Applied Problems of Nonlinear Oscillation and Wave Theory

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Analytical method of optical wave behavior studying in nonlinear medium with periodically arranged conducting nanofilms

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Abstract. The *purpose* of this work is to build the analytical model of the behavior of a harmonic wave in a nonlinear optical medium with periodically arranged nanofilms. *Methods.* The modernized method is presented of non-smooth transformation of the argument to eliminate the Dirac functions on the right side of the non-linear inhomogeneous differential equation describing linear polarized wave behavior within a non-linear optical medium with periodically arranged conducting nanofilms. Small parameter methods, in particular, the averaging method, is also used to find an approximate analytical solution. *Results.* The fully analytical model of the behavior of a linear polarized harmonic wave within a nonlinear optical medium with periodically arranged conducting nanofilms is constructed. *Conclusion.* For the case of propagation of a linearly polarized harmonic wave in a nonlinear optical medium with periodically arranged conducting nanofilms, the mathematical model based on the non-smooth argument transformation method is constructed. The model is fully analytical, all expressions are obtained directly from Maxwell's equations by identical transformations. The limits of its applicability are determined by the limits of application of the wave theory of light.

Keywords: nonlinear optical medium, periodic structure, Dirac function, non-smooth argument transformation, solution stability.

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Introduction

The study of various periodic structures in optics has attracted the attention of scientists for more than a hundred years [1–4]. One-dimensional [5,6], two-dimensional [7,8], three-dimensional [9,10] structures were studied. From the very beginning to the present, special attention has been drawn to various linear layered media, both isotropic [3] and anisotropic [4]. The main tasks in this direction are to study the areas of passage and non-passage of the wave (the problem of solution stability), finding solutions in the areas of passage, including periodic solutions.

With the development of laser technology, interest has also appeared in nonlinear periodic optical structures [11, 12]. The main areas of research are the generation of higher harmonics, self-focusing, etc. In this case, nonlinear medium can be used as high-quality filters. Since the beginning of the XXI century, the study of various nanostructures [13, 14], including nanocoats and nanofilms, has become relevant. The theoretical study of nanofilms is usually carried out using a quantum optics apparatus. Further, to analyze structures including nanocoats and nanolayers, it is necessary to jointly solve both the Maxwell equations and the Schredinger equation [13–15]. Experimental results of measuring the conductivity of nanofilms of various thicknesses from various materials are presented in [16].

We have considered a one-dimensional nonlinear structure with a periodic arrangement of conductive nanofilms. The mechanisms of current formation in conductive nanofilms are taken into account in Maxwell's equations in the form of δ functions within the framework of the nonlinear wave theory of light. This approach is simplified. But it greatly facilitates the solution of the problem of analyzing optical devices on nonlinear structures, including periodically arranged nanofilms. The authors constructed an analytical model using the method of non-smooth argument transformation, the averaging method, and Lyapunov stability conditions. An important point in the presented model is the exclusion of the δ function in the original wave equation by converting the argument.

1. Setting the task

In this paper, we consider the behavior of a plane electromagnetic wave in a one-dimensional infinite nonlinear dielectric medium described by the magnetic permeability μ and the dielectric constant of the form [12]

$$\varepsilon(E) = \varepsilon_0 + \varepsilon_2 E^2. \tag{1}$$

The structure contains conductive nanofilms with a period of Λ (Fig. 1). In this case, currents J_x in neighboring nanofilms flow along the x axis in opposite directions. This is possible when external voltages of opposite polarity are applied to adjacent films. The case of the propagation of a linearly polarized wave along the z axis is considered (Fig. 1). Here z_1^+, z_1^- are coordinates of the location of conductive nanofilms with positive and negative current directions.

The aim of the work is to develop a fully analytical mathematical method describing the behavior of a plane harmonic wave in the structure under consideration and allowing periodic solutions to be found in the nonlinear structure under consideration.

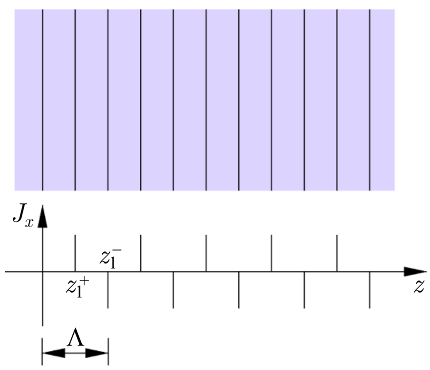

Fig 1. Geometry of the problem

2. The wave equation

Solving Maxwell's equations

$$\begin{aligned} \operatorname{rot}\vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \\ \operatorname{rot}\vec{H} &= \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \end{aligned} \tag{2}$$

in the case of a harmonic linearly polarized wave for a given structure as a result of transformations leads to a nonlinear inhomogeneous differential equation of the form

$$\frac{d^2E_x(z)}{dz^2} + \frac{\omega^2\varepsilon_0\mu}{c^2}E_x(z) + \frac{3\omega^2\varepsilon_2\mu}{c^2}|E_x(z)|^2E_x(z) = j\frac{4\pi\omega\mu}{c^2}J_x, \tag{3}$$

where $J_x = \sigma(z)E_x$ is current density, $\sigma(z)$ is specific conductivity of a nonlinear dielectric medium or nanofilms depending on the coordinate z, from (3) we obtain

$$\frac{d^2E_x(z)}{dz^2} + \frac{\omega^2\varepsilon_0\mu}{c^2}E_x(z) + \frac{3\omega^2\varepsilon_2\mu}{c^2}|E_x(z)|^2E_x(z) = j\frac{4\pi\omega\mu}{c^2}\sigma(z)E_x(z). \tag{4}$$

Since the nonlinear medium in the problem under consideration is dielectric, the $\sigma(z)$ in the intervals between nanofilms is zero. Approximately considering the thickness of nanofilms tending to zero in comparison with the size of the period of the structure, the expression on the right side can be written using the Dirac δ function. Thus, for currents flowing in the positive direction of the x axis, we have

$$J_x^+ = j\frac{4\pi\omega\mu}{c^2}\sigma_{\text{nf}}(z)E_x(z)\delta(z - z_k^+), \tag{5}$$

where z_k^+ is the spatial coordinate of the corresponding nanofilm. For currents flowing in the negative direction of the x axis, we have

$$J_x^- = -j\frac{4\pi\omega\mu}{c^2}\sigma_{\text{nf}}(z)E_x(z)\delta(z - z_k^-), \tag{6}$$

where z_k^- is the spatial coordinate of the corresponding nanofilm. Here, the current densities are the same in absolute value and their amplitudes are equal. As a result, the equation for a plane harmonic wave in a nonlinear medium with periodically arranged nanofilms (Fig. 1) takes the form

$$\begin{aligned} &\frac{d^2E_x(z)}{dz^2} + \frac{\omega^2\varepsilon_0\mu}{c^2}E_x(z) + \frac{3\omega^2\varepsilon_2\mu}{c^2}|E_x(z)|^2E_x(z) = \\ &= j\frac{4\pi\omega\mu}{c^2}\sigma_{\text{nf}}(z)E_x(z)\sum_{k=-\infty}^{\infty}\left[\delta(z - z_k^+) - \delta(z - z_k^-)\right], \end{aligned} \tag{7}$$

where σ_{nf} is the specific conductivity of the nanofilm. Next, for the convenience of the solution, we introduce the notation

$$\begin{aligned} p &= \frac{\omega^2\varepsilon_0\mu}{c^2}, \\ \varepsilon &= \frac{3\omega^2\varepsilon_2\mu}{c^2}, \\ q &= j\frac{4\pi\omega\mu}{c^2}\sigma_{\text{nf}}. \end{aligned} \tag{8}$$

Here ε is a small parameter, because for real environments ε_2 has the order of $10^{-11}...10^{-20}$, $c \approx 3 \cdot 10^8$. Now the equation (7) takes the form

$$\frac{d^2E_x(z)}{dz^2} + pE_x(z) + \varepsilon|E_x(z)|^2E_x(z) = -2qE_x \sum_{k=-\infty}^{\infty} \left[\delta(z - z_k^+) - \delta(z - z_k^-)\right]. \tag{9}$$

3. Solving the differential equation

This section presents a fully analytical model for analyzing the behavior of a plane harmonic wave in an isotropic nonlinear structure with periodic inclusion of conductive nanofilms. The model is based on the method of non-smooth argument transformation and the averaging method.

3.1. Non-smooth argument transformation. According to the previous section, the right side of the equation (9) contains a sequence of δ functions. This is due to the fact that two nanofilms with the opposite direction of current flow are located on the same period. The system is solved for equidistant nanofilms, that is, for the so-called equidistant system. From a mathematical point of view, the difficulty lies in the non-smoothness of the corresponding dynamic process.

Modeling the effect of thin nanofilms (so-called pulse effects) can be performed in several ways. The first way is to use the generalized functions [16]. However, this approach requires additional mathematical justification for each specific system.

Another approach is to solve the problem at each of the intervals between nanofilms, followed by combining solutions. Thus, instead of a single task, they solve an entire sequence of tasks [17].

In this paper, the method of non-smooth argument transformation is used to model the behavior of the wave in the structure under consideration. This method [18–20] allows you to build a fully analytical mathematical model of the behavior of a wave in a nonlinear medium, which contains δ Dirac functions, and obtain a solution for the period in the form of an analytical expression.

According to the method, the solution (9) is searched for in the form

$$E_x = X(\tau) + Y(\tau)\frac{d\tau}{dz} \tag{10}$$

where (Fig. 2)

$$\tau = \begin{cases} -4z/\Lambda + 1 + 2k, & k\Lambda \leqslant z \leqslant (k + 1/2)\Lambda, \\ 4z/\Lambda - 3 - 2k, & (k + 1/2)\Lambda \leqslant z \leqslant (k + 1)\Lambda, \end{cases} \tag{11}$$

where $k = 0, \pm 1, \pm 2, ...$

The essence of the method of non-smooth argument conversion when replacing the variable (10) is illustrated in Fig. 2. In accordance with the method, such a piecewise linear function $\tau(z)$, was introduced that the second derivative of it was equal to the sequence of δ functions in (9):

$$\frac{d^2\tau(z)}{dz^2} = \sum_{k=-\infty}^{\infty} \left[\delta(z - z_k^+) - \delta(z - z_k^-)\right]. \tag{12}$$

Let's take the first derivative of the function $E_x(z)$:

$$\frac{dE_x}{dz} = \frac{dX(\tau)}{dz}\frac{d\tau}{dz} + \frac{dY(\tau)}{dz}\left(\frac{d\tau}{dz}\right)^2 + Y(\tau)\frac{d^2\tau}{dz^2}. \tag{13}$$

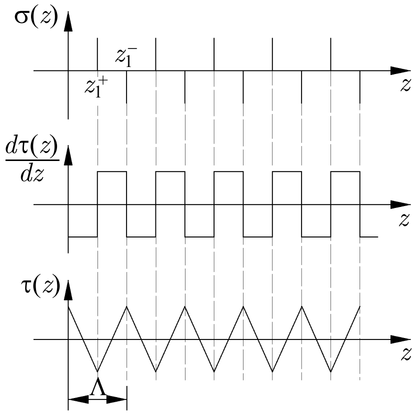

Fig 2. Non-smooth argument conversion

It is obvious that for equidistant nanofilms with the opposite direction of currents [18] $(d\tau/dz)^2 = 1$. Then

$$\frac{dE_x}{dz} = \frac{dX(\tau)}{dz}\frac{d\tau}{dz} + \frac{dY(\tau)}{dz} + Y(\tau)\frac{d^2\tau}{dz^2}. \tag{14}$$

The term containing δ functions in (14) is excluded due to the boundary condition [19, 20]:

$$\begin{aligned} &Y|_{\tau=(k+1/2)/\Lambda} = 0,\\ &Y|_{\tau=k\Lambda} = 0. \end{aligned} \tag{15}$$

Then we will rewrite the first derivative in (14) as

$$\frac{dE_x}{dz} = \frac{dX(\tau)}{dz}\frac{d\tau}{dz} + \frac{dY(\tau)}{dz}. \tag{16}$$

Taking into account the above justification, the second derivative of the function $E_x(z)$ has the form

$$\frac{d^2E_x}{dz^2} = \frac{d^2X(\tau)}{dz^2}\frac{d^2Y(\tau)}{dz^2}\frac{d\tau}{dz} + \frac{dX(\tau)}{dz}\frac{d^2\tau}{dz^2}. \tag{17}$$

Substitute the expressions (10) and (17) into (9) and get a differential equation of the form

$$\begin{aligned} &\frac{d^2X(\tau)}{dz^2}\left(\frac{d\tau}{dz}\right)^2 + \frac{d^2Y(\tau)}{dz^2}\frac{d\tau}{dz} + \frac{dX(\tau)}{dz}\frac{d^2\tau}{dz^2} = \\ &= -p\left(X(\tau) + Y(\tau)\frac{d\tau}{dz}\right) - \varepsilon\left(X(\tau) + Y(\tau)\frac{d\tau}{dz}\right)^3 - \\ &- 2q\left(X(\tau) + Y(\tau)\frac{d\tau}{dz}\right)\sum_{k=-\infty}^{\infty}\left[\delta(z - z_k^+) - \delta(z - z_k^-)\right]. \end{aligned} \tag{18}$$

Thus, we got the expression (18) containing δ functions in the left and right parts. These singular terms should be excluded from (18) by equating the coefficients for the same orders of derivatives τ. In this case, the influence of currents in nanofilms is manifested in the boundary conditions discussed below.

3.2. Building periodic solutions. Let's perform algebraic transformations of the expression (18) and get

$$\frac{d^2X(\tau)}{dz^2}\left(\frac{d\tau}{dz}\right)^2+\frac{d^2Y(\tau)}{dz^2}\frac{d\tau}{dz}+\frac{dX(\tau)}{dz}\frac{d^2\tau}{dz^2}=-pX(\tau)-pY(\tau)\frac{d\tau}{dz}-$$
$$-2qX(\tau)\sum_{k=-\infty}^{\infty}\left[\delta(z-z_k^+)-\delta(z-z_k^-)\right]-$$
$$-2qY(\tau)\frac{d\tau}{dz}\sum_{k=-\infty}^{\infty}\left[\delta(z-z_k^+)-\delta(z-z_k^-)\right]- \tag{19}$$
$$-\varepsilon\left(X^3(\tau)+3X^2(\tau)Y(\tau)\frac{d\tau}{dz}+3X(\tau)\left(Y(\tau)\frac{d\tau}{dz}\right)^2+\left(Y(\tau)\frac{d\tau}{dz}\right)^3\right).$$

Let's take into account that in (19) [18]

$$\begin{aligned}&\frac{d\tau}{dz}\frac{d^2\tau}{dz^2}=0,\\&\left(\frac{d\tau}{dz}\right)^2=1\end{aligned} \tag{20}$$

and we will get

$$\frac{d^2X(\tau)}{dz^2}\left(\frac{d\tau}{dz}\right)^2+\frac{d^2Y(\tau)}{dz^2}\frac{d\tau}{dz}+\frac{dX(\tau)}{dz}\frac{d^2\tau}{dz^2}=-pX(\tau)-pY(\tau)\frac{d\tau}{dz}-$$
$$-2qX(\tau)\sum_{k=-\infty}^{\infty}\left[\delta(z-z_k^+)-\delta(z-z_k^-)\right]-$$
$$-2qY(\tau)\frac{d\tau}{dz}\sum_{k=-\infty}^{\infty}\left[\delta(z-z_k^+)-\delta(z-z_k^-)\right]- \tag{21}$$
$$-\varepsilon\left(X^3(\tau)+3X^2(\tau)Y(\tau)\frac{d\tau}{dz}+3X(\tau)\left(Y(\tau)\frac{d\tau}{dz}\right)^2+Y^3(\tau)\frac{d\tau}{dz}\right).$$

In accordance with the above, we equate the coefficients at τ^0, $d\tau/dz$, $d^2\tau/dz^2$ to zero. We will get a system that describes the behavior of the wave in the structure under consideration in a general way:

$$\begin{aligned}&\frac{d^2X(\tau)}{dz^2}+pX(\tau)=-\varepsilon(3X^2(\tau)Y(\tau)+Y^3(\tau)),\\&\frac{d^2Y(\tau)}{dz^2}+pY(\tau)=-\varepsilon(3X(\tau)Y^2(\tau)+X^3(\tau))\end{aligned} \tag{22}$$

on condition

$$\begin{aligned}&Y|_{\tau=(k+1/2)/\Lambda}=0,\\&Y|_{\tau=k\Lambda}=0,\\&\left(\frac{dX(\tau)}{dz}+qX(\tau)\right)\Big|_{\tau=(k+1/2)\Lambda}=0,\\&\left(\frac{dX(\tau)}{dz}+qX(\tau)\right)\Big|_{\tau=k\Lambda}=0.\end{aligned} \tag{23}$$

From a mathematical point of view, the equations (22), (23) represent a boundary value problem for determining the functions X and Y. Despite the formally complex form, the main advantage of the resulting system (22), (23) is the absence of singular terms in it.

The presence of a small parameter ε in the model (22), (23) makes it possible to use the Poincare scheme to construct periodic solutions ([21]). Let's imagine the solution as a decomposition over a small parameter of nonlinearity ε

$$\begin{aligned} X &= X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + ..., \\ Y &= Y_0(\tau) + \varepsilon Y_1(\tau) + \varepsilon^2 Y_2(\tau) + ..., \\ p &= \gamma^2 + \varepsilon p_1 + \varepsilon^2 p_2 + ..., \end{aligned} \tag{24}$$

where $X_0(\tau)$, $X_1(\tau)$,...; $Y_0(\tau)$, $Y_1(\tau)$,... and γ, p_1, p_2, ... are functions to be defined. This approach leads to splitting the original equation describing the model into a recurrent sequence of boundary value problems in the interval $k\Lambda \leqslant z \leqslant (k+1)\Lambda$. The generating one is a linear ($\varepsilon = 0$) disconnected with respect to X, Y-component eigenvalue problem

$$\begin{aligned} &\frac{d^2 X_0(\tau)}{dz^2} + \gamma^2 X_0(\tau) = 0, \\ &\frac{d^2 Y_0(\tau)}{dz^2} + \gamma^2 Y_0(\tau) = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} &\frac{dX_0(\tau)}{dz} + qX_0(\tau)|_{z=(k+1/2)\Lambda} = 0, \\ &Y_0(\tau)|_{z=(k+1/2)\Lambda} = 0, \\ &\frac{dX_0(\tau)}{dz} + qX_0(\tau)|_{z=k\Lambda} = 0, \\ &Y_0(\tau)|_{z=k\Lambda} = 0. \end{aligned} \tag{26}$$

Solving the problem (25), (26) on eigenvalues, we define the desired functions X_0 and Y_0

$$\begin{aligned} X_{0i} &= A_0 \varphi_i(\tau), \\ Y_{0i} &= C_0 \psi_i(\tau), \end{aligned} \tag{27}$$

where A_0 and C_0 are constants defined by initial conditions. In this case, $C_0 = -A_0$ for $k\Lambda \leqslant z \leqslant (k+1/2)\Lambda$ and $C_0 = A_0$ for $(k+1/2)\Lambda \leqslant z \leqslant (k+1)\Lambda$.

Depending on the values of the parameter γ_i, there are two different types of eigenforms of vibrations

$$\begin{aligned} &\varphi_i(\tau) = \sqrt{\frac{2}{q^2 + \gamma_i^2}} (q \cos(\gamma_i \tau) + \gamma_i \sin(\gamma_i \tau)), \\ &\psi_i(\tau) = \sqrt{2} \cos(\gamma_i \tau) \end{aligned} \tag{28}$$

for $\gamma_i = i\pi/2$, where $i = 1, 3, 5, ...$ and

$$\begin{aligned} &\varphi_i(\tau) = -\sqrt{\frac{2}{q^2 + \gamma_i^2}} (q \sin(\gamma_i \tau) - \gamma_i \cos(\gamma_i \tau)), \\ &\psi_i(\tau) = \sqrt{2} \sin(\gamma_i \tau) \end{aligned} \tag{29}$$

for $\gamma_i = i\pi$, where $i = 1, 2, 3, ...$ Here the functions $\varphi_i(\tau)$ and $\psi_i(\tau)$ are represented as follows:

$$\langle\varphi_n(\tau), \varphi_m(\tau)\rangle = \frac{1}{2}\int\limits_{k\Lambda}^{(k+1/2)\Lambda} \varphi_n(z), \varphi_m(\tau)d\tau = \delta_{nm}, \quad (30)$$
$$\langle\psi_n(\tau), \psi_m(\tau)\rangle = \delta_{nm},$$

where δ_{ij} is the Kronecker symbol, $n = 1, 2, 3, ...,$ $m = 1, 2, 3, ...$ Then, after a series of mathematical transformations and substituting (28) and (29) into (27), and then into (10), we get

$$E_x(z) = -A_0\sum_{i=1}^{\infty}\left[\frac{1}{\sqrt{q^2+(i\pi)^2}}\left(\frac{i\pi}{\Lambda}\cos\left(\frac{4i\pi}{\Lambda}z\right) - q\sin\left(\frac{4i\pi}{\Lambda}z\right)\right) - \frac{4}{\Lambda}\sin\left(\frac{4i\pi}{\Lambda}z\right)\right] +$$
$$+A_0\sum_{i=2l-1}^{\infty}\left[\frac{1}{\sqrt{q^2+(i\pi)^2}}\left(q\cos\left(\frac{2i\pi}{\Lambda}z\right) + \frac{i\pi}{2\Lambda}\sin\left(\frac{2i\pi}{\Lambda}z\right)\right) - \frac{4}{\Lambda}\cos\left(\frac{2i\pi}{\Lambda}z\right)\right] + O(\varepsilon) \quad (31)$$

for $k\Lambda \leqslant z \leqslant (k+1/2)\Lambda$. Here $l = 1, 2, 3, ...$.

$$E_x(z) = -A_0\sum_{i=1}^{\infty}\left[\frac{1}{\sqrt{q^2+(i\pi)^2}}\left(\frac{i\pi}{\Lambda}\cos\left(\frac{4i\pi}{\Lambda}z\right) - q\sin\left(\frac{4i\pi}{\Lambda}z\right)\right) + \frac{4}{\Lambda}\sin\left(\frac{4i\pi}{\Lambda}z\right)\right] +$$
$$+A_0\sum_{i=2l-1}^{\infty}\left[\frac{1}{\sqrt{q^2+(i\pi)^2}}\left(q\cos\left(\frac{2i\pi}{\Lambda}z\right) + \frac{i\pi}{2\Lambda}\sin\left(\frac{2i\pi}{\Lambda}z\right)\right) + \frac{4}{\Lambda}\cos\left(\frac{2i\pi}{\Lambda}z\right)\right] + O(\varepsilon) \quad (32)$$

for $(k + 1/2)\Lambda \leqslant z \leqslant (k + 1)\Lambda$. It is important to note that the solution obtained in the form (31), (32), does not contain the δ Dirac function. The expressions (31), (32) represent a zero approximation of the solution. Similarly, the first, second and other approximations are found, depending on the required accuracy of the solution.

Conclusion

In this paper, the behavior of a linearly polarized harmonic wave in an optical nonlinear medium in the presence of periodically arranged conductive nanofilms is considered. The effect of nanofilms on the behavior of the wave is taken into account by the introduction of δ functions. A non-smooth transformation method is proposed to find periodic solutions to the differential equation describing the behavior of the wave. It allows you to find an analytical solution for the case under consideration. The advantage of the constructed model is to find an analytical periodic solution on a period and the absence of δ functions in finite expressions. The correctness of the constructed model is confirmed by the fact that it is completely analytical, and the final expressions are obtained by identical transformations. The results presented in this paper can be used for any medium and wave parameters within the framework of the wave theory of light.

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