as

$$\bar{\pi}^A(i, j) = \pi_1^A(j) \frac{i}{N} + \pi_2^A(j) \frac{N-i}{N}. \quad (4)$$

$$\bar{\pi}^B(i, j) = \pi_1^B(i) \frac{j}{N} + \pi_2^B(i) \frac{N-j}{N}. \quad (5)$$

Then, for example, the frequency-dependent probability to choose for reproduction in the population of males of the player with the first strategy

$$P_1^A(i, j) = \frac{i}{N} \cdot \frac{1-w+w\pi_1^A(j)}{1-w+w\bar{\pi}^A(i, j)}, \quad (6)$$

where $1-w+w\pi_s^A(j)$ is the reproductive fitness of male with the first strategy, $1-w$ is the baseline fitness, and the small parameter w is called the selection strength [13]. When $w=0$, the probability of a player choice depends only on the frequency of the strategies. As w increases, the dependence of fitness on average payoffs becomes more and more significant.

Thus, according to the Moran process, an individual with a (currently) more successful strategy (that is, with a strategy that has a larger average payoff) will most likely be selected for the game and further reproduction.

Single game trajectories $i(m), j(m), m = 1, 2, 3, \dots$ are determined by simulating the Moran process. For multiple simulation of trajectories, the Monte Carlo approach is used. The game process starts from random initial conditions $i_0, j_0 \in \{1, \dots, N - 1\}$ and is considered until absorption or during a limited number of rounds M .

1.2. Determining the frequency and phase of oscillations. Stochastic cyclic oscillations of i and j are observed in the game «Battle of the Sexes» even that the respective mean field equations display a stable stationary state [14]. Such oscillations are transient, until a trajectory hits one of the absorbing boundaries and fixation of strategy happens. At the same time, the considerable duration of transient oscillations allows for defining their frequency and phase.

The mean oscillation period is defined as

$$\langle T \rangle = \frac{1}{R} \sum_{r=1}^R \langle T \rangle_r = \frac{1}{R} \sum_{r=1}^R \frac{m_{n_r} - m_{1_r}}{n_r - 1}, \quad (7)$$

where $\langle T \rangle_r$ is the mean period of game trajectory r ; m_{1_r} and m_{n_r} are the first and last time that trajectory r crossed the secant line $j^* = N/2$ from top to bottom; n_r is the number of returns of the trajectory r to the secant during the observation time $M = 500N$; R is the number of stochastic trajectories taken for averaging.

The mean oscillation frequency is then calculated as

$$\langle \Omega \rangle = \frac{2\pi}{\langle T \rangle}. \quad (8)$$

The instantaneous phase of oscillations of one game trajectory is determined as

$$\Phi(m) = 2\pi \frac{m - m_k}{m_{k+1} - m_k} + 2\pi k, \quad (9)$$

where $m_k \leq m < m_{k+1}$, $k = 1, 2, \dots$ and m_k is the time of the k -th top-down crossing of the secant line $j^* = N/2$.

1.3. System of coupled evolutionary games. We consider a system of coupled communities «Battle of the Sexes» located in the neighboring, but spacially distinct regions and interacting with each other through the migration of players. Now a single round of the game consists of two steps. In the first step, the birth-death Moran process is carried out independently in each community. At the second step, random players migrate between communities.

The coupling strength $0 \leq p \leq 1$ between communities determines probability of player migration. If $p = 0$, then there is no migration and the communities evolve independently. When $p = 1$, then the migration of players between communities occurs on each game round.

At each round, with probability p , only one player can migrate from the population of one community to the corresponding population of another community. We impose an additional condition to keep the population size constant throughout the game: if some player migrates from community C_1 to community C_2 , then some player from community C_2 must migrate to community C_1 .

Both males and females migrate with the same probability p . However, since the migration process is random, for some game round, migration can happen to for one of the populations (males or females only).

We consider non-identical migration-related communities C_1 and C_2 with the same population size N but different selection strength w , and explore 1:1 synchronization. The presence of

mutual synchronization is identified by two criteria [1]. First, the region of frequency locking is determined when the difference between the frequencies of two communities is close to zero

$$\langle \Omega_1 \rangle - \langle \Omega_2 \rangle \approx 0. \quad (10)$$

Second criterion, the phase locking condition is checked

$$|\varphi(m)| = |\Phi_1(m) - \Phi_2(m)| < \text{const}, \quad (11)$$

where $\varphi(m)$ is the phase difference (relative phase).

In a stochastic system, the phase difference fluctuates, therefore, the existence of phase synchronization in a statistical sense is indicated by the appearance of a peak in the distribution of the cyclic relative phase

$$\Psi(m) = \varphi(m) \bmod 2\pi. \quad (12)$$

Synchronization can also be characterized by other quantitative measures [15]. Here we calculate the first Fourier mode of the distribution (12):

$$\gamma = \sqrt{\langle \cos \Psi(m) \rangle^2 + \langle \sin \Psi(m) \rangle^2}, \quad (13)$$

where brackets denote averaging over time and over game trajectories. If the phases are not synchronized, then $\gamma = 0$, and in the case of complete synchronization in a system without noise, $\gamma = 1$. In stochastic systems, γ remains close to 1 in the synchronization regime and decreases with loss of synchronization.

We also consider spatially ordered ensembles of more than two coupled «Battle of the Sexes» communities, taking chains with open and periodic boundary conditions. In both cases, the same coupling strength p is established between the communities. Synchronization is assessed through the dependences of the mean frequencies on the coupling strength, more precisely, when they start coinciding.

2. Results

The transitional dynamics of the game «Battle of the Sexes» demonstrates stochastic cyclical oscillations in the number of players with the first strategy around the Nash equilibrium state [14]. An example of oscillations in the game with a population size of $N = 200$ is shown in Fig. 1, *a*, *b*. Within the framework of evolutionary game theory, the Nash equilibrium can be

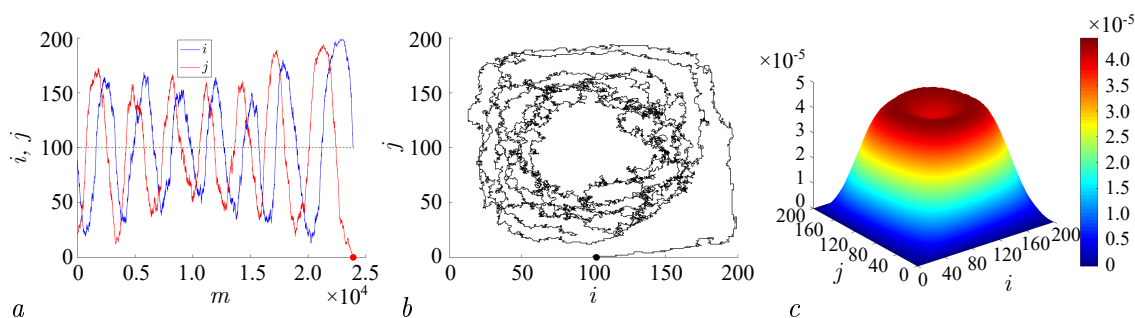


Fig 1. Transitional dynamics in the game «Battle of the Sexes»: *a* — change in the number of players with the first strategy in the male (i) and female (j) populations over time (m is the number of the game round), the dot denotes the exit of the system to the absorbing boundary; *b* — the game trajectory on the plane of quantities (i, j); *c* — quasi-stationary probability density distribution $d(i, j)$; parameters: $N = 200$, $w = 0.3$, $R = 10^5$

interpreted as a potential stable point of a dynamic game process [16]. For the considered model, given by the payoff bimatrix (1), the Nash equilibrium state is (0.5, 0.5). In this case the number of players that stick to the first and second strategies is the same in both populations (males and females) and equals $(\frac{N}{2}, \frac{N}{2})$.

Fig. 1, *c* shows the quasi-stationary probability density distribution $d(i, j)$, referring to the transient dynamics of the game. The distribution is obtained by multiple sampling of stochastic game trajectories that start from a random point and are simulated during $M = 500N$ rounds. The distribution is non-unimodal and crater-shaped, centered about the Nash equilibrium.

Amplitude, frequency, and lifetime of oscillations depend on the populations size N and the selection strength w . The oscillation frequency increases as the selection strength increases (Fig. 2), for example, in a game with a population size $N = 200$ the mean frequency for $w = 0.3$ is $\langle \Omega \rangle = 0.002$, and for $w = 0.4$ is $\langle \Omega \rangle = 0.003$. The lifetime of oscillations (the number of oscillation periods before absorption) also increases with selection strength.

Next, we address a system of migration-coupled communities that evolve according to the rules of the «Battle of the Sexes» game. We consider two non-identical communities C_1 and C_2 with the same population size ($N_1 = N_2 = 200$) but different selection strength ($w_1 = 0.3$, w_2 varies), which is equivalent to different natural oscillation frequencies. To determine mutual synchronization, we calculated the mean observed frequencies of communities $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$, and also investigated the difference in instantaneous phases $\varphi(m) = \Phi_1(m) - \Phi_2(m)$.

The resulting relative frequency difference $(\langle \Omega_1 \rangle - \langle \Omega_2 \rangle) / \langle \Omega_1 \rangle$ versus w_2 for different values of the coupling strength p are shown in Fig. 3, *a*. Its approach to zero corresponds to

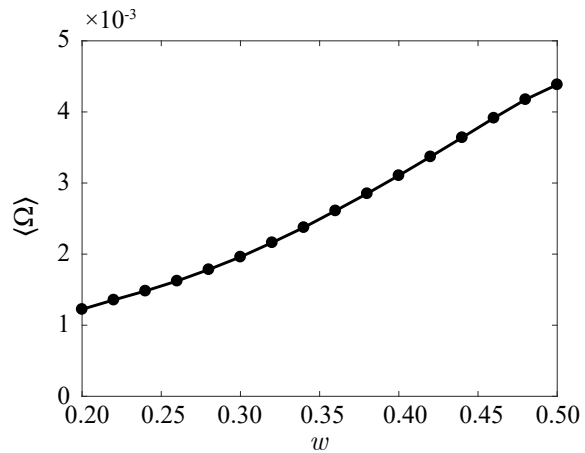


Fig 2. Dependence of the mean oscillation frequency on the value of the selection strength in the game «Battle of the Sexes»; parameters: $N = 200$, $R = 10^3$

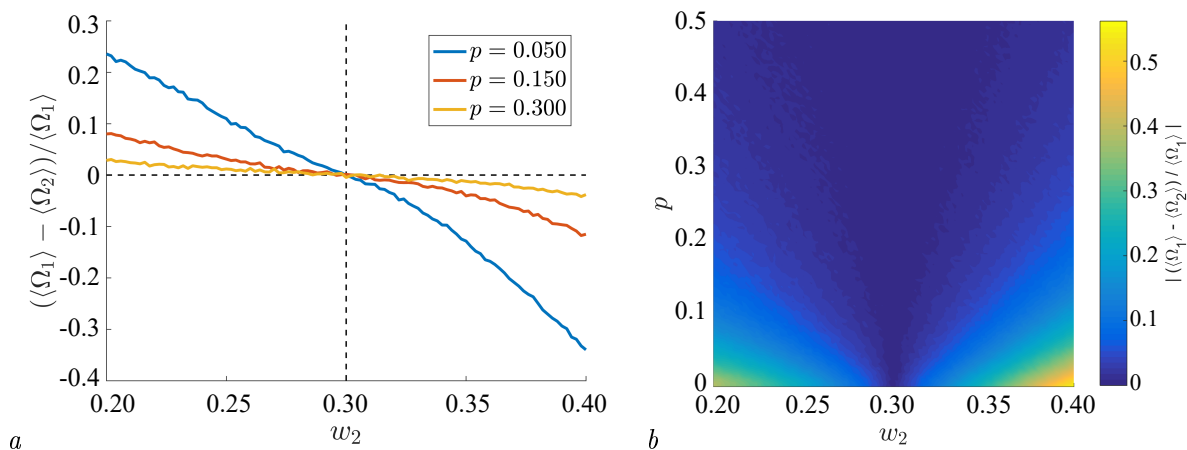


Fig 3. Frequency synchronization observed in the system of two coupled communities «Battle of the Sexes» C_1 and C_2 : *a* — the relative difference of the frequencies of C_1 and C_2 as a function of w_2 for a fixed value of $w_1 = 0.3$; *b* — the synchronization region (Arnold tongue); parameters: $N = 200$, $R = 10^3$ (color online)

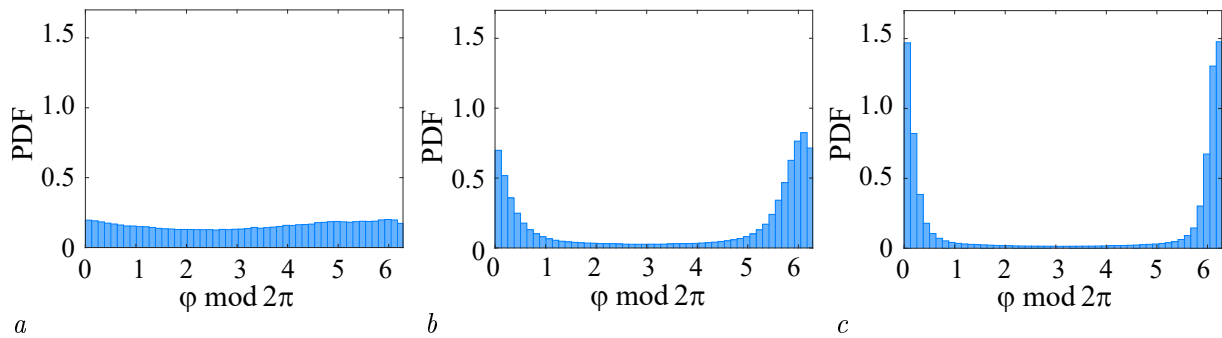


Fig 4. The distribution of phase difference between two coupled communities «Battle of the Sexes»: $a - p = 0.005$; $b - p = 0.1$; $c - p = 0.5$; parameters: $N = 200$, $w_1 = 0.3$, $w_2 = 0.32$, $R = 10^3$

synchronization (frequency locking). Fig. 3, *b* shows the synchronization region (Arnold tongue), the absolute value of relative frequency difference is color coded. Much as for the classical synchronization, the frequency locking region increases with the coupling strength.

Next, we investigate the dynamics of phases. The distributions of phase difference between communities are presented in Fig. 4. When the coupling strength is low, the distribution is broad and close to uniform (Fig. 4, *a*), typical of desynchronization. With an increase in the coupling strength (Fig. 4, *b, c*), it manifests a peak, which indicates phase locking.

To quantify phase synchronization, we calculated the first Fourier mode γ of the distribution of the cyclic phase difference. Different curves in Fig. 5 correspond to different pairs of coupled communities. Index γ is greater for communities with closer values of the selection strength ($w_1 = 0.3$, $w_2 = 0.32$, red curve), hence closer natural oscillation frequencies. Nevertheless, greater coupling strength ($p > 0.3$) provides a sufficiently high synchronization index ($\gamma > 0.7$) for more different communities ($w_1 = 0.3$, $w_2 = 0.22$ and $w_1 = 0.3$, $w_2 = 0.38$).

Fig. 6 shows examples of oscillations in the number of males with the first strategy in two coupled communities with $w_1 = 0.3$ and $w_2 = 0.32$. It can be seen that with an increase in the coupling strength between the communities, the oscillations begin to adjust, and with a sufficiently large strength, almost complete synchronization occurs.

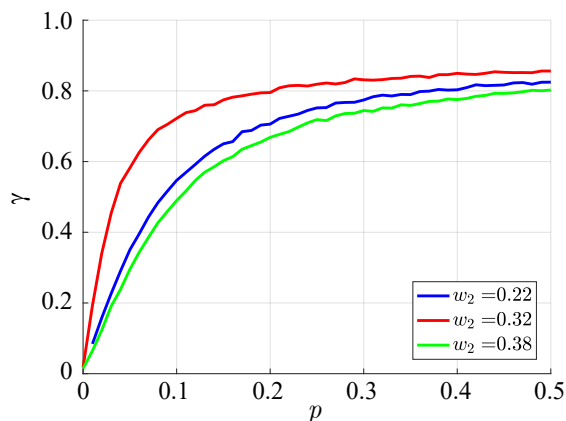


Fig 5. Dependence of the quantitative measure of phase synchronization (the first Fourier mode of the distribution of the cyclic phase difference) on the coupling strength between two communities «Battle of the Sexes»; parameters: $N = 200$, $w_1 = 0.3$, $R = 10^3$ (color online)

While the phase and frequency effects are quite in line with the classical results, the amplitude and lifetime dependences on coupling and synchronization represent the features, specific to the finite size evolutionary game. As specific trajectories demonstrate, the amplitude of oscillations decreases with increasing coupling and the onset of synchronization.

Whereas an analytical description this phenomenon is yet unclear, the numerical results can elucidate it in detail. We constructed quasi-stationary probability density distributions on the set of game states (i, j) for a number of coupling strengths and show its sections at $d(\frac{N}{2}, j)$ in Fig. 7 for two coupled communities with $w_1 = 0.3$ and $w_2 = 0.32$.

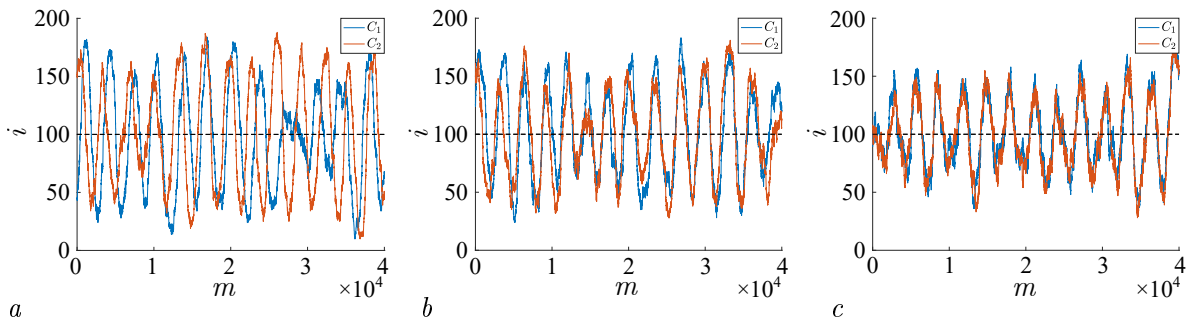


Fig 6. Change in the number of males with the first strategy in two coupled communities «Battle of the Sexes» over time (m is the number of the game round): $a - p = 0.005$; $b - p = 0.1$; $c - p = 0.5$; parameters: $N = 200$, $w_1 = 0.3$, $w_2 = 0.32$ (color online)

First, one observes that the well in the distribution becomes more shallow as the coupling strength increases to $p = 0.005$, although synchronization is not reached yet (cf. also Fig. 6, a). The onset of synchronization at greater p lead to (i) the two distributions becoming almost identical, and (ii) their radius decreases, indicative of a decrease in the oscillation amplitude.

The effect on the lifetime of oscillations is even more dramatic, as it increases in more than an order of magnitude (Fig. 8). Even a low coupling strength ($p = 0.01$), at which synchronization is not yet observed, increases the lifetime of the transient dynamics by about 6 times compared to a system without coupling ($p = 0$). Thus, the spatial migration of individuals protects populations from extinction. It should also be noted that the lifetime depends on the selection strength w , and synchronization effectively favors the quicker fixating community to keep on oscillating longer. Even, if extinction would occur in one of the communities, due to its persistence in the other community both strategies of behavior can coexist for longer time.

Ensembles of more than two coupled communities «Battle of the Sexes» have also been investigated. Arrays with two types of boundary conditions were considered, giving a chain and a ring. In the case of a chain structure, each community (except the first and last) is interacted with two nearest neighbors. The first and last communities are connected to only one neighbor.

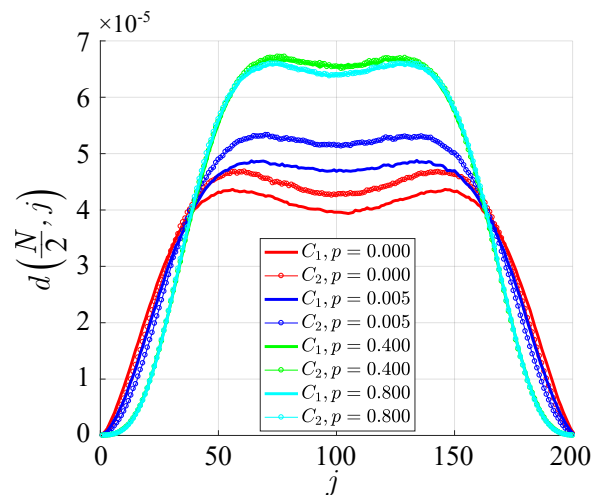


Fig 7. Quasi-stationary probability density distributions for two coupled communities «Battle of the Sexes»; parameters: $N=200$, $w_1=0.3$, $w_2=0.32$, $R=10^5$

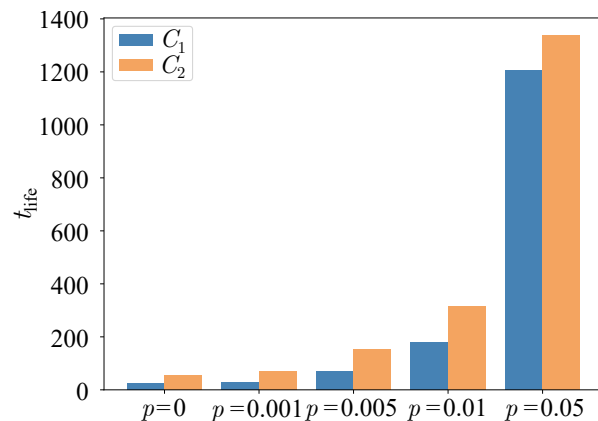


Fig 8. The lifetime of transient oscillations in the system of two coupled communities «Battle of the Sexes» depending on the coupling strength; parameters: $N=200$, $w_1=0.3$, $w_2=0.32$, $R=10^3$

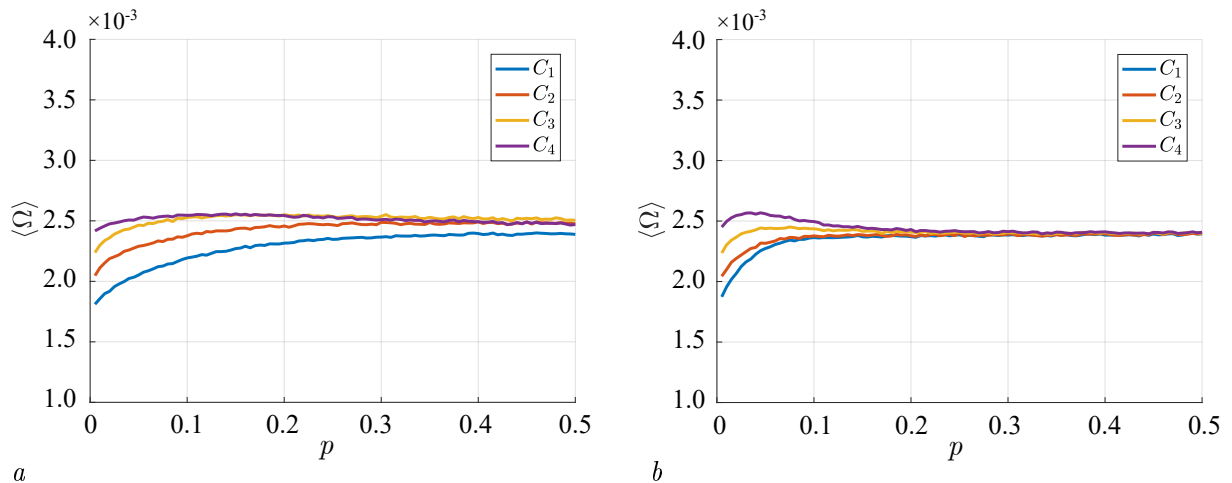


Fig 9. Dependence of the observed mean frequency of oscillations in the system of four coupled communities «Battle of the Sexes» on the coupling strength: *a* – chain structure; *b* – ring structure; parameters: $N = 200$, $w_1 = 0.28$, $w_2 = 0.3$, $w_3 = 0.32$, $w_4 = 0.34$, $R = 10^3$ (color online)

In the ring structure, one realizes periodic boundary conditions by an additional coupling between the first and the last community.

The results for an ensemble of four communities (C_1 with $w_1 = 0.28$, C_2 with $w_2 = 0.3$, C_3 with $w_3 = 0.32$ and C_4 with $w_4 = 0.34$) are shown in Fig. 9. The figure illustrates the dependence of the mean observed frequencies of communities on the coupling strength, which is the same between each pair of communities.

Notably, even the strongest coupling strength ($p = 0.5$) cannot synchronize all four communities in the chain (Fig. 9, *a*). In a particular system, a slight coupling allowed communities C_3 and C_4 to adjust their rhythms. A further increase in the coupling value led to the synchronization of the three subsystems (C_2 , C_3 and C_4), but the oscillation frequency of community C_1 remained different.

Converting it to the ring structure makes it possible to overcome this feature (Fig. 9, *b*). In this case, an increase in the coupling strength leads to synchronization of all four communities, despite their initial frequency detunings.

With an increase in the coupling strength in both structures, the phenomenon of cluster synchronization is observed. For example, in a ring structure, four communities are divided into two subgroups, called clusters, so that the communities within one cluster oscillate at the same frequency, but the frequencies of the two clusters differ. In a specific example, clusters C_1 and C_2 are formed, as well as C_3 and C_4 . However, an even greater increase in the coupling strength leads to almost complete synchronization of communities.

Conclusions

We investigated the mutual synchronization of transient oscillations in the system of discrete population evolutionary games, coupled through migration. Beside classical manifestations of synchronization, we demonstrated marked alterations in quasi-stationary distributions, amplitude and lifetime effects.

In particular, for a system of two coupled communities, the frequency and phase locking region was determined. The great coupling strength allows even considerably different communities to be synchronized. A similar result was obtained for an ensemble of more than two coupled

subsystems. With an increase in the coupling strength in the ring of mutually connected communities, we observed a transition from cluster synchronization to an almost complete synchronization. Adjustment of rhythms also took place in the chain of coupled communities, however, even the strong coupling strength did not allow all subsystems to oscillate synchronously.

A concluding remark concerns the other types of player migration, that we also addressed in numerics. In particular, we considered the case when the players with a less successful strategy are more likely to migrate. There an increase in the coupling strength between communities leads to a more rapid absorption, leading to the cessation of transient oscillations. In the opposite case, when players with a more successful strategy are more likely to migrate, the results are more similar to those reported in the manuscript for the random player migration. However, it should be noted that the synchronization region, as well as the lifetime of oscillations, become larger than for random migration.

The interaction of population communities through spatial migration can be important from an ecological point of view. Even if, due to random fluctuations, a population from a certain community is on the verge of extinction, the processes of migration and synchronization support the disappearing population. Thus, our results show that synchronization is important for maintaining the stability and coexistence of all phenotypes (game strategies). The conclusions obtained can be used in the study of biological rhythms of real world populations.

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