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Dynamics of full-coupled chains of a great number of oscillators with a large delay in couplings

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Abstract. *The subject* of this work is the study of local dynamics of full-coupled chains of a great number of oscillators with a large delay in couplings. From a discrete model describing the dynamics of a great number of coupled oscillators, a transition has been made to a nonlinear integro-differential equation, continuously depending on time and space variable. A class of full-coupled systems has been considered. The main assumption is that the amount of delay in the couplings is large enough. This assumption opens the way to the use of special asymptotic methods of study. The parameters under which the critical case is realized in the problem of the equilibrium state stability have been distinguished. It is shown that they have infinite dimension. The analogues of normal forms — nonlinear boundary value problems of Ginzburg–Landau type have been constructed. In some cases, these boundary value problems contain integral components too. Their nonlocal dynamics describes the behavior of all solutions of the original equations in the balance state neighbourhood. *Methods.* As applied to the considered problems, methods of constructing quasinormal forms on central manifolds are developed. An algorithm for constructing the asymptotics of solutions based on the use of quasinormal forms for determining slowly varying amplitudes has been created. *Results.* Quasinormal forms that determine the dynamics of the original boundary value problem have been constructed. The dominant terms of asymptotic approximations for solutions of the considered chains have been obtained. On the basis of the given statements, a number of interesting dynamical effects have been revealed. For example, an infinite alternation of direct and reverse bifurcations when the delay coefficient increases. Their distinguishing feature is that they have two or three spatial variables.

Keywords: boundary value problem, dynamics, delay, oscillators, normal form, stability.

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Introduction

First, let's consider a chain of N connected second-order oscillators with delayed connections

$$\ddot{u}_j + a\dot{u}_j + u_j + F(u_j, \dot{u}_j) = \sum_{k=1}^{N} a_k u_{j+k}(t-T), \tag{1}$$

$F(u,v)$ is a sufficiently smooth nonlinear function having an order of smallness above the first one at zero, $T > 0$ is delay, the index j varies from 1 to N and for any integer k the values of $u_{k+N}(t)$ are identified with $u_k(t)$. This type of model occurs in many applied problems of radiophysics [1–8], laser physics [9–13], mathematical ecology [14, 15], theory of neural networks [16–21] and see, for example, [22].

It is convenient to associate the values of $u_j(t)$ with the values of the function of two variables $u(t, x_j)$, where the points x_j $(j = 1, \ldots, N)$ are uniformly distributed on some circle.

We assume that the number of elements of the chain N is quite large:

$$N \gg 1. \tag{2}$$

This condition gives a reason to move from discrete with respect to $u(t, x_j)$ systems (1) to an equation with parameters distributed on the segment $[0, 2\pi]$ for the function $u(t, x)$

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u + F\left(u, \frac{\partial u}{\partial t}\right) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(s) u(t-T, x+s)ds \tag{3}$$

with periodic boundary conditions

$$u(t, x+2\pi) = u(t, x). \tag{4}$$

Function $\Phi(s)$ defines the couplings between the elements and, generally speaking, depends on the parameter ε. For example, in the case of one-sided [17] and diffusive [2] connections, the values of $\Phi(s)$ are concentrated in the vicinity of one or more points of the segment $[0, 2\pi]$. Here we assume that the function $\Phi(s)$ does not depend on the ε parameter. In this case, the chains are called full-coupled.

The main assumption that opens the way to the application of asymptotic methods is that the delay parameter T in (3) is large enough:

$$\varepsilon = T^{-1} \ll 1. \tag{5}$$

In the boundary value problem (3), (4), we will replace the time $t \to Tt$. As a result , we come to a singularly perturbed boundary value problem

$$\varepsilon^2\frac{\partial^2 u}{\partial t^2} + \varepsilon a\frac{\partial u}{\partial t} + u + F\left(u, \varepsilon\frac{\partial u}{\partial t}\right) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(s) u(t-1, x+s)ds, \tag{6}$$

$$u(t, x+2\pi) \equiv u(t, x). \tag{7}$$

Let us set the task of investigating the behavior of all solutions from some sufficiently small and independent of ε neighborhood of the zero equilibrium state in (6), (7) for sufficiently small values ε. When studying the local dynamics of (6), (7), the central place is occupied by the analysis of the stability of solutions of the linearized equation at zero

$$\varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \varepsilon a \frac{\partial u}{\partial t} + u = \frac{1}{2\pi} \int\limits_0^{2\pi} \Phi(s) u(t-1, x+s) ds \tag{8}$$

with boundary conditions (7).

Consider the characteristic equation for (8), (7)

$$\varepsilon^2 \lambda^2 + \varepsilon a \lambda + 1 = f_k \exp(-\lambda), \tag{9}$$

where f_k are the Fourier coefficients of the function $\Phi(s)$:

$$f_k = \frac{1}{2\pi} \int\limits_0^{2\pi} \Phi(s) \exp(-iks) ds, \quad k = 0, \pm 1, \pm 2, \ldots.$$

In the case when all the roots of the equation (9) for all $k = 0, \pm 1, \pm 2, \ldots$ have negative and real parts separated from zero for $\varepsilon \to 0$, solutions of the boundary value problem (8), (7) are asymptotically stable, and solutions (6), (7) with sufficiently small and independent of ε (according to the norm $C_{[0,2\pi]} \times C(R^2)$) initial conditions tend to zero at $t \to \infty$. If the equation (9) has a root with a positive and separated from zero at $\varepsilon \to 0$ real part, then the solutions (8), (7) are unstable and the problem of dynamics (6), (7) becomes non-local.

Here we will consider the critical case when in (9) there are no roots with a positive and zero-separated real part, but there are roots that tend to the imaginary axis at $\varepsilon \to 0$. Note that in the case of a finite dimension of the critical case, the methodology for studying local dynamics is well known. It relies on the method of integral manifolds and the method of normal forms (see, for example, [23–25]). A characteristic feature of all the problems considered below is the fact that they implement infinite-dimensional critical cases when infinitely many roots of the characteristic equation tend to the imaginary axis at $\varepsilon \to 0$. Therefore, the methods of integral manifolds and normal forms are not directly applicable. The approach developed in [17, 26–30] is substantially used here, related to the construction of infinite-dimensional quasi-normal forms.

The sections 1–3 present studies of the most important critical cases. As the main results, the so-called quasi-normal forms are constructed, the nonlocal dynamics of which determines the behavior of all solutions of the original boundary value problem (6), (7) in a small neighborhood of the equilibrium state.

The section 4 considers a significantly more complex case compared to the sections 1–3, which is implemented for a slightly different, but close to (6), (7) edge task

$$\varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \varepsilon a \frac{\partial u}{\partial t} + u + F\left(u, \varepsilon \frac{\partial u}{\partial t}\right) = \frac{1}{2\pi} \int\limits_0^{2\pi} \Phi(s) \Big[u(t-1, x+s) - \\ - u(t-1, x) \Big] ds, \quad u(t, x+2\pi) \equiv u(t, x). \tag{10}$$

The content of the section 5 is devoted to generalizing the results of the previous sections to a more general boundary value problem

$$\varepsilon^2\frac{\partial^2 u}{\partial t^2}+\varepsilon a\frac{\partial u}{\partial t}+u+F\left(u,\varepsilon\frac{\partial u}{\partial t}\right)=\frac{1}{2\pi}\int\limits_0^{2\pi}\Phi_1(s)u(t-1,x+s)ds+$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi}\Phi_2(s)\frac{\partial u(t-1,x+s)}{\partial t}ds, \qquad (11)$$

$$u(t,x+2\pi)\equiv u(t,x). \qquad (12)$$

Let's make one simplifying assumption about the nonlinear function $f(u,\dot u)$. It was said above that this function is quite smooth and has a zero order of magnitude higher than the first. It is technically easier to use the nonlinearity of $f(u,\dot u)$ of the third order of smallness, that is, for simplicity, we consider below that in(1)

$$f(u,\dot u)=b_1u^3+b_2u^2\dot u+b_3u\dot u^2+b_4\dot u^3. \qquad (13)$$

We will describe here the structure of all the following sections. It is the same for all of them. First, the parameters of the problem are distinguished, at which a critical case is observed in the problem of the stability of the zero state of equilibrium. Then the linearized boundary value problem is considered and its characteristic equation is given. After that, the asymptotics of all those roots of the characteristic equation whose real part tends to zero when the small parameter ε tends to zero is investigated. There are infinitely many such roots. On their basis, a set of special solutions is constructed for a linearized problem. Such solutions can be written in a form that allows their use for the analysis of solutions (with unknown amplitudes) of the original nonlinear boundary value problem. It is possible to determine an explicit form for the main approximation (by the parameter ε) of the corresponding solution. Let 's conditionally denote it here by εU_1. We then look for solutions to the nonlinear boundary value problem in the form

$$u(t,x,\varepsilon)=\varepsilon U_1+\varepsilon^3U_3+\dots.$$

Note that the absence of ε quadratic coefficients here is due to the fact that there is no quadratic nonlinearity in the original equation. With respect to U_3, it is known in advance that it is periodic by several of its arguments. Substituting the above expression instead of $u(t,x,\varepsilon)$, we come to a special linear inhomogeneous boundary value problem to determine U_3. The solvability condition of this boundary value problem in the specified class of functions allows us to write out an equation for unknown amplitudes included in U_1. Obtaining such equations is the ultimate goal. The nonlocal dynamics of these equations, also called quasi-normal forms, allows us to describe the local behavior of solutions to the original boundary value problem. Note that the fulfillment of the solvability conditions of the equations for U_3 allows us to explicitly define this function. Below we will use the function U_3, but sometimes we will not give formulas for this function for brevity.

1. Description of critical cases

Consider the characteristic equation (9). By f we denote the largest of the numbers $|f_k|$:

$$f=\max_{-\infty<k<\infty}|f_k|. \qquad (14)$$

Let's use the methodology of [26–28, 31]. Let us introduce into consideration the value

$$\gamma_0 = \begin{cases} 1, & \text{if } a^2 \geqslant 2, \\ \dfrac{a^2}{4}\left(4-a^2\right)^{1/2}, & \text{if } a^2 < 2. \end{cases}$$

Note that

$$\gamma_0 \leqslant 1 \text{ and } 0 < \gamma_0 < 1 \text{ for } a^2 < 2$$

and

$$\gamma_0 = 0 \text{ for } a = 0.$$

In [28] it is shown that the following statements hold.

Lemma 1. *Let the inequality be satisfied*

$$f < \gamma_0.$$

Then for all sufficiently small ε, all the roots of (9) have negative and real parts separated from zero for $\varepsilon \to 0$.

Lemma 2. *Let the inequality be satisfied*

$$f > \gamma_0.$$

Then for all sufficiently small ε, the equation (9) has a root with a positive and separated from zero at $\varepsilon \to 0$ real part.

Note that in the condition of the lemma 1, all solutions of the boundary value problem (8), (7) and (6), (7) from a small and independent of ε neighborhood of zero tend to zero at $t \to \infty$. If the conditions of Lemma 2 are met, then the problem of dynamics (6), (7) is non-local: there cannot be stable solutions in the neighborhood of the zero solution.

In the future, it is assumed that there is a critical case: for an arbitrary fixed γ_1, either the equality is fulfilled for the parameter f

$$f = \gamma_0 + \varepsilon^2\gamma_1, \tag{15}$$

or

$$f = -\gamma_0 - \varepsilon^2\gamma_1. \tag{16}$$

Thus, the boundary value problems (8), (7) and (6), (7) are investigated in the critical case.

Consider the question of the asymptotics of all those roots of the equation (9), the real parts of which tend to zero at $\varepsilon \to 0$. The corresponding asymptotic expansions are fundamentally different for the case when $a_0^2 > 2$ and for the case when $a_0^2 < 2$.

Lemma 3. *Let the condition be fulfilled*

$$a_0^2 > 2. \tag{17}$$

Then $\gamma_0 = 1$ and for those roots $\lambda_k(\varepsilon)$ $(k = 0, \pm 1, \pm 2, \ldots)$ the equations (9), whose real parts tend to zero at $\varepsilon \to 0$, have asymptotic equalities:

1) in case

$$f = 1 + \varepsilon^2\gamma_1 \tag{18}$$

we have

$$\lambda_k(\varepsilon) = 2\pi ik + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + o\big(\varepsilon^3\big), \quad k = 0, \pm1, \pm2, \ldots,$$

where

$$\lambda_{k1} = -2\pi ika, \quad \lambda_{k2} = -2\pi^2k^2\big(a^2-2\big) + 2\pi ika^2 + \gamma_1;$$

2) in case

$$f = -1 - \varepsilon^2\gamma_1 \tag{19}$$

we have

$$\lambda_k(\varepsilon) = i\pi\big(2k+1\big) + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + o\big(\varepsilon^3\big), \quad k = 0, \pm1, \pm2, \ldots,$$

where

$$\lambda_{k1} = -\pi\big(2k+1\big)a, \quad \lambda_{k2} = -\frac{1}{2}\pi^2\big(2k+1\big)^2\big(a^2-2\big) + i\pi\big(2k+1\big)a^2 + \gamma_1.$$

Let us then consider the case when

$$0 < a^2 < 2. \tag{20}$$

Let's introduce a few notations. Let $\theta = \theta(\varepsilon) \in [0, 1)$ be a value that complements the expression to an integer $\omega_0\varepsilon^{-1}$, where

$$\omega_0 = \begin{cases} 0, & \text{if } a^2 > 2, \\ \left(1 - \dfrac{a^2}{2}\right)^{1/2}, & \text{if } a^2 < 2. \end{cases}$$

Let's put

$$R_1 = (ia - 2\omega_0)\gamma_0^{-1}\exp\big(-i\Omega_0\big),$$

$$R_2 = \frac{1}{2}R_1^2 + \gamma_0^{-1}\exp\big(-i\Omega_0\big),$$

$$R_3 = \big(a - 2i\omega_0\big)R_1\gamma_0^{-1}\exp\big(-i\Omega_0\big).$$

Lemma 4. *Let the inequalities (20) be satisfied. Then for those roots $\lambda_k(\varepsilon)$ $(k = 0, \pm1, \pm2, \ldots)$ the equations (9), whose real parts tend to zero at $\varepsilon \to 0$, have asymptotic equalities*

$$\lambda_k(\varepsilon) = i\big(\omega_0\varepsilon^{-1} + \theta - \Omega_0 + 2\pi k\big) + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + o\big(\varepsilon^3\big),$$

where

$$\lambda_{k1} = R_1\big(\theta - \Omega_0 + 2\pi k\big),$$

$$\lambda_{k2} = R_2\big(\theta - \Omega_0 + 2\pi k\big)^2 + R_3\big(\theta - \Omega_0 + 2\pi k\big) + \gamma_1\exp\big(-i\Omega_0\big),$$

moreover , the ratios are fulfilled

$$\operatorname{Re} R_1 = 0, \quad \operatorname{Re} R_2 < 0. \tag{21}$$

2. Critical case on zero mode

Here we assume that

$$\gamma_0 = |f_0| \quad \text{and} \quad |f_0| > |f_k| \quad (k = \pm 1, \pm 2, \ldots). \tag{22}$$

We will conduct the study separately for the cases (17) and (20). We should immediately note that the corresponding results for them are significantly different. Under the condition (17), solutions are formed at frequencies of the order of 1, we will call them slowly oscillating. Under the condition (20), the solutions contain frequencies of the order of ε^{-1}, so let's call them rapidly oscillating.

2.1. Slowly oscillating solutions. Let the inequality (17) be satisfied first, that is, $a_0^2 > 2$ and $\gamma_0 = 1$. Then the roots of $\lambda_k(\varepsilon)$, which were discussed in the lemma 3, correspond to the solutions of $v_k(t, \varepsilon)$ of the linear boundary value problem (8), (7) and $v_k(t, \varepsilon) = \exp(\lambda_k(\varepsilon)t)$, which means that the solution of (8), (7) are the functions

$$v(t, \varepsilon) = \sum_{k=-\infty}^{\infty} \xi_k \exp\big(\lambda_k(\varepsilon)t\big),$$

where ξ_k are arbitrary complex constants. For the cases (15) and (16), this expression can be represented as

$$v(t, \varepsilon) = \sum_{k=-\infty}^{\infty} \xi_k(\tau) \exp\big(2\pi i k y\big) = \xi(\tau, y)$$

and

$$v(t, \varepsilon) = \sum_{k=-\infty}^{\infty} \xi_k(\tau) \exp\big(\pi i (2k+1) y\big) = \xi(\tau, y),$$

where $\tau = \varepsilon^2 t$, $y = (1 - \varepsilon a)t$, $\xi_k(\tau) = \xi_k \exp\big((\lambda_{k2} + o(\varepsilon))\tau\big)$.

Solutions $u(t, y, \varepsilon)$nonlinear boundary value problem (6) and (7), "close to critical" solutions of $v(t, \varepsilon)$ linear boundary value problem (8), (7), we are looking for in the form

$$u(t, y, \varepsilon) = \varepsilon^{1/2}\xi(\tau, y) + \varepsilon^{3/2}u_3(\tau, y) + o(\varepsilon^2), \tag{23}$$

where $\xi(\tau, y)$ is an unknown real function for which the conditions are met:

1) in case (15) — periodicity condition by y

$$\xi(\tau, y + 1) \equiv \xi(\tau, y); \tag{24}$$

2) in case (16) — antiperiodicity condition by y

$$\xi(\tau, y + 1) \equiv -\xi(\tau, y). \tag{25}$$

Let's make a formal series (23) into the boundary value problem (8), (7) and equate the coefficients at the same degrees ε. Then, collecting the coefficients for $\varepsilon^{3/2}$, we get the equation for $u_3(\tau, y)$. From the condition of its solvability, we conclude that $\xi(\tau, y)$ is the solution of the boundary value problem

$$\frac{\partial \xi}{\partial \tau} = \Big(\frac{a^2}{2} - 1\Big)\frac{\partial^2 \xi}{\partial y^2} - a^2 \frac{\partial \xi}{\partial y} + \gamma_1 \xi - b_1 \xi^3 \tag{26}$$

with boundary conditions, respectively (24) or (25). Let's formulate the final result.

Theorem 1. *Let the conditions (17), (18)((17), (19)) and $b_1 \neq 0$ be met. Let the function $\xi(\tau, y)$ be bounded at $\tau \to \infty$, $y \in [0,1]$ by the solution of the boundary value problem (26), (24)((26) and (25)). Then the function*

$$u(t, y, \varepsilon) = \varepsilon\xi(\tau, y) + \varepsilon^3 u_3(\tau, y) \tag{27}$$

for $\tau = \varepsilon^2 t$, $y = (1 - \varepsilon a)t$ satisfies the boundary value problem (6), (7) up to $o(\varepsilon^3)$.

It follows from this theorem that under the formulated conditions, the constructed boundary value problems (26), (24) and (26), (25) play the role of normal forms for the boundary value problem (6), (7). Note that if $b_1 = 0$, that is, the condition of the theorem 1 is not satisfied, and $b_2 \neq 0$, then the changes are small. The last term in (26) is replaced by $-b_2\xi^2\partial\xi/\partial y$ and the formula (27) takes the form

$$u(t, y, \varepsilon) = \varepsilon^{1/2}\xi(\tau, y) + \varepsilon^{3/2}u_3(\tau, y).$$

2.2. Fast oscillating solutions. Let the inequality (20) be satisfied, that is, $a^2 < 2$. The roots of $\lambda_k(\varepsilon)$ $(k = 0, \pm1, \pm2, \ldots)$, which were mentioned in the lemma 4, correspond to the solutions of $v_k(t, \varepsilon)$ of the linear boundary value problem (8), (7) $v_k(t, \varepsilon) = \exp(\lambda_k(\varepsilon)t)$. Hence the function

$$v(t, \varepsilon) = \sum_{k=-\infty}^{\infty} \xi_k \exp(\lambda_k(\varepsilon)t), \tag{28}$$

where ξ_k are arbitrary complex constants also satisfy the boundary value problem (8), (7). Given the asymptotic formulas for $\lambda_k(\varepsilon)$ presented in the lemma 4, the expression (28) can be written as

$$v(t, \varepsilon) = E(t, \varepsilon)\xi(\tau, y).$$

Here $E(t, \varepsilon) = \exp\left[\big(i(\omega_0\varepsilon^{-1} + \theta - \Omega_0) + \varepsilon R_1(\theta - \Omega_0)\big)t\right]$,

$$\xi(\tau, y) = \sum_{k=-\infty}^{\infty} \xi_k(\tau)\exp(2\pi iky),\ \tau = \varepsilon^2 t,\ y = (1 + i\varepsilon R_1)t,$$

$\xi_k(\tau) = \xi_k \exp\big((\lambda_{k2} + o(\varepsilon))\tau\big)$. Recall that according to (21), the value of iR_1 is real.

Solutions $u(t, y, \varepsilon)$ of the nonlinear boundary value problem (6), (7) in the case under consideration, we are looking for in the form of a formal series

$$u(t, y, \varepsilon) = \varepsilon\big(\xi(\tau, y)E(t, \varepsilon) + \bar{\xi}(\tau, y)\bar{E}(t, \varepsilon)\big) + \varepsilon^3 u_3(t, \tau, y) + \ldots. \tag{29}$$

In this expression, $\xi(\tau, y)$ is an unknown complex function to be defined, which is 1-periodic in the spatial variable y:

$$\xi(\tau, y + 1) \equiv \xi(\tau, y), \tag{30}$$

and the dependence on the argument t of the function u_3 is periodic.

Put (29) in (6), (7) and we will equate the coefficients at the same degrees ε. Then, collecting the coefficients for ε^3, we come to the equation with respect to u_3. The condition for the solvability of this equation in the class of periodic t functions consists in the fulfillment of the equality

$$\frac{\partial\xi}{\partial\tau} = -R_2\frac{\partial^2\xi}{\partial y^2} + R_4\frac{\partial\xi}{\partial y} + R_5\xi - \xi^2\frac{\partial\bar{\xi}}{\partial y} - \\ - \& - \big(3b_1 + i\omega_0 b_2 + \omega_0^2 b_3 + 3i\omega_0^3 b_4\big)\exp(-i\Omega_0)\xi|\xi|^2, \tag{31}$$

in which $R_4 = -i(2R_2(\theta - \Omega_0) + R_3)$, $R_5 = R_2(\theta - \Omega_0)^2 + R_3(\theta - \Omega_0) + \gamma_1 \exp(-i\Omega_0)$.

Let's introduce the notation. Here and below via $\varepsilon_n = \varepsilon_n(\theta_0)$ we will denote such a sequence $\varepsilon_n \to 0$ for which $\theta(\varepsilon_n) = \theta_0$. Let's formulate the main result.

Theorem 2. *Let the conditions (20) and (15) be met and let for an arbitrarily fixed value $\theta_0 \in [0, 2\pi)$ the boundary value problem (31), (30) has a bounded at $\tau \to \infty$, $y \in [0, 1]$ solution $\xi(\tau, y)$. Then on the sequence $\varepsilon_n = \varepsilon_n(\theta_0)$ function*

$$u(t, y, \varepsilon_n(\theta_0)) = \varepsilon_n(\theta_0)\big[\xi(\tau, y)E\big(t, \varepsilon_n(\theta_0)\big) + \bar{\xi}(\tau, y)\bar{E}\big(t, \varepsilon_n(\theta_0)\big)\big] + \varepsilon_n^3(\theta_0)u_3(t, \tau, y)$$

for $\tau = \varepsilon_n^2(\theta_0)t$, $y = (1 - \varepsilon_n(\theta_0)a)t$ satisfies the boundary value problem (6), (7) up to $o(\varepsilon_n^3(\theta_0))$.

Thus, the boundary value problem (31), (30) is a quasi-normal form for (6), (7) in the critical case under consideration. Due to the conditions (21), this boundary value problem is parabolic. The structure of its solutions can be complex. This boundary value problem defines the main terms of asymptotic approximations of solutions of the original boundary value problem (6), (7). It follows that the local dynamics (6), (7) can also be complex. Note also that the cubic nonlinearities of the equation (31) are more complex in form than in the classical Ginzburg-Landau equation.

2.3. A system with small coupling coefficients. It was shown above that the critical case for the boundary value problem (6), (7) at $a = 0$ is realized at $f = 0$. Here we assume that the coefficients a and f are sufficiently small, that is, for some a_1 and γ_1 we have the relations

$$a = \varepsilon a_1, \quad f = \varepsilon\gamma_1 \quad (0 < \varepsilon \ll 1). \tag{32}$$

Below we consider a problem of exactly this type, that is, with boundary conditions (7) we investigate the equation

$$\frac{\partial^2 u}{\partial t^2} + \varepsilon a_1 \frac{\partial u}{\partial t} + u + f(u, \dot{u}) = \frac{\varepsilon}{2\pi}\int_0^{2\pi} \Phi(s)u\big(t - \varepsilon^{-1}, x + s\big)ds. \tag{33}$$

For a linearized equation at zero , the characteristic quasipolynomial has the form

$$\lambda^2 + \varepsilon a_1\lambda + 1 = \varepsilon f_k \exp\big(-\lambda\varepsilon^{-1}\big).$$

Infinitely many roots $\lambda_k(\varepsilon)$ $(k = 0, \pm 1, \pm 2, \ldots)$ of this characteristic equation tends to the imaginary axis at $\varepsilon \to 0$ and there is no root with a positive and separated from zero at $\varepsilon \to 0$ real part. Thus, in the problem of the stability of the null solution (33) a critical case of infinite dimension is realized. For $\lambda_k(\varepsilon)$ asymptotic representations take place

$$\lambda_k(\varepsilon) = i + \varepsilon\lambda_{k1} + \ldots,$$

and λ_{k1} is the root of a quasi-polynomial

$$\lambda_{k1} + \frac{1}{2}a_1 = -\frac{i}{2}\gamma_1 \exp\big(i\theta - T_1\lambda_{k1}\big). \tag{34}$$

The value $\theta = \theta(\varepsilon) \in [0, 2\pi)$ appearing in (34) complements the value of $T\varepsilon^{-1}$ to an integer multiple of 2π. Note that the quasi-polynomial (34) has infinitely many roots.

We are looking for solutions to a nonlinear equation in the form of a formal series

$$u = \varepsilon^{1/2}\big(\xi(\tau, x)\exp(it) + \bar{\xi}(\tau, x)\exp(-it)\big) + \varepsilon^{3/2}u_3(t, \tau, x) + \ldots, \tag{35}$$

in which the dependence on the argument $t-2\pi$ is periodic. Substitute (35) into (33). Performing standard actions, we come to the equation for u_3. From the condition of its solvability in the class of 2π-periodic t functions, we obtain an equation with a fixed delay for determining an unknown complex amplitude $\xi(\tau, x)$:

$$\frac{\partial \xi}{\partial \tau} = -\frac{1}{2} a_1 \xi - \frac{i}{2} \exp(i\theta) \cdot \frac{1}{2}\pi \int\limits_0^{2\pi} \Phi(s) \xi\big(\tau - 1, x + s\big) ds + \\ + \frac{1}{2}\big(3ib_1 - b_2 + ib_3 - 3b_4\big) \xi |\xi|^2. \tag{36}$$

Let's formulate the main result.

Theorem 3. *Let the condition (32) be met and for an arbitrarily fixed value $\theta_0 \in [0, 2\pi)$ by the equation (36) has a solution $\xi(\tau, x)$ bounded at $\tau \to \infty$. Then on the sequence $\varepsilon_n = \varepsilon_n(\theta_0)$ function*

$$u(t, x, \varepsilon_n) = \varepsilon_n^{1/2}\big(\xi(\tau, x)\exp(it) + \bar{\xi}(\tau, x)\exp(-it)\big) + \varepsilon_n^{3/2} u_3(t, \tau, x)$$

for $\tau = \varepsilon_n t$ satisfies the equation (33) with an accuracy of $o(\varepsilon_n^{3/2})$.

It follows from this theorem that the distributed equation (36) is a quasi-normal form in the case under consideration. For the equation (36), it is simple to investigate questions about the existence and stability of the simplest cycles of the form $\rho \exp(i\sigma\tau)$. We do not dwell on this in more detail.

3. A critical case on a non-zero mode

Here we assume that the largest modulo the Fourier coefficient f_k of the function $\Phi(s)$ has a non-zero number k_0, that is

$$f = \max_{-\infty < k < \infty} |f_k| = |f_{k_0}|, \quad k_0 \neq 0$$

and

$$|f_{k_0}| > |f_k| \quad \text{at} \quad k \neq \pm k_0.$$

Consider the critical case in the problem of stability of the zero equilibrium state of the boundary value problem (6), (7), when for some δ

$$f_{k_0} = \big(\gamma_0 + \varepsilon^2 \gamma_1\big) \exp(i\delta). \tag{37}$$

We give asymptotic formulas for all those roots of the characteristic equation (9) whose real parts tend to zero at $\varepsilon \to 0$.

Recall that the characteristic equation (9) is obtained by substituting Euler solutions in (8)

$$u_k^{\pm}(t, x, \varepsilon) = \exp\big[\pm ikx + \lambda_k(\varepsilon) t\big]. \tag{38}$$

To find the roots of $\lambda_k(\varepsilon)$, we come to the equation

$$\varepsilon^2 \lambda^2 + \varepsilon a \lambda + 1 = \big(\gamma_0 + \varepsilon^2 \gamma_1\big) \exp\big(\pm i\delta - \lambda\big). \tag{39}$$

From here we get the asymptotics $\lambda_k^{\pm}(\varepsilon)$:

1) for $a^2 > 2$ we have $\lambda_k^-(\varepsilon) = \bar{\lambda}_k^+(\varepsilon)$ and $\lambda_k^+(\varepsilon) = i\delta + 2\pi ki + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + \ldots$, where

$$\lambda_{k1} = -ia\big(2\pi k + \delta\big),$$
$$\lambda_{k2} = \frac{1}{2}\big(2 - a^2\big)\big(\delta + 2\pi k\big)^2 + \gamma_1 + ia^2\big(\delta + 2\pi k\big);$$

2) for $a^2 < 2$ we have $\lambda_k^{\pm}(\varepsilon) = i(\omega_0/\varepsilon + \theta \pm \delta - \Omega_0 + 2\pi k) + \varepsilon\lambda_{k1}^{\pm} + \varepsilon^2\lambda_{k2}^{\pm} + \ldots$, where

$$\lambda_{k1}^{\pm} = -i\big(\gamma_0 \exp(i\Omega_0)\big)^{-1}\big(2i\omega_0 - a\big)K_k^{\pm}, \quad K_k^{\pm} = \theta \pm \delta - \Omega_0 + 2\pi k,$$
$$\lambda_{k2}^{\pm} = \frac{1}{2}\big(\lambda_{k1}^{\pm}\big)^2 - \gamma_1\gamma_0^{-1} - \big(R_k^{\pm}\big)^2 - a\lambda_{k1}^{\pm} - 2i\omega_0\lambda_{k1}^{\pm} = A^{\pm}\big(K^{\pm}\big)^2 + B^{\pm}K^{\pm} + \gamma_1\gamma_0^{-1},$$
$$K^{\pm} = \theta - \Omega_0 + 2\pi k \pm \delta,$$
$$A^{\pm} = \Big(\gamma_0 \exp\big(i(\Omega_0 \mp \delta)\big)\Big)^{-1} - \frac{1}{2}\Big((a - 2i\omega)\big(\gamma_0 \exp[i(\Omega_0 \mp \delta)]\big)^{-1}\Big)^2,$$
$$B^{\pm} = -i\big(a^2 + 4\omega^2\big)\big(\gamma_0 \exp[i(\Omega_0 \mp \delta)]\big)^{-1}.$$

Note that $\operatorname{Re}\lambda_{k1}^{\pm} = \operatorname{Re}\lambda_{k1} = 0$, $\operatorname{Re}\lambda_{k2} < 0$ and $\operatorname{Re}\lambda_{k2}^{\pm} < 0$.

3.1. The case $a^2 > 2$. In this case, the constructions in a critical situation on a non-zero mode differ little from the constructions of the section 2. We are looking for solutions to the nonlinear boundary value problem (6), (7) in the form

$$u(t,x,\varepsilon) = \varepsilon\Big(\xi(\tau,y)\exp\big[i\delta(1-\varepsilon a)t + ik_0x\big] + \bar{\xi}(\tau,y)\exp\big[-i\delta(1-\varepsilon a)t - ik_0x\big]\Big) + \varepsilon^3 u_3(t,\tau,x,y) + \ldots, \quad (40)$$

where by t, x and y the dependence is periodic, $\tau = \varepsilon^2 t, y = (1-\varepsilon a)t$. Substituting (40) into (6), (7) and performing standard actions, we get the following result.

Theorem 4. *Let $k_0 \neq 0$ and $a^2 > 2$ and let $\xi(\tau, y)$ be bounded at $\tau \to \infty, y \in [0,1]$ by the solution of the boundary value problem*

$$\frac{\partial\xi}{\partial\tau} = \frac{1}{2}(a^2 - 2)\frac{\partial^2\xi}{\partial y^2} + \big(a^2 + i\delta(a^2 - 2)\big)\frac{\partial\xi}{\partial y} + \Big(\frac{1}{2}\delta^2(2 - a^2) + \gamma_1 + ia^2\delta\Big)\xi - 3b_1\xi|\xi|^2, \quad (41)$$

$$\xi(\tau, x + 2\pi, y) \equiv \xi(\tau, x, y) \equiv \xi(\tau, x, y + 1). \quad (42)$$

Then the function

$$u(t,x,\varepsilon) = \varepsilon\Big(\xi(\tau,y)\exp\big[i\delta(1-\varepsilon a)t + ik_0x\big] + \bar{\xi}(\tau,y)\exp\big[-i\delta(1-\varepsilon a)t - ik_0x\big]\Big) + \varepsilon^3 u_3(t,\tau,x,y) \quad (43)$$

satisfies the boundary value problem (6), (7) up to $o(\varepsilon^3)$.

Note that for $b_1 = 0$ and $b_2 \neq 0$, the last term in the equation (41) is replaced by

$$- - b_2\Big[i\delta\xi|\xi|^2 + 3\xi^2\frac{\partial\bar{\xi}}{\partial y} + 2|\xi|^2\frac{\partial\xi}{\partial y}\Big],$$

and the asymptotics of solutions in (43) goes not by integer powers of ε, but by degrees $\varepsilon^{1/2}$.

3.2. The case $a^2 < 2$. Here, unlike the results of the section 2.2, not one, but two chains of roots are involved $\lambda_k^+(\varepsilon)$ and $\lambda_k^-(\varepsilon)$. First we introduce the notation. Let's put

$$E^{\pm} = \exp\left[ik_0x + i(\omega_0\varepsilon^{-1} + \theta - \Omega \pm \delta)t\right].$$

We are looking for solutions to the nonlinear boundary value problem (6), (7) in the form

$$u(t,x,\varepsilon) = \varepsilon\big(\xi^+(\tau,y)E^+ + \bar{\xi}^+(\tau,y)\bar{E}^+ + \xi^-(\tau,y)E^- + \bar{\xi}^-(\tau,y)\bar{E}^-\big)+ \\ +\varepsilon^3 u_3(t,\tau,x,y) + \ldots, \tag{44}$$

where the dependence on t, x and y is periodic. After substituting (44) into (6) and after standard actions, we obtain a parabolic boundary value problem for determining unknown amplitudes $\xi^{\pm}(\tau,y)$

$$\frac{\partial\xi^{\pm}}{\partial\tau} = -A^{\pm}\frac{\partial^2\xi^{\pm}}{\partial y^2} - i\big(2A^{\pm}(\theta - \Omega_0 \pm \delta) + B^{\pm}\big)\frac{\partial\xi^{\pm}}{\partial y} + \\ + A^{\pm}(\theta - \Omega_0 \pm \delta)^2 - B^{\pm}(\theta - \Omega_0 \pm \delta) + \gamma_1\gamma_0^{-1} + \\ + 3\xi^{\pm}\big(|\xi^{\pm}|^2 + 2|\xi^{\mp}|^2\big)\big(b_1 - \omega_0^2 b_3 - ib_4\big) + i\omega_0 b_2\xi^{\pm}|\xi^{\pm}|^2 \tag{45}$$

with boundary conditions

$$\xi^{\pm}(\tau, y+1) \equiv \xi^{\pm}(\tau,y). \tag{46}$$

We formulate the obtained statement in the form of a theorem.

Theorem 5. *Let $0 < a^2 < 2$ and let for an arbitrarily fixed value $\theta_0 \in [0, 2\pi)$ the boundary value problem (45), (46) has a bounded at $\tau \to \infty$, $y \in [0,1]$ solution $\xi(\tau,y)$. Then on the sequence $\varepsilon_n = \varepsilon_n(\theta_0)$ function*

$$u(t,x,\varepsilon_n) = \varepsilon_n\big(\xi^+(\tau,y)E^+ + \bar{\xi}^+(\tau,y)\bar{E}^+ + \xi^-(\tau,y)E^- + \bar{\xi}^-(\tau,y)\bar{E}^-\big) + \varepsilon_n^3 u_3(t,\tau,x,y),$$

where $\tau = \varepsilon_n^2 t$, $y = (1 - \varepsilon_n a)t$, satisfies the boundary value problem (6), (7) up to $o(\varepsilon_n^3)$.

4. Critical cases in the boundary value problem (10)

Fundamentally new effects can occur in a situation where critical cases are realized simultaneously on an infinite set of modes. Let's consider such a situation on one of the most common examples, when in the boundary value problem (10) we have

$$\Phi(s) \equiv \text{const} \equiv \mathrm{f}_0. \tag{47}$$

We linearize this boundary value problem at zero. As a result , we obtain a linear equation

$$\varepsilon^2\frac{\partial^2 u}{\partial t^2} + \varepsilon a\frac{\partial u}{\partial t} + u = f_0\left[\frac{1}{2\pi}\int\limits_0^{2\pi} u(t-1, x+s)ds - u(t-1,x)\right] \tag{48}$$

with periodic boundary conditions

$$u(t, x+2\pi) \equiv u(t,x). \tag{49}$$

The characteristic equation for (48), (49) has the form

$$\varepsilon^2\lambda^2 + \varepsilon a\lambda + 1 = f_k \exp(-\lambda), \quad k = 0, \pm 1, \pm 2, \ldots, \tag{50}$$

where

$$f_k = \begin{cases} 0, & \text{if } k = 0, \\ -f_0, & \text{if } k = \pm 1, \pm 2, \ldots . \end{cases} \tag{51}$$

We fix arbitrarily γ_1 and assume

$$f_0 = \gamma_0 + \varepsilon^2\gamma_1. \tag{52}$$

Under this condition, an infinite-dimensional critical case takes place for all but one of the equations in (50). Following the methodology developed above, we are looking for solutions to the nonlinear boundary value problem (10) in the form of a formal series:

1) provided $a^2 > 2$ we have $\gamma_0 = 1$ and

$$u(t, x, \varepsilon) = \varepsilon\xi(\tau, x, y) + \varepsilon^3 u_3(\tau, x, y) + \ldots \tag{53}$$

2) provided $a^2 < 2$ we have

$$\gamma_0 = \frac{a^2}{4}(4 - a^2)^{1/2}$$

and

$$\begin{aligned} u(t, x, \varepsilon) \ &= \varepsilon\Big(\xi(\tau, x, y) \exp\big[i(\omega_0\varepsilon^{-1} + \theta + \delta - \Omega_0)t\big] + \\ &+ \bar{\xi}(\tau, x, y) \exp\big[-i(\omega_0\varepsilon^{-1} + \theta + \delta - \Omega_0)t\big]\Big) + \\ &+ \varepsilon^3 u_3(t, \tau, x, y) + \ldots \end{aligned} \tag{54}$$

In (53) and(54) $\tau = \varepsilon^2 t$, $\ y = (1 - \varepsilon a)t$, the dependence on x —2π is periodic, from y — 1-antiperiodic, from t to (54) – $2\pi/\omega_0$ is periodic, where $\omega_0 = (1 - a^2/2)^{1/2}$. By virtue of the condition (51), a condition must be imposed on the function $\xi(\tau, x, y)$

$$M_x(\xi) = 0, \quad \text{где} \quad M_x(\xi) = \frac{1}{2\pi}\int\limits_0^{2\pi} \xi(\tau, x, y)dx. \tag{55}$$

Substitute the expression (54) into (10). Performing standard actions, we obtain an equation with respect to u_3. From the condition of its solvability in the specified class of functions , we obtain the equality:

1) for $a^2 > 2$

$$\frac{\partial\xi}{\partial\tau} = \left(\frac{a^2}{2} - 1\right)\frac{\partial^2\xi}{\partial y^2} - a^2\frac{\partial\xi}{\partial y} + \gamma_1\xi - b_1\big(\xi^3 - M_x(\xi^3)\big), \tag{56}$$

$$\xi(\tau, x, y + 1) \equiv -\xi(\tau, x, y), \quad \xi(\tau, x + 2\pi, y) \equiv \xi(\tau, x, y); \tag{57}$$

2) for $a^2 < 2$ (in the notation of the section 2.2)

$$\frac{\partial\xi}{\partial\tau} = -R_2\frac{\partial^2\xi}{\partial y^2} + R_4\frac{\partial\xi}{\partial y} + R_5\xi - \xi^2\frac{\partial\bar{\xi}}{\partial y} - 2|\xi|^2\frac{\partial\xi}{\partial y} - M\left(\xi^2\frac{\partial\bar{\xi}}{\partial y} + 2|\xi|^2\frac{\partial\xi}{\partial y}\right) \tag{58}$$

with conditions (57).

Here are the final statements.

Theorem 6. *1) Let $a^2 > 2$ and let $\xi(\tau, x, y)$— bounded at $\tau \to \infty, x \in [0, 2\pi], y \in [0, 1]$ be the solution of the boundary value problem (56), (57). Then the function*

$$u(t, x, \varepsilon) \;=\; \varepsilon\xi(\tau, x, y) + \varepsilon^3 u_3(t, \tau, x, y)$$

satisfies the boundary value problem (48), (49) for $\tau = \varepsilon^2 t, y = (1 - \varepsilon a)t$ up to $o(\varepsilon^3)$.

2) Let $0 < a^2 < 2$ and let $\xi(\tau, x, y)$— bounded at $\tau \to \infty, x \in [0, 2\pi], y \in [0, 1]$ be the solution of the boundary value problem (58), (57). Then the function

$$u(t, x, \varepsilon) \;= \varepsilon\Big(\xi(\tau, x, y) \exp\big[i(\omega_0\varepsilon^{-1} + \theta + \delta - \Omega_0)t\big]+ \\ + \bar{\xi}(\tau, x, y) \exp\big[-i(\omega_0\varepsilon^{-1} + \theta + \delta - \Omega_0)t\big]\Big) + \varepsilon^3 u_3(t, \tau, x, y)$$

satisfies the boundary value problem (48), (49) for $\tau = \varepsilon^2 t, y = (1 - \varepsilon a)t$ up to $o(\varepsilon^3)$.

Thus, the boundary value problems (56), (57) and (58), (57) are quasi-normal forms for (48), (49). Note that the presence in the equations (56) and (58) of integral terms in a spatial variable allows us to explicitly find solutions smooth in τ and y and stepwise in x. These issues have been studied in detail in [32–34], so we do not dwell on them here.

5. About one generalization of the results

Let us briefly consider the more general boundary value problem (11), (12). We write out the characteristic equation for its linearization at zero

$$\varepsilon^2\lambda^2 + \varepsilon a\lambda + 1 = (f_{k1} + \varepsilon\lambda f_{k2}) \exp(-\lambda), \quad k = 0, \pm 1, \pm 2, \dots, \tag{59}$$

where f_{k1} and f_{k2} are the coefficients of the decomposition of the functions $\Phi_1(s)$ and $\Phi_2(s)$, respectively, in the Fourier series. Put in (59) $\lambda = i\omega\varepsilon^{-1}$. As a result , we come to the equation

$$\big(1 - \omega^2 + ia\omega\big)\big(f_{k1} + i\omega f_{k2}\big)^{-1} = \exp\big(-i\omega\varepsilon^{-1}\big). \tag{60}$$

Let's study the question of the existence of a real root ω_0 in the equation (60). Denote by $p_k(\omega)$ function

$$p_k(\omega) = |1 - \omega^2 + ia\omega| \cdot |f_{k1} + i\omega f_{k2}|^{-1}.$$

This function grows indefinitely at $\omega \to \pm\infty$. Therefore, there is such a ω_0 that

$$p_k(\omega_0) = \min_{-\infty<k,\omega<\infty} p_k(\omega).$$

Let's formulate one simple result.

Lemma 5. *Let $p(\omega_0) < 1$. Then, for all sufficiently small values of ε, all the roots of the equation (59) have negative and real parts separated from zero at $\varepsilon \to 0$. If $p(\omega_0) > 1$, then for all sufficiently small values of ε, the equation (59) has a root with a positive and separated from zero at $\varepsilon \to 0$ real part.*

Thus, the critical case in the stability problem is realized under the condition $p(\omega_0) = 1$. After that, the above method is transferred to the boundary value problem (11), (12).

Conclusion

The question of the local dynamics of a fully connected system of oscillators is considered. Critical cases in the problem of stability of the equilibrium state are highlighted. It is shown that they have infinite dimension. The main results are that an algorithm has been developed for constructing special nonlinear boundary value problems — quasinormal forms. Their nonlocal dynamics determines the asymptotics of all solutions of the original equation in the vicinity of the equilibrium state.

Quasinormal forms are spatially distributed nonlinear boundary value problems, for example, the classical Ginzburg-Landau equations, so we can conclude that the class of problems considered here is characterized by complex and irregular oscillations.

Depending on the value of the parameter a — coefficient at $\dot{u}$ in (1) — solutions are either slowly oscillating (at $a^2 > 2$) or rapidly oscillating (at $a^2 < 2$) with asymptotically high frequency. The quasi-normal forms from the section 4 contain another spatial variable. This, of course, leads to a complication of the dynamic properties of solutions.

It is shown that the number of related equations in quasi-normal form is determined by the number of modulo equal Fourier coefficients of the function $\Phi(s)$. An important role belongs to the value of the argument δ for the corresponding coefficients $\Phi(s)$.

It should be particularly noted that in a number of quasinormal forms there are integral terms of a nonlinear function with respect to a spatial variable. This leads to the fact that solutions of quasi-normal forms can become structurally more complicated. For example, it is explicitly possible to define solutions that are smooth and periodic in time, and stepwise in a spatial variable, and in some cases to investigate their stability.

Note that in the case of slowly oscillating solutions, only the nonlinearity $b_1 u^3$ of the function $f(u, \dot{u})$ is involved in constructing quasi-normal forms. If $b_1 = 0$ and $b_2 \neq 0$, then there is a structural complication of solutions and a change in the asymptotics of solutions of the original equation. For fast oscillating solutions, all coefficients of the function are involved $f(u, \dot{u})$.

The dynamic properties of the original boundary value problem (3), (4) at $a^2 < 2$ are sensitive to parameter changes. This conclusion follows from the fact that, firstly, in the quasi-normal form there is a value $\theta = \theta(\varepsilon) \in [0, 2\pi)$, which, with $\varepsilon \to 0$, runs through all values from 0 to 2π. Secondly, at different values of θ, the dynamics of the quasi-normal form can differ [35], which means that at $\varepsilon \to 0$, an unlimited process of forward and reverse bifurcations can occur.

An important conclusion concerns the role of a large delay in the chains under consideration. On the one hand, all analytical constructions for large T allow us to explicitly identify critical cases, find the asymptotics of the roots of characteristic equations and obtain asymptotic formulas for solutions. On the other hand, for $T \gg 1$, all quasi-normal forms contain another spatial variable, so we can conclude that the dynamics of properties becomes more complicated.

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