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Identification and dynamics prediction of a plane vortex structure based on a mathematical model of a point vortices system

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Abstract. The *aim* of the article is developing and analyse an algorithmic method for solution finding of one inverse problem of 2d vortex fluid dynamics. It is identification and prediction of the flow structure evolution of the based on the data on fluid velocity vectors in a set of reference points. Theoretical analysis of convergence and adequacy of the method is difficult due to the ill-posedness typical of inverse problems, these issues studied experimentally. *Methods.* The proposed method uses a mathematical model of a point vortex dynamics system for identification and prediction flow structures. The parameters of the model system are found by minimising the functional that evaluates the closeness of the original and model vectors fields at the reference points. The prediction of the vortex structure dynamics is based on the solution of the Cauchy problem for a system of ordinary differential equations with the parameters found in the first stage. *Results.* As a result of the calculations, we found it out: the algorithm converges to the desired minimum from a wide range of initial approximations; the algorithm converges in all cases when the identified structure consists of sufficiently distant vortices; the forecast of the development of the current gives good results with a steady flow; if the above conditions are violated, the part of successful calculations decreases, false identification and an erroneous forecast may occur; with the convergence of the method, the coordinates and circulation of the eddies of the model system are close to the characteristics of the eddies of the test configurations; the structures of the streamlines of the flows are topologically equivalent; convergence depends more on location than on the number of vectors used for identification. *Conclusion.* An algorithm for solving the problem of identifying and the evolution forecast of a 2d vortex flow structure is proposed when the fluid velocity vectors in a finite set of reference points are known. The method showed its high efficiency when using from 40 to 200 reference points. The results of the study make it possible to recommend the proposed algorithm for identifying flat vortex structures, which consist of vortices separated from each other.

Keywords: vortex structures, identification algorithm, systems of point vortices, minimization.

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Introduction

Algorithms and methods for identifying areas of vortex flows and predicting their development are in demand when solving many problems of hydroaerodynamics. These include the tasks of forecasting synoptic flows based on satellite images of the seas, oceans and atmosphere [1, 2], processing the results of physical and numerical experiments [3–5], visualization of currents [6], blood flow [7], design and optimization of technical devices [8–10]. To solve these problems, methods are needed to search for and recognize vortices and their characteristics at a fixed point in time and predict changes in the vortex structure in time and space. Such opportunities can be provided by methods of mathematical modeling and analysis, which include two stages. At the first stage, it is assumed that the vortex is identified by known information about the flow, and non-stationary equations of hydrodynamics can be used for forecasting (see, for example, [11–13]).

To identify the vortex configuration, it is necessary to develop and use reverse approaches when the flow characteristics at some points or areas are known, and the number, location, size, and intensity of vortices are unknown. In recent decades, a number of vortex identification methods have been proposed, most of which can be divided into two classes. The first and most widespread approach consists of mathematical criteria based on local physical quantities determined by the flow, such as the velocity field, pressure, vorticity and their derivatives. Examples are the Q method, λ_2 method, Ω method, and others. A description and comparative analysis of some of the listed methods is given in the articles [14–17]. In this case, the vortex is defined as an associated region with a high "density" of one of the characteristic quantities. Most of these methods are mathematically rigorous and physically consistent, but require detailed information about the flow. The second class consists of methods using the topological properties of the flow in the entire flow region, or its subdomains. Such characteristics may be the closure, or helicity, of the flow lines of liquid particles, the presence of special points and separatrices in the flow structure [18, 19].

After successfully solving the inverse problem, it becomes possible to predict the development of the structure based on solving the nonstationary Euler, Navier-Stokes equations, other equations of mathematical hydrodynamics. Direct methods for solving such problems are well developed and widely used. The result of calculations in this case is the dynamics of the velocity field and other characteristics of the flow under study, which makes it possible to analyze and predict the flow structure over time [20–22].

Significant difficulties in identifying the vortex structure may arise with limited information about the flow. In this case, the listed methods, based on sufficiently detailed information about the course, fail and other approaches are required. A possible solution in this case may be the use of mathematical models of vortex dynamics. The article [23] proposes an algorithm based on the use of a small set of known flow velocity vectors and a mathematical model of a system of point vortices on a plane. Especially attractive in this way is that the mathematical model can also be used to predict the development of a vortex configuration over time. In the presented article, the development of an algorithm for both vortex identification and prediction of two-dimensional vortex dynamics is proposed, an experimental study of the applicability, adequacy and effectiveness of such an approach is carried out. As a meaningful example, numerical solutions of two-dimensional Euler equations of an ideal liquid with an initial condition in the form of three vortex spots in a rectangular region with impenetrable boundaries are used. The choice of this configuration is due to the fact that the system of three point vortices is the simplest, demonstrating nontrivial dynamics [24].

1. Problem statement and method of identification and prediction of vortex dynamics

A vortex is one of the fundamental hydrodynamic objects, which can empirically be described as a flow region in which particles rotate around a common center or axis. There is currently no generally accepted mathematical definition of a vortex, and its formulation remains a matter of debate. The vortex identification problem can have different formulations and is not mathematically correct. A possible formulation used in the article may be a description of a plane vortex flow using the coordinates and intensities of the vortex centers, as well as the structures of the flow lines of liquid particles based on information about the velocity of liquid particles in a finite (possibly small) set of reference points. Mathematical models can be used to find the characteristics of a vortex structure, the simplest of which is a system of point vortices. In this case, the intensities of distributed vortices and their centers can be described using the properties of point vortices.

1.1. Description of the identification and prediction method. Let the velocity vectors of the vortex flow be known at some point in time in a set of N reference points

$$U = \left\{ \left[x^{(j)}, y^{(j)}, \mathbf{u}^{(j)} = \left(u_1^{(j)}, u_2^{(j)}\right)\right], j = 1, \ldots, N \right\}, \tag{1}$$

where $(x^{(j)}, y^{(j)})$ are coordinates of the reference points, and $\mathbf{u}^{(j)}$ is corresponding velocity vector. It is necessary to determine the intensities and coordinates of the centers of vortices forming this flow. To describe the vortex configuration, we will use a well-studied system of point vortices. The choice of this mathematical model is due to the following reasons: relative simplicity, qualitatively correct description of many real flows [25, 26], topological equivalence of current lines near the point and the vicinity of the core of many distributed vortices. At the same time, a significant disadvantage of the model is the degeneracy of the velocity field in a point vortex, which should be taken into account when constructing an identification algorithm.

The motions of K point vortices on the plane are described by the following system of ordinary differential equations:

$$\omega_i \dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \omega_i \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad H = -\frac{1}{4\pi} \sum_{i,k=1,\, i\neq k}^{K} \omega_i \omega_k \ln(r_{ik}), \quad i = 1, \ldots, K. \tag{2}$$

Here (x_i, y_i) are coordinates of the vortex with the number i on the plane, $r_{ik} = (x_i - x_k)^2 + (y_i - y_k)^2$, and ω_i is its intensity (circulation). Obviously, H is the Hamiltonian and the first integral of the system (2), that is, the system is conservative. The initial values determine the values of the integrals of the point vortex system, which means that the invariant subspaces on which the dynamics occurs. This means the absence of attractors and other dynamic properties characteristic of conservative systems.

The current function of the velocity field generated by the point vortex system has the form

$$\Psi = -\frac{1}{4\pi} \sum_{i=1}^{K} \omega_i \ln\left[(x - x_i)^2 + (y - y_i)^2\right], \tag{3}$$

where (x, y) are coordinates on the plane. Then the dynamics of a passive particle is described by a system of two ordinary differential equations

$$\begin{aligned} \dot{x} &= v_1(x, y) = \frac{\partial \Psi}{\partial x} = -\sum_{i=1}^{K} \frac{\omega_i}{2\pi} \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2}, \\ \dot{y} &= v_2(x, y) = -\frac{\partial \Psi}{\partial y} = \sum_{i=1}^{K} \frac{\omega_i}{2\pi} \frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2}. \end{aligned} \tag{4}$$

System (4) sets the model velocity field used in the identification of the vortex structure $\mathbf{v}(x, y) = (v_1(x, y), v_2(x, y))$ at any point in the plane. The field $\mathbf{v}(P, x, y)$ is determined by the configuration parameters of point vortices $P = \{K, (x_i, y_i, \omega_i), i = 1, \dots, K\}$, that is, the number of vortices K, their intensities ω_i and coordinates (x_i, y_i) in the flow region. The idea of the proposed identification algorithm is based on the search for the listed system parameters (2), (4) using known vectors (1). In this case, the centers and intensity of the vortices of the flow are approached by point vortices, despite the above-mentioned drawback of the mathematical model.

It is necessary to formulate a condition that will mean that the system of point vortices (2) for some parameter values $\hat{P} = \left\{\hat{K}, (\hat{x}_i, \hat{y}_i, \hat{\omega}_i), i = 1, \dots, \hat{K}\right\}$ qualitatively describes the initial vortex configuration, due to the existing set of U. To do this, consider a set of vectors of the model system (4) at the same reference points $(x^{(j)}, y^{(j)})$, $j = 1, \dots, N$, что и (1):

$$V(P) = \left\{\left[x^{(j)}, y^{(j)}, \mathbf{v}^{(j)} = \left(v_1^{(j)} = v_1(P, x^{(j)}, y^{(j)}), v_2^{(j)} = v_2(P, x^{(j)}, y^{(j)})\right)\right], j = 1, \dots, N\right\}. \tag{5}$$

If $U \equiv V(P)$, then it is natural to assume that the system of point vortices (2) fully describes the vortex flow at parameter values P from the available set U.

Two vectors $\mathbf{u}, \mathbf{v}$ can be compared using two values:

$$d(\mathbf{u}, \mathbf{v}) = \frac{\|\mathbf{u} - \mathbf{v}\|}{\|\mathbf{u}\| + \|\mathbf{v}\|}, \quad \phi(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\left(1 - \frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right). \tag{6}$$

The value $d(\mathbf{u}, \mathbf{v})$ in (6) characterizes the lengths of the vectors, and $\phi(\mathbf{u}, \mathbf{v})$ are the angles between them. Both characteristics take values in the range $[0, 1]$, equal to zero for $\mathbf{u} = \mathbf{v}$ and one if $\mathbf{u} = -\mathbf{v}$.

To compare two sets of vectors in U and $V(P)$ in a given set of reference points, we use the expression

$$\sigma(U, V(P)) = \sum_{j=1}^{N}\left[c_j\, d(\mathbf{u}^{(j)}, \mathbf{v}^{(j)}(P)) + C_j\, \phi(\mathbf{u}^{(j)}, \mathbf{v}^{(j)}(P))\right]. \tag{7}$$

In the expression (7) c_j and C_j are the weight coefficients that are taken as positive constants during calculations in the article, and $c_j = 1 - C_j$. If $U \equiv V(P)$, then it is obvious that $\sigma(U, V(P)) = 0$. Due to the idealization of the mathematical model, the coincidence of the sets of vectors (1) and (5) is unlikely. It is natural to assume that the smaller the value of $\sigma(U, V(P))$, the closer the two sets of vectors are, and the mathematical model qualitatively describes the structure of the vortex configuration better. In other words, we assume that the original and model sets of vectors U and $V(\hat{P})$ are closest in this sense if (7) reaches a minimum in the parameter space of the system when $P = \hat{P}$.

Thus, the task of identifying a vortex structure in the considered formulation is reduced to finding the parameters of a system of point vortices that minimize (7)

$$\hat{P} = \min_{P} \sigma(U, V(P)). \tag{8}$$

To numerically search for the minimum (8), a combined algorithm based on Newton's methods and gradient descent is used in the work.

Note that the formulated formulation of the vortex structure identification problem has many disadvantages is not mathematically correct, is not always solvable, the solution may not be the only one, the result may strongly depend on the choice of reference points generating a set of U flows, etc. For example, it is obvious that for $N = 1$ identification by one vector is impossible. The analytical analysis of the proposed approach is very difficult, however, the initial results of the work of [23] demonstrated the effectiveness of the method for identifying some model flows. Further, in the section 2 of the article, an experimental study of the method is carried out using the example of a vortex structure of three vortices distributed in space in a flat square container.

If the desired set of parameters $\hat{P}$ is found, then the structure of the flow lines can be described using (3) when $\omega_i = \hat{\omega}_i$, $x_i = \hat{x}_i$, $y_i = \hat{y}_i$, $i = 1, \dots, \hat{K}$. The prediction of the vortex structure development can be carried out by solving the Cauchy problem for the system (2), (4) with initial data for point vortices, set by $\hat{P}$, and coordinates $x(0) = x_0$, $y(0) = y_0$ for calculating particle trajectories.

1.2. An algorithm for identifying and predicting the vortex configuration. Here is a variant of the algorithm used in this paper, based on the method described above. The first stage is the identification of the vortex structure, which consists of the following steps.

1. Initialization. This includes: determining the reference points and vectors of the set (1), the number of K used to describe point vortices, the initial approximation $P^{(m)}, m = 0$, for the parameters of the system (2), (4), the initial step h and the accuracy ε of the minimization method, the step size δ for numerical differentiation according to the parameters of the model system using finite differences, the number of steps of the gradient descent method M, the maximum number of steps of the algorithm $M_{\max}$, the minimum step size $h_{\min}$.
2. The step of the minimum search method.

$$P^{(m+1)} = \begin{cases} P^{(m)} - h\,\nabla\sigma\left(U, V(P^{(m)})\right), & m \leqslant M, \\ P^{(m)} - h\,\left[\nabla^2\sigma\left(U, V(P^{(m)})\right)\right]^{(-1)}\nabla\sigma\left(U, V(P^{(m)})\right), & m > M, \end{cases}$$

where $\nabla\sigma$ and $\nabla^2\sigma$ are, respectively, the gradient and the Hesse matrix of the expression (7) according to the parameters of the model configuration P. Approximations by central finite differences are used to calculate derivatives.
3. Checking the condition

$$\left\|\nabla\sigma\left(U, V(P^{(m+1)})\right)\right\| < \varepsilon. \tag{9}$$

If the condition (9) *is met*, then an approximation of the vortex configuration $\hat{P} = P^{(m+1)}$ is found, and *the algorithm stops working*. If not, then proceed to the next step of the algorithm.
4. Checking the condition $\left\|\nabla\sigma\left(U, V(P^{(m+1)})\right)\right\| < \left\|\nabla\sigma\left(U, V(P^{(m)})\right)\right\|$. When it is executed, $h = h \cdot .01$, and $h = h \cdot .5$, $P^{(m+1)} = P^{(m)}$, if the condition is not met.
5. $m = m + 1$. Checking the conditions $m < M_{\max}$ and $h > h_{\min}$. If both conditions are met — go to the 2 point of the algorithm, and "emergency" shutdown otherwise.

The prediction of the dynamics of the vortex structure (the second stage of the algorithm) can be realized only with the successful identification of the vortex structure (the first stage) and the found minimizing (7) set of parameters of the model system $\hat{P}$. This stage consists in solving the Cauchy problem for the system of equations (2), (4), for the number of point vortices $\hat{K}$ with intensities $\hat{\omega}_i$, $i = 1, \dots, \hat{K}$. The initial conditions for $t = 0$ are: $x_i(0) = \hat{x}_i$, $y_i(0) = \hat{y}_i$ and $x(0) = x_s$, $y(0) = y_s$, where (x_s, y_s) are coordinates of the passive particle intended for prediction at $t = 0$. To solve the Cauchy problem on the interval $t \in [0, T]$, the Runge–Kutta method is used, T is forecast duration.

2. Identification and prediction of the dynamics of test configurations

As a meaningful example for the identification and prediction of a vortex structure, we use solutions to the problem of the dynamics of an inviscid incompressible fluid in a square container. In terms of the current function $\psi(t,x,y)$ and vorticity $\omega(t,x,y)$, the dynamics of a plane vortex flow is described by a system of Euler equations

$$\frac{D\omega}{Dt} \equiv \omega_t + \omega_x\psi_y - \omega_y\psi_x = 0, \tag{10}$$

$$\psi_{xx} + \psi_{yy} = -\omega. \tag{11}$$

Here $\frac{D}{Dt}$ is the material derivative. The subscript indicates the partial derivative of the corresponding variable.

Fluid velocity $\mathbf{u} = (u_1, u_2)$ is expressed in terms of a current function

$$u_1 = \psi_y, \quad u_2 = -\psi_x, \tag{12}$$

The flow is considered in the square area of $D : \{0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant a\}$. The following boundary conditions are set on the D boundary:

$$\psi(t,0,y) = \psi(t,a,y) = \psi(t,x,0) = \psi(t,x,a) = 0. \tag{13}$$

The equations and boundary conditions must be supplemented with an initial condition

$$\omega(0,x,y) = \Omega(x,y), \tag{14}$$

where $\Omega(x,y)$ is some function defined in D.

It is known that the problem (10)–(14) is solvable, its solutions describe many real flows when the influence of liquid viscosity is negligible. A grid-free spectral method was used for the numerical solution. A detailed description of the method for calculating flows in closed and flowing areas can be found in [21, 27], and in open in [28].

2.1. Setting up computational experiments. For an experimental study of the effectiveness of the proposed identification method, consider a square area D with a side $a = 8$. As the initial vorticity distribution (14), we will use a configuration of three identical vortices with centers located at the vertices of an equilateral triangle with a centroid in the middle of D (point $(4,4)$), 1 away from the centroid vertices $\left(x_c^{(i)}, y_c^{(i)}\right)$, $i = 1, 2, 3$:

$$\Omega(x,y) = \sum_{i=1}^{3} W\left(x_c^{(i)}, y_c^{(i)}, x, y\right), \tag{15}$$

where $W(x,y)$ is the vorticity distribution function in a distributed vortex.

Let's consider two qualitatively different examples, differing in the initial distribution of vorticity in vortices:

$$W\left(x_c^{(i)}, y_c^{(i)}, x, y\right) = \begin{cases} G \cdot e^{-10\cdot\left(x_c^{(i)}-x\right)^2 - 10\cdot\left(y_c^{(i)}-y\right)^2}, & r \leqslant R, \\ 0, & r > R, \end{cases} \tag{16}$$

$$W\left(x_c^{(i)}, y_c^{(i)}, x, y\right) = \begin{cases} G \cdot \left(R^2 - r^2\right)^2, & r \leqslant R \\ 0, & r > R, \end{cases} \tag{17}$$

where $r = \sqrt{\left(x_c^{(i)} - x\right)^2 + \left(y_c^{(i)} - y\right)^2}$, and the coefficient G is chosen so that The intensity (circulation) of the entire vortex spot was equal to one. For both initial states (15), (16) and (15), (17), the total vorticity of the entire configuration is 3.

The empirical law of interaction of vortex spots of the same orientation (direction of rotation) is known from physical and computational experiments. At large distances between vortices, they rotate around a common center of vorticity. If the distance between two vortices is less than some critical one, then they merge (see [29] and links in this article). The magnitude of the critical distance depends on the distribution and intensity of vorticity in vortices. Computational experiments with two initial configurations have shown that in the case of (15), (16), a quasi-stationary vortex structure is formed (Fig. 1). At the initial distribution of (15), (17), all three vortices merge into one (Fig. 2). The figures show: the distribution of vorticity (in shades of gray), marker particles (for each initial vortex in its own color) and flow lines. As characteristics of vortices, we use the coordinates of the vorticity centers of vortices and their circulation [12]. The proposed method of identification and prediction is applicable to the obtained vortex structures.

2.2. Successful identification and prediction. With the initial vortex configuration (15), (16), three vortex spots remain throughout the considered time interval $t \in [0.1000]$, which rotate around a common center and exchange filaments at $t > 50$ (see Fig. 1). In this case, for any t, as a result of the first stage of the algorithm, when the method converges, a structure of three point vortices is identified, whose coordinates and intensities are close to the characteristics of vortex spots of the configuration distributed in space. The second stage of the algorithm

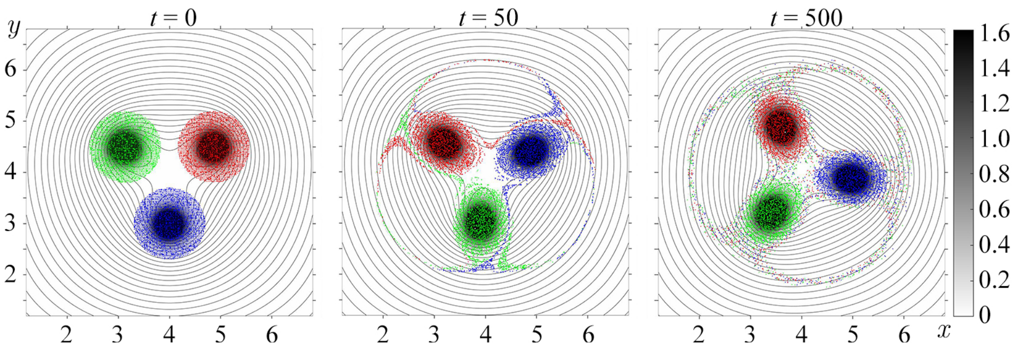

Fig 1. Three vortices at different times (the formation of a quasi-stationary structure) for the initial configuration (15), (16) (color online)

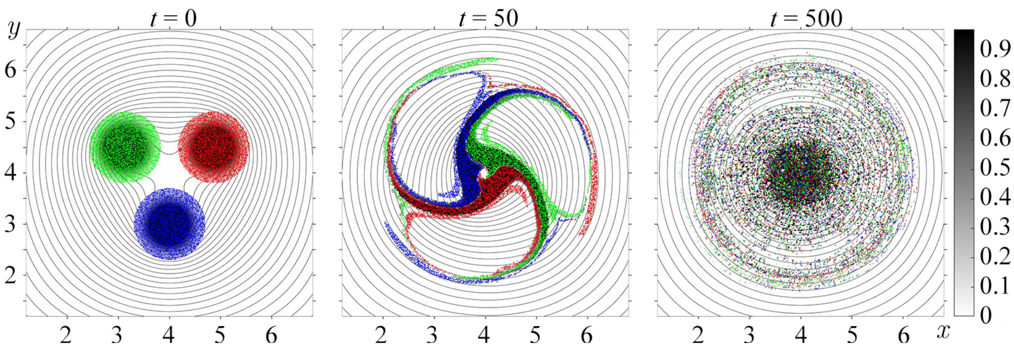

Fig 2. Three vortices at different times (vortex merging) for the initial configuration (15), (17) (color online)

demonstrates a good forecast over a time interval corresponding to several turns of vortices.

Here is an example of successful identification and prediction (Fig. 3 and Fig. 4). As vectors of the U set in (1), 40 randomly distributed vectors of the (12) field obtained as a result of solving the problem (10)–(14) by the grid-free spectral method at $t = 200$ were set. The vectors of the set of U for calculation are shown as a dotted line in Fig. 3, a. Coordinates of the vorticity centers of the vortex spots of the test configuration at $t = 200$: $(3.694, 3.104)$, $(3.373, 4.710)$, $(4.926, 4.193)$, and their circulations are approximately equal to one. A system of $K = 3$ point vortices was indicated as an initial approximation for the algorithm. As a result of the algorithm, it was found

$$\hat{P} = \{\hat{K} = 3, (\hat{x}_1 = 3.687,\ \hat{y}_1 = 3.079,\ \hat{\omega}_1 = 0.956), (\hat{x}_2 = 3.514,\ \hat{y}_2 = 4.756,\ \hat{\omega}_2 = 0.830), (\hat{x}_3 = 4.953,\ \hat{y}_3 = 4.117,\ \hat{\omega}_3 = 0.949)\}.$$

It can be seen that the initial and obtained model characteristics are close. The streamlines of the identified and model fields are qualitatively identical (Fig. 3, b, c), and the model set of vectors V differs slightly from U (Fig. 3, a).

The results of the second stage of the algorithm (prediction) in the interval $t \in [0.400]$ from the moment of identification are shown in Fig. 4. It can be seen that the structure of the trajectories of the vorticity centers of the original test system (Fig. 4, a) and the vortices of the model system found as a result of the identification algorithm (Fig. 4, b) are qualitatively identical. Moreover, there is also a good quantitative coincidence of the trajectories of the center of vorticity and the point vortex (Fig. 4, c). That is, for this example, we can state a reliable forecast of the development of the vortex structure using the proposed algorithm.

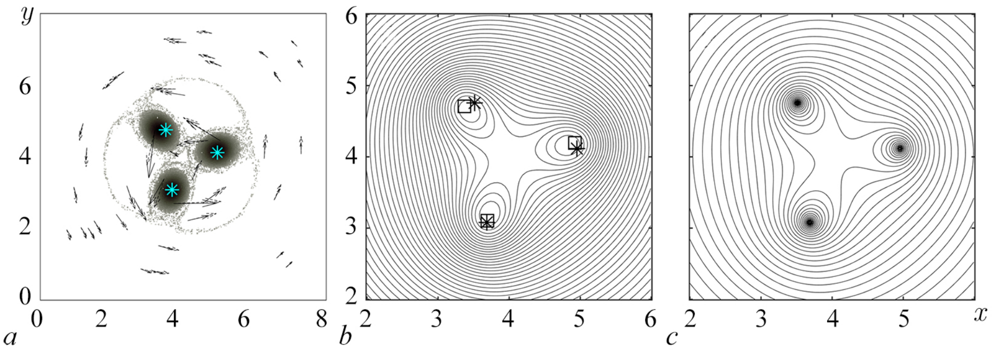

Fig 3. The result of the first stage of the algorithm (identification) for the configuration fig. 1 at $t = 200$. a — Test configuration vorticity distribution (shades of grey), vectors of initial U (dotted line) and found model V (solid lines) sets of vectors; b — streamlines of the initial flow, vorticity centres of spots (squares) and model vortices (asterisks); c — streamlines of the model system (color online)

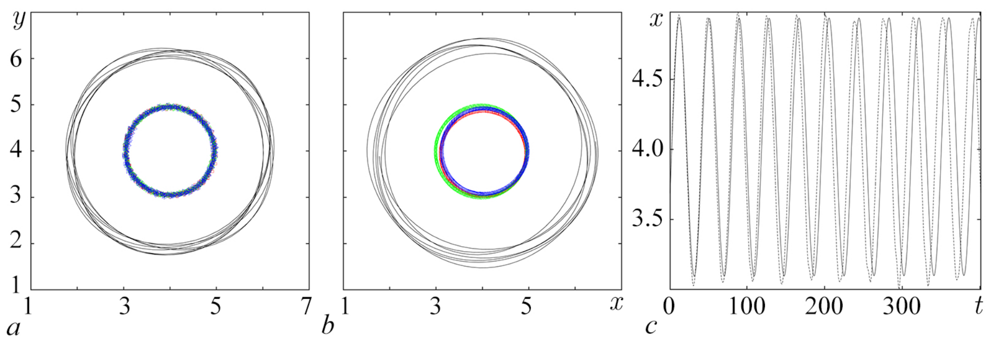

Fig 4. The result of the forecast of the development of the vortex configuration fig. 1 for $t > 200$. a — Trajectories of vorticity centres of vortex patches (symbols) and one passive particle; b — trajectories of point vortices of the model system (symbols) and passive particle (solid line); c — time dependence of the coordinate x of the center of vorticity (dotted line) and the corresponding point vortex of the model system (solid line) (color online)

2.3. Examples of successful identification and incorrect prediction. The application of the algorithm for a vortex structure with an initial distribution (15), (17) does not give such good results. In the considered time interval $t \in [0, 1000]$, the topological structure of the flow is not preserved (see Fig. 2). At the first stage, three vortices converge, then they merge to form thin structures, and the final state is one vortex. The quality of identification depends on its moment.

At the initial moment $t = 0$, and before the interaction of the vortices of the test flow begins, the first stage of the algorithm gives the correct result. In Fig. 5 an example of successful operation of the first stage of the algorithm using $N = 40$ vectors in a set of U is given. Coordinates of the vorticity centers of the vortices of the test configuration at $t = 0$: $(3.134, 4.5)$, $(4, 3)$, $(4.866, 4.5)$, and their intensities are equal to one. The result of the algorithm:

$$\hat{P} = \big\{\hat{K} = 3, (\hat{x}_1 = 3.140,\ \hat{y}_1 = 4.430,\ \hat{\omega}_1 = 0.993), (\hat{x}_2 = 3.927,\ \hat{y}_2 = 3.056,\ \hat{\omega}_2 = 0.824),$$
$$(\hat{x} = 4.874_3,\ \hat{y}_3 = 4.424,\ \hat{\omega}_3 = 0.970)\big\}.$$

That is, the coordinates and intensities of the point vortices are quite close. It is also seen that the structures of the current lines of the test (Fig. 5, *b*) and model flows (Fig. 5, *c*) are qualitatively the same, the point vortices of the model system are close to the centers of vorticity distributed vortices, as well as vectors of the sets U and V (Fig. 5, *a*). In this case, the forecast stage is correct only for short periods. So, for the example presented, this is only one turn of the structure (Fig. 6). This is due to the fact that the model system of point vortices does not describe the processes of vortex fusion, which occurs starting from $t \approx 50$.

Until the moment of merging the vortex spots of the test structure, the algorithm either does not converge or gives an erroneous result, see the following subsection. After the formation of one vortex, the structure of the test flow does not change, the model system allows you to identify the structure of the flow. At the same time, when using three point vortices $K = 3$, the algorithm converges to a qualitatively incorrect structure. When using a single model vortex, this effect disappears (Fig. 7), but the forecast loses its meaning due to the triviality of the dynamics of the model system. The result of the algorithm at $t = 500$ is: $\hat{P} = \big\{\hat{K} = 1,\ (\hat{x}_1 = 4.038,\ \hat{y}_1 = 3.916,\ \hat{\omega}_1 = 2.880)\big\}$.

2.4. An example of the erroneous operation of the algorithm. The application of the algorithm for transient (not time-stable) flows with a complex structure can lead to false identification and incorrect prediction of its development. Here is an example of such a calculation (Fig. 8). It can be seen that the coordinates of the found point vortices differ from the coordinates

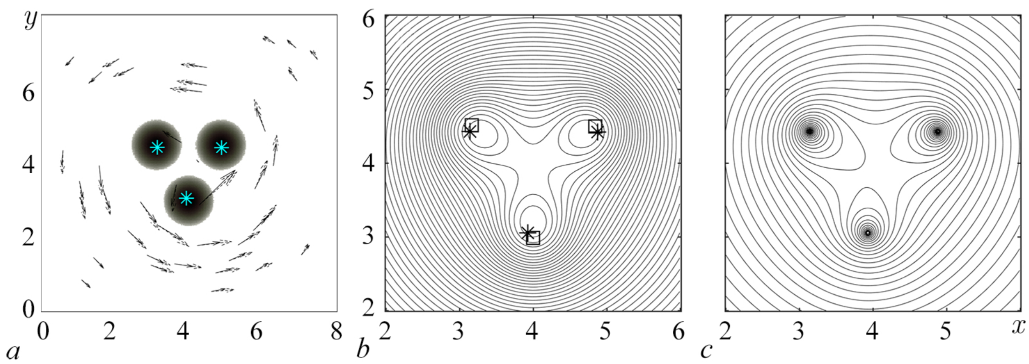

Fig 5. Result of configuration identification fig. 2 at $t = 0$. Explanations for the figure see in fig. 3 (color online)

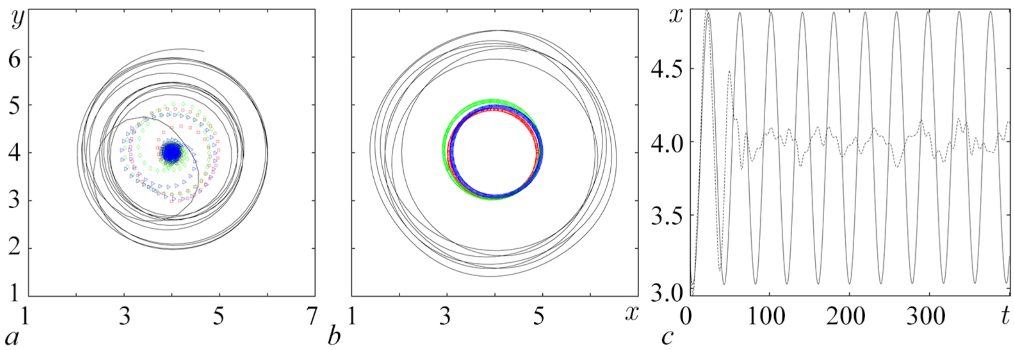

Fig 6. The result of predicting the dynamics of the vortex configuration fig. 2 for $t > 0$. Explanations in fig. 4 (color online)

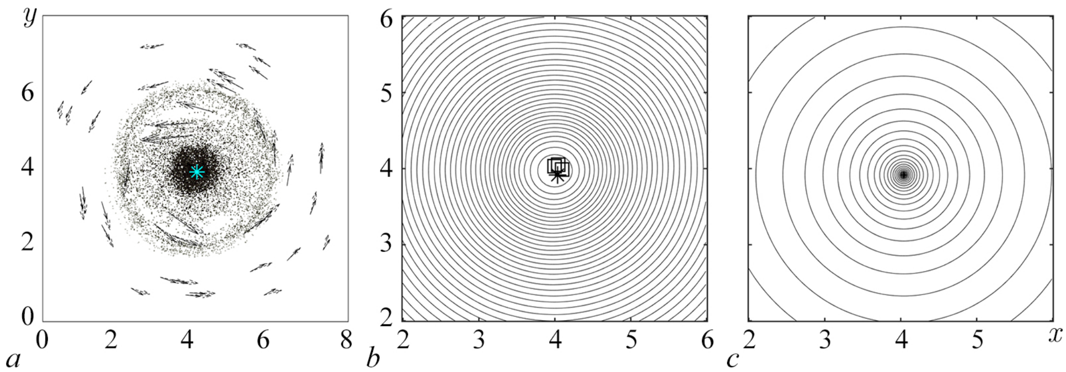

Fig 7. Result of configuration identification fig. 2 at $t = 500$. Explanations see in fig. 3 (color online)

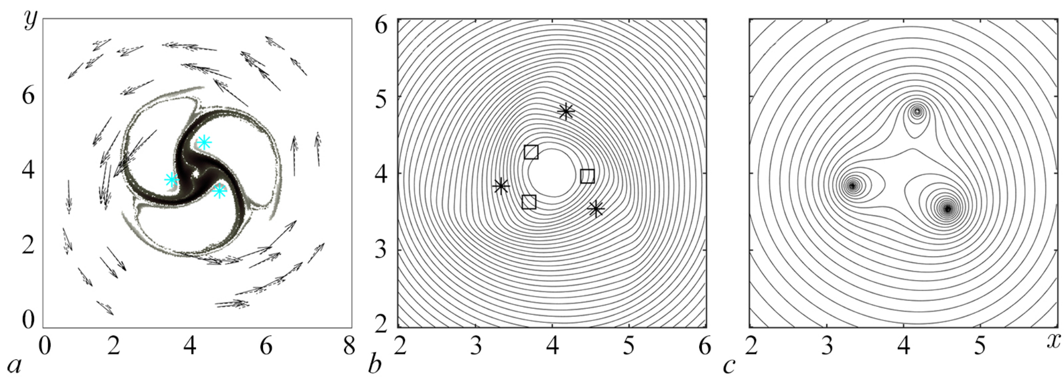

Fig 8. Result of configuration identification fig. 2 at $t = 50$. Explanations see in fig. 3 (color online)

of the centers of vortices distributed. The discrepancy between these characteristics is generally acceptable, but there is also a qualitative difference in the structures of the current lines of the test and model systems (Fig. 8, *b* and Fig. 8, *c*). Despite this, the sets of reference and model vectors turned out to be close (Fig. 8, *a*). That is, in this case, the minimization of (9) led to qualitatively incorrect conclusions about the flow structure. Apparently, this is a consequence of the possible non-uniqueness of solutions to the minimization problem.

3. Experimental analysis of algorithms

Due to the incorrectness of the inverse identification problem, the theoretical analysis of the convergence conditions of the algorithm and the adequacy of the proposed forecast is difficult. To study the dependence of the algorithm on the initial information U, see (1), and the initial approximation $P^{(0)}$, section 1.2, two series of computational experiments were conducted. As a test, a spatially distributed vortex structure was used, calculated with the initial vorticity distribution (15), (16) at $t = 100$ (see Fig. 1). Coordinates of the vorticity centers of distributed vortices: $(3.396, 4.734), (3.644, 3.117), (4.972, 4.1522)$.

In the first series of experiments with a fixed initial approximation

$$P^{(0)} = \Big\{K = 3, (x_1^{(0)} = 3, y_1^{(0)} = 5, \omega_1^{(0)} = 0.75), (x_2^{(0)} = 5, y_2^{(0)} = 4, \omega_2^{(0)} = 0.75), \\ (x_3^{(0)} = 4, y_3^{(0)} = 3, \omega_3^{(0)} = 0.75)\Big\} \tag{18}$$

the number of reference points N in the set U has changed. For each N, 100 calculations were performed, the location of the reference points was set randomly in D. The results are shown in the table. 1. The number of reference points is indicated in the first column, the percentage of successful identification is indicated in the second column, and the average coordinates and intensities of model vortices are indicated in the rest. Successful identification refers to the proximity of point vortices to the characteristics of distributed and topological equivalence of current lines. N $N < 10$ the set U contains insufficient information about the flow structure, which is the reason for the low probability of successful identification of the structure. With the growth of N, the percentage of successful calculations increases, already at $N = 10$ there are about half of them. The maximum for successful identification is achieved at $N = 100$, but over the entire range of $N \in [20, 200]$, the probability of successful identification is more than 80%. That is, for the algorithm to work effectively, it is enough to have information in a small set of reference points.

In the second series of computational experiments, not only the set of U was randomly changed, but also the initial approximation of the algorithm $P^{(0)}$. The initial approximations for the coordinates of point vortices were chosen randomly in a single circle centered at the points of the set (18), and the approximations for intensities from the interval $[0.5, 2]$. The results are given in the table. 2. It can be seen that the random perturbation of the initial approximation slightly reduced the probability of successful identification for almost all N, but this decrease is insignificant, which demonstrates the effectiveness of the algorithm in a wide range of initial approximations. Note that the averaged coordinates of the point vortices and their intensities are close to the corresponding characteristics of the distributed vortices of the test configuration, see both tables.

Table 1

N	%	$\tilde{x}_1$	$\tilde{y}_1$	$\tilde{\omega}_1$	$\tilde{x}_2$	$\tilde{y}_2$	$\tilde{\omega}_2$	$\tilde{x}_3$	$\tilde{y}_3$	$\tilde{\omega}_3$
5	27	3.462	4.774	1.332	3.745	3.153	1.296	4.814	4.272	1.232
10	51	3.283	4.631	0.930	3.764	3.066	0.948	4.936	4.300	0.983
15	70	3.272	4.671	0.949	3.805	3.091	0.987	4.947	4.314	0.899
25	85	3.283	4.636	0.930	3.812	3.087	0.963	4.909	4.301	0.936
50	92	3.308	4.647	0.954	3.798	3.087	0.945	4.898	4.296	0.926
100	95	3.307	4.636	0.924	3.805	3.100	0.953	4.899	4.295	0.926
200	92	3.308	4.635	0.948	3.806	3.091	0.933	4.898	4.287	0.927

Table 2

N	%	x_1	y_1	ω_1	x_2	y_2	ω_2	x_3	y_3	ω_3
5	24	3.490	4.628	1.169	3.965	3.350	1.198	4.754	4.180	1.046
10	43	3.341	4.670	0.989	3.756	3.118	1.065	4.886	4.270	1.042
15	70	3.274	4.655	0.950	3.756	3.134	1.027	4.924	4.309	0.924
25	75	3.275	4.647	0.927	3.817	3.078	0.946	4.893	4.278	0.982
50	87	3.298	4.652	0.919	3.787	3.098	0.949	4.896	4.285	0.938
100	93	3.316	4.618	0.943	3.792	3.076	0.919	4.900	4.282	0.947
200	92	3.318	4.637	0.936	3.801	3.090	0.939	4.906	4.290	0.921

Conclusion and discussion

The article proposes an algorithmic method for solving the inverse problem of identifying and predicting the development of a plane vortex flow when the velocity vectors of the liquid in a finite set of reference points are known. The method is based on the use of a mathematical model of a system of point vortices and minimization of the target functional, estimating the proximity of sets of velocity vectors of the initial and model flows. Due to the inherent incorrectness of inverse problems and the strong dependence of the results on the initial information, the theoretical analysis of the methods of their solution is very difficult. This also applies to the problem under consideration, therefore, the applicability and effectiveness of the algorithm in the article is investigated experimentally using a sufficiently meaningful example - the dynamics and interaction of three distributed vortices.

Numerical experiments have been carried out to identify test flows. The sets of vectors used for identification were formed in randomly distributed sets of reference points and determined by the values calculated as a result of solving the non-stationary problem for Euler equations. The number of reference points ranged from 5 to 200. The algorithm demonstrated effective convergence in the case when the vortex structure consisted of distributed vortices sufficiently distant from each other. The calculated coordinates of the vortices of the model system and their intensity turned out to be close to the characteristics of the test spatially distributed vortices in all cases when the algorithm successfully converged. In addition, the structures of the known test and model current lines are topologically equivalent. The dynamics forecast carried out for this case using the solution of the Cauchy problem for the model system showed a good coincidence at times of the order of several turns of the structure and a correct qualitative description at long times. The results deteriorated significantly with the complication of the vortex configuration and with the interaction of test distributed vortices. Numerical analysis of the algorithm has shown that the convergence of the method strongly depends on the location of the reference points where the vectors of the original set are specified. For successful convergence in the considered examples, a small (on the order of tens) number of vectors used for identification is sufficient. In addition, the algorithm converges to the desired minimum from a wide range of initial approximations.

Many problems and inaccuracies that arise when applying the algorithm are related to the disadvantages of the simplest vortex dynamics model system used. Its main disadvantage is the degeneracy of the model vector field directly in the point vortex and, as a result, the high value of velocity in their vicinity. This contradicts real two-dimensional vortices, in the center of the core of which the velocity of the liquid is zero and small in its vicinity. This defect in the mathematical model can lead to incorrect operation of the algorithm in the presence of reference points near the vortex cores. In addition, point vortex systems do not describe processes such as interactions and mergers of distributed vortices, the presence of background currents, etc. Despite this, the advantages of the simplest model are its relative mathematical simplicity and an adequate description of the current lines of many vortex flows, which made it possible to successfully identify test vortex structures consisting of non-interacting vortices and predict their dynamics.

There are two ways to get rid of the disadvantages of the model system do not take into account the reference vectors from the vicinity of the vortex nuclei, as well as from the vicinity of the point vortices of the model system. Another, and more promising, is the use of mathematical models of vortex dynamics that adequately describe the velocity field in the entire flow region and dynamic vortex processes. This is a further direction of our research in the near future. The results obtained in the article using the simplest model demonstrated the effectiveness and feasibility of the proposed approach to solving the considered inverse problem.

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