Applied problems of nonlinear oscillation and wave theory

Izv.VUZ «AND», vol.6, № 4, 1998

PHASE SYNCHRONIZATION OF SWITCHINGS IN STOCHASTIC AND CHAOTIC BISTABLE SYSTEMS

V.S. Anishchenko, A.B. Neiman, A.N. Silchenko, I.A. Khovanov

We study synchronization of switching processes in stochastic and chaotic bistable the systems driven by a periodic signal in terms of phase synchronization. Introducing instantaneous phases of transitions between metastable states and the periodic forcing we show explicitly the effect of phase locking. The dynamics of phase difference appears to be qualitatively equivalent to that of a synchronized classical self-sustained oscillator. We have found that the degree of phase coherence between the input signal and the response estimated employing the effective diffusion constant is maximal at an optimal noise level in stochastic bistable system or at an optimal value of a control parameter in a purely deterministic case. We also consider the effect of mutual synchronization of the switching processes in coupled stochastic and chaotic bistable systems.

I. Introduction

Starting with the work of C.Huygens [1] synchronization phenomena attracted great attention of researchers from many different fields of science [2]. According to [3], «to synchronize» means to concur or agree in time, to proceed or to operate at exactly the same rate, to happen at the same time. Synchronization occurs in nonlinear self-sustained oscillators driven by external periodic force or coupled with each other [4,5]. In the absence of periodic force the system should possess a stable limit cycle in the phase space which reflects stable oscillations occurring in the system. It is important to note, that the properties of these oscillations, i.e., the natural frequency and the amplitude, are determined by the internal dynamics and do not depend (within reasonable ranges) on the initial conditions. In general, synchronization can be treated as the appearance of some functionals characterizing the correlations in temporal behavior of two or more processes [2]. The instanteneuos phase plays the role of such functional in classical theory of oscillations. Synchronization defines as the locking of instantaneous phases $\Phi(t)$ of a state variable of the self-sustained oscillator and of the external periodic force $\Psi(t)=\Omega_0 t$: $|n\Phi(t)-m\Psi(t)|<\mathrm{const}$, or by a weaker requirement of frequency locking $\Omega=\dot{\Phi}=(m/n)\Omega_0$. Here m, n are integer numbers. These requirements are fulfilled in finite regions of the parameter space of the system which called Arnold tongues. Recently, classical approach to synchronization based on the notion of instantaneuos phase of oscillations was generalized on the cases of non-autonomous and interacting chaotic systems [6,7].

The problem of noise influence on the synchronization effect of self-sustained oscillators was first raised by S.M. Rytov [8] and then studied in detail by R.L. Stratonovich [9]. Gaussian noise leads to amplitude and phase fluctuations [9]. As a result, the phase difference $\phi(t)=\Phi(t)-\Psi(t)$ also fluctuates and, in the assumption of constant amplitude, its slow dynamics can be described by the stochastic differential equation (SDE):

$$\dot{\phi} = \Delta - \varepsilon G(\phi) + \xi(t), \qquad (1)$$

where $\Delta=\Omega-\Omega_0$ is the frequency mismatch, $G(\phi)$ is a 2π periodic function, ε is the parameter of nonlinearity and $\xi(t)$ is Gaussian noise. In the case of Van der Pol oscillator studied by R.L.Stratonovich [9] $G(\phi)\equiv\sin\phi$ and the phase difference performs overdamped Brownian motion in the tilted periodic potential $U(\phi)=-\Delta\phi-\varepsilon\cos\phi$. If $\Delta<\varepsilon$ and the noise strength is small, then the phase difference fluctuates for a long time inside a well of the potential $U(\phi)$ (the phase locking) and rarely makes jumps from one potential well to another (i.e., displays phase slips).

As is well known, the effect of noise on the synchronized self-sustained oscillator is negative: the increase of noise intensity leads to the loss of phase coherence (phase slips become more frequent) and shrinks Arnold tongues [10,11]. However, there are qualitatively different situations when additive noise plays the constructive role and causes phase transitions [12]. One of the typical examples of the positive effect of noise is stochastic resonance (SR).

The phenomenon of stochastic resonance [13] has been extensively studied over the last two decades [13]. SR occurs in a wide class of nonlinear systems driven simultaneously by noise and a signal. The necessary property which a nonlinear system should possess to be able to demonstrate SR is the existence of a noise-controlled time scale.

Recent investigations have shown that SR also takes place in multistable dynamical systems which have two co-existing attractors in phase space [16-19]. The effect of noise on such systems leads to the random transitions of the phase trajectory between different attractors and causes the appearance of a new noise-controlled time scale. It should be noted that SR takes place independently on the type of the co-existing attractors which can be regular or chaotic [16]. Moreover, as was shown in works [16,20], SR also can be observed in purely deterministic case when the regime of intermittency of «chaos-chaos» type takes place and hoppings between different chaotic attractors caused by the internal dynamics of the system.

The traditional description of SR defines this effect as the amplification of a weak signal applied to the input of the system by tuning the noise intensity. SR manifests itself in the existence of a bell shaped maximum in the dependence of the spectral power amplification (SPA) [21] or signal-to-noise ratio (SNR) [22] versus noise intensity. For extremely weak signals SR is correctly described by the linear response theory [23,24]. In this case a stochastic resonator might be thought as an equivalent filter with a noise tuned transfer function determined by the linear susceptibility of the system. In order to calculate the response of the system we have to know its statistical property in an unperturbed (i.e. in the absence of the signal) equilibrium state (or more generally, in a stationary state). From this point of view, the structure of the weak signal is immaterial: the signal can be harmonic, quasiperiodic [25] or even aperiodic broad-band noisy [26,27].

Another way for description of SR, based on the statistics of residence times, has been proposed by Gammaitoni et. al. [28,29]. In the absence of periodic excitation the residence time distribution possesses an exponential shape. However, when the periodic signal is switched on, the residence time distribution becomes structurized and contains series of peaks centered at the odd multiples of the half period of the signal the strengthes

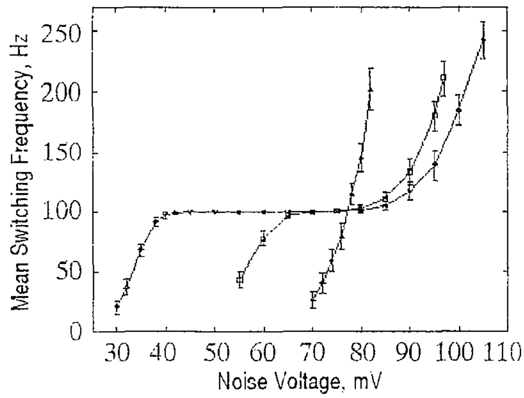

Fig. 1. The dependence of mean switching frequency of Shmitt trigger versus noise intensity for different values of the amplitude of periodic force: A=0 mV, A=60 mV, A=100 mV. Frequency of input signal is Ω_0=100 Hz, threshold of Shmitt trigger V_{th}=150 mV

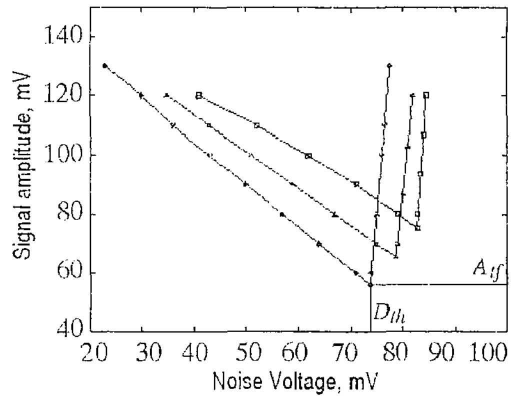

Fig. 2. The regions of synchronization for different values of the frequency of periodic force: Ω_0=100 Hz, Ω_0=250 Hz, Ω_0=500 Hz, threshold of Shmitt trigger V_{th}=150 mV

of which decrease according to the exponential law. At an optimal noise level the peak at the half driving period becomes dominant and the probability concentrated within this peak passes through a maximum by varying both the noise intensity and the driving frequency [30]. This picture clearly displays synchronization features of SR. These features manifest themselves more brightly in a situation when the amplitude of the periodic force is large enough, although it is insufficient to cause the switchings in the absence of noise. As shown in work [31], which main results are presented in Figs. 1,2, the effect of mean switching frequency locking in Shmitt trigger driven simultaneuosly by noise and periodic signal takes place. Moreover, there are regions of synchronization of switchings on the parameter plane «noise intensity - amplitude of periodic force» (see Fig. 2) in which the mean frequency of switchings coincides with the frequency of the periodic input signal.

The effect of mutual synchronization of switching processes in symmetrically coupled stochastic bistable systems was discovered in work [32]. In this case there are no deterministic time scales in the system and synchronization of switchings caused by the interaction of statistical time scales of subsystems which are defined as the moments of a stationary probability density. The mean frequencies of switchings of the subsystems are drawn close to one another with the increase of the coupling parameter and coincide at the moment of synchronization.

As follows from these results, SR-systems can demonstrate synchronization-like phenomena which are similar to classical synchronization mentioned above. The description of these phenomena by means of the residence time distribution and the mean frequency of switchings has one disadvantage - it provides no information about instantaneous matching of output switching events with the input signal and does not allow to answer the questions: how long the switchings between metastable states are in synchrony with the input signal? Is it possible to observe phase locking and locking of the mean frequency in finite regions in the parameter space of the system as it is in classical self-sustained oscillators? It is important, that such a formulation of the problem is just the same as in the classical theory of oscillations, where synchronization is understood originally as instantaneous matching of input / output phases.

The goal of the present study is to bridge a classical notion of synchronization as the instantaneous phase locking effect [2,4,5] and synchronization-like effects occurring in stochastic and chaotic bistable systems [31-33]. For this purpose we first discuss two definitions of the instantaneuos phase for a periodically driven noisy bistable system in

Sec. II. The effects of phase and frequency locking are discussed in Sec. III. Sec. IV devotes to the effect of synchronization of switchings in a periodically driven chaotic bistable system with discrete time. In Sec. V we consider synchronization of the switching processes in coupled stochastic and chaotic bistable systems in terms of the phase synchronization theory. Our conclusions are given in Sec. VI.

II. Instantaneous phase for noisy bistable system driven by periodic signal

We treat as the model an overdamped stochastic bistable system driven by a periodic signal, which is governed by the following SDE:

$$\dot{x} = - dU(x)/dx + 2D^{1/2}\xi(t) + A\cos(\Omega_0 t+\psi_0), \tag{2}$$

where $U(x)=(-\alpha/2)x^2+(\beta/4)x^4$ is the symmetric potential with $\alpha,\beta>0$, $\xi(t)$ is white Gaussian noise, ψ_0 is the initial phase of the signal. We set $\psi_0=0$ for convenience. This system has no deterministic oscillation frequency. Instead it possesses a noise controlled time scale represented by Kramers time or mean escape time from a potential well and has essentially relaxation features. In the frequency domain this time scale determines the mean switching frequency (MSF) of the system.

It should be noted that the amplitude of the periodic forcing is always small: the signal alone can not switch the system from one state to another. For the low frequency modulation this requires that

$$A < A_0 = {}^2/_3[\alpha^3/(3\beta)]^{1/2}. \tag{3}$$

In order to study synchronization in the above described classical sense we need to introduce an instantaneous phase of the system. For this purpose we have used the formal but general definition of instantaneous phase which is based on the concept of analytic signal introduced by Gabor [35]. The analytic signal $w(t)$ is a complex function of time defined as

$$w(t) = x(t) + iy(t) = a(t)\exp(i\Phi(t)), \tag{4}$$

where $y(t)$ is the Hilbert transform (HT) of original process $x(t)$:

$$y(t) = 1/\pi \int_{-\infty}^{\infty} x(\tau)/(t-\tau)d\tau. \tag{5}$$

In the last expression the integral is taken in the sense of Cauchy principal value. Instantaneous amplitude $a(t)$ and phase $\Phi(t)$ of $x(t)$ are unambiguously defined through this concept as:

$$\Phi(t) = \arctan[y(t)/x(t)], \qquad a^2(t) = x^2(t) + y^2(t), \tag{6}$$

as well as instantaneous frequency $\omega(t)=\dot{\Phi}(t)$

$$\omega(t) = [1/a^2(t)][x(t)\dot{y}(t) - y(t)\dot{x}(t)]. \tag{7}$$

The mean frequency $\langle\omega\rangle$ is then defined as

$$\langle\omega\rangle = \lim_{T\to\infty}(1/T)\int_0^T \omega(t)dt. \tag{8}$$

The concept of analytic signal is widely used in the theory of non-linear oscillations and waves [36,37]. Recently the definition of instantaneous phase through this concept has been applied to study phase synchronization of chaotic systems [6].

The phase difference between response and input signal is defined by the following expression:

$$\phi(t) = \Phi(t) - \Omega_0 t. \tag{9}$$

The analytic signal concept can be applied directly to SDE (2) to derive explicit SDEs for the instantaneous amplitude and the phase difference. During the transformations we have used the remarkable property of any analytic signal: its Fourier transform vanishes for negative frequencies. SDE for the analytic signal is

$$\dot{w} = \alpha w - (\beta/4)(3a^2 w + w^3) + \Xi(t) + A\exp(i\Omega_0 t), \tag{10}$$

The analytic noise $\Xi(t)=\xi(t)+i\eta(t)$, where $\eta(t)$ is the Hilbert transformation of $\xi(t)$. From Eq.(10) we derive SDEs for the instantaneous amplitude and phase:

$$\begin{aligned} \dot{a} &= \alpha a - (\beta/2)a^3[1+\cos^2(\phi+\Omega_0 t)] + A\cos\phi + \xi_1(t), \\ \dot{\phi} &= -\Omega_0 - (\beta/4)a^2\sin[2(\phi+\Omega_0 t)] - (A/a)\sin\phi + (1/a)\xi_2(t), \end{aligned} \tag{11}$$

where noise sources $\xi_{1,2}(t)$ are defined as

$$\xi_1(t) = \xi(t)\cos\Phi + \eta(t)\sin\Phi,$$

$$\xi_2(t) = \eta(t)\cos\Phi - \xi(t)\sin\Phi.$$

Note that the equations (11) are exact and similar to those for amplitude and phase fluctuations of the Van der Pol oscillator [9]. This similarity is indeed due to the structure of nonlinear transformation which we have used to derive equations for the amplitude and phase. However, there is also an important difference. The distinction appears in the second equation for the phase. In the case of Van der Pol oscillator there is an additional term in the r.h.s, Ω, which refers to the natural frequency of the oscillator. The absence of this term in (11) reflects the fact that the overdamped oscillator has no deterministic natural frequency, e.g. there is no rotational term in the equation for the phase for the unperturbed system (A=0).

Although both the amplitude and the phase are defined at any instant moment of time, they have integral character [37]. This follows from the fact that in order to calculate them we have to perform the Hilbert transform (5), which implies knowledge of the system behavior at all time axis from $-\infty$ to $+\infty$.

The exact SDEs (11) are highly nonlinear with multiplicative noise. For computational reasons it is more convenient to integrate original SDE (2) numerically and then perform HT by the well established technique (see, for example, [39]).

Indeed, other definitions of the phase are possible. As known, the dynamics of bistable systems includes the fast intrawell motions and slow switchings between metastable states. We can introduce the instantaneous phase of oscillations basing on the moment of switching times. For this purpose we map continuous stochastic process $x(t)$ into a stochastic point process $\{t_k\}$, where t_k are the moments of time of successive level crossing $x=\pm x_m=(\alpha/\beta)^{1/2}$ (see for detail [28]). The residence time between two subsequent switching events is then $T(t)=t_{k+1}-t_k$, $t_k<t<t_{k+1}$. In this case phase $\Phi(t)$ is defined as

$$\Phi(t) = 2\pi(t-t_k)/(t_{k+1}-t_k) + 2\pi k, \qquad t_k<t<t_{k+1}. \tag{12}$$

The phase defined in this way is a piecewise-linear function of time. In the case of purely periodic switching process, when transitions between metastable states are fully synchronized with the period $2\pi/\Omega_0$, this definition gives the exact phase $\Omega_0 t$. The instantaneous frequency $\omega(t)=2\pi/T(t)$ is constant during waiting periods inside potential wells, while the mean frequency for this definition is equivalent to the mean switching frequency of the system:

$$\langle\omega\rangle = \lim_{M\to\infty}(1/M)\Sigma_{k=1}^{M}2\pi/(t_{k+1}-t_k) \quad (13)$$

and can be calculated also via residence time distribution.

Note, that the first definition of the phase bears both inter- and intrawell motions, while the second one takes into account only global switching dynamics. Nevertheless, both definitions display equivalent averaged behavior up to a constant phase shift [24]. This coincidence is not by chance. The analytic signal concept makes automatically a separation of different time-scales [37]. This follows from the property of the Hilbert transformation to froze slow variables. The global dynamics of the system, e.g. transitions between metastable states, gives the main contribution to the phase dynamics, while the short-time fluctuations inside a potential well are immaterial for the global phase dynamics [42].

IV. Noise enhanced phase coherence

The results of calculations of phase difference (9) by using the Hilbert transformation are presented in Fig.3. As it can be clearly seen from this figure, within some region of noise intensity the phase coherence becomes amenable to be observed. At an optimal noise level the phase is locked in course of observation time [41,42]. With deviations of noise intensity from this optimal value the phase slips appear, so that we can speak about partially synchronized dynamics. It is remarkable that the dynamics of the phase difference $\phi(t)$ is very similar to that of a synchronized self-sustained oscillator and can be qualitatively described by SDE (1). In the limit cases of small or large noise the switching process and the periodic force are incoherent: the mean frequency of switchings becomes smaller (larger) than the driving frequency and the phase difference monotonically decreases (increases) with time and is represented by a straight line with negative (positive) slope (not shown). The same behavior has been observed for the phase determined via switching times (12). Fig. 3 clearly shows the effect of synchronization: the phases of the switching process and of the input signal are *instantaneously* locked at an optimal noise level. It is also seen from this figure that tuning noise we can increase the duration of locking time intervals.

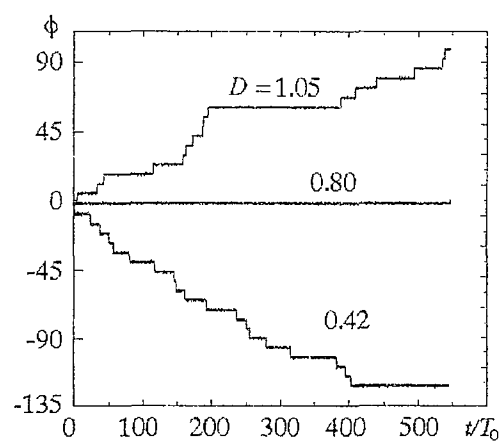

Fig. 3. The instantaneous phase difference calculated using the analytic signal approach for indicated values of noise intensity. Other parameters are A=3, α=5, β=1, Ω_0=0.01

The dependence of the mean frequency, determined via the analytic signal concept (8), and of the mean switching frequency, calculated by averaging return times (13), versus noise intensity is shown in Fig. 4 for different values of the driving amplitude. This figure displays once more the effect of the mean switching frequency locking reported first in [31]. For a weak signal the mean frequency follows the Kramers law and raises exponentially with increasing noise strength. However, for a large enough A the mean frequency matches with the driving frequency in a finite region of noise

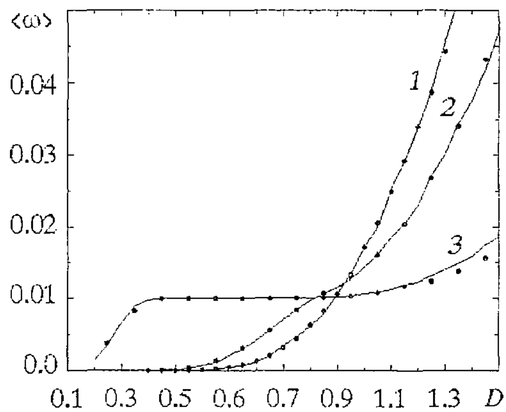

Fig. 4. Mean frequency (8) (solid line) and the mean switching frequency (13) (symbols) versus noise intensity for different values of driving amplitude: A=0 (1), A=1 (2) and A=3 (3). Other parameters are the same as in previous figure

intensity. Note, that the behavior of the mean frequencies calculated using two different definitions of instantaneous phase is nearly the same. It is important to mention that the effect of mean frequency locking occurs in a finite region of noise intensity. The width of this region depends on driving amplitude and frequency [31].

As it was mentioned above, the presence of noise makes obscure the classical definition of synchronization. That is why for noisy systems we must use the notion of *effective synchronization* [40]. The definition of effective synchronization can be made via imposing restrictions to (i) phase fluctuations, (ii) frequency fluctuations and (iii) output signal-to-noise ratio [40]. In our study we consider the strongest definition of effective synchronization based on statistics of phase fluctuations. The stochastic system can be considered as effectively synchronized by external periodic force if the mean time in course of which the instantaneous phase of the system is locked, is much larger than the driving period.

Although the effect of phase and mean frequency locking already indicates a synchronization-like behavior we need to calculate second-order statistical quantities to determine synchronization according to the definition given above. The quantity related to this definition which can be used as a measure of the phase coherence is the effective diffusion constant D_{eff} defined as

$$D_{eff} = {}^1/_2\, (d/dt)[\langle\phi^2(t)\rangle - \langle\phi(t)\rangle^2]. \tag{14}$$

This value characterizes the spreading of an initial distribution of the phase difference along the potential profile. It can be shown that the effective diffusion constant is proportional to the mean escape rate r from a well of the potential $U(\phi)$ (1): $D_{eff}=4\pi^2 r$ [9], i.e. D_{eff} is inverse proportional to the mean time interval of phase locking. Thus, we can use this quantity to answer the question: for how long the phase at the output still locked by the signal?

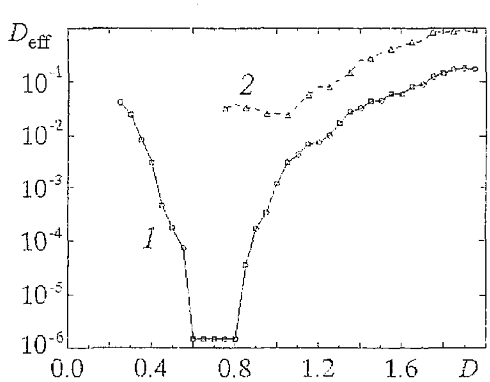

Fig. 5. The effective diffusion constant versus noise intensity for indicated values of driving amplitude: A=3.0 (1), A=1.0 (2). Other parameters are: α=5, β=1, Ω_0=0.01

The dependence of the effective diffusion constant (14) vs noise intensity is shown in Fig. 5 for different values of driving amplitude. In contrast to classical oscillators, where D_{eff} monotonically increases, here the effective diffusion constant passes through a minimum. This means that the phase proves to be locked for longer time intervals with the increase of noise intensity. In other words, we can enhance the phase coherence by increasing noise level in the system. As clearly seen from Fig. 5, the diffusion of the phase difference is extremely slow for a strong enough signal in a finite region of noise intensity so that we can define a region of effective synchronization. The condition determining a region of effective synchronization can be expressed as

$$D_{\text{eff}} \leq 2\pi\Omega_0/n, \quad (15)$$

where n>>1 is a number of periods of external force. In our concrete case we took n=100, e.g. the system (2) is to be effectively synchronized by an external periodic signal if its instantaneous phase is locked in course of at least 100 periods of the signal. The synchronization region is shown in Fig. 6 on the parameter plane A-D and as seen, it posses a tongue-like shape. Also, the threshold-like character of the synchronization effects is clearly seen: a periodic force with the amplitude less than a threshold value can not synchronize the bistable system in the above defined sense. Recall, that experimentally obtained «Arnold tongues» of periodically driven noisy Schmitt trigger (see Fig. 2) also have the same threshold feature. With the increase of the driving frequency the threshold value of the driving amplitude also increases and effective synchronization regions shrink (not shown). This feature is determined by the low-frequency character of SR in bistable systems.

It is important to underline that phase and frequency locking effects occur for a strong enough signal only. It can be seen clearly in Figs 4,5. For weak signals, both the diffusion coefficient and frequency fluctuations are large, so that the phase is not locked for long periods. For a weak signal the system is only partially synchronized even in the case when the mean switching frequency equals exactly the driving frequency. This situation is shown in Fig. 7. Although there are relatively short locking segments, the difference displays a random walk-like behavior with zero slope. The dependence of the effective diffusion constant has not a minimum and increases with the growth of noise intensity as well as in the classical case.

The case of a weak input signal can be correctly described in the frame of a qualitatively different approach for investigation of SR based on the calculation of the residence time distribution which was given in studies of Gammaitoni et al. [28,30].

The residence times distribution gives a weaker definition of synchronization in SR systems based on the restriction imposed to frequency fluctuations. This approach defines synchronization in an averaged sense. Really, the existence of the peak at the half driving period indicates that the number of residence times which are near the half driving period is much larger than the whole number of switchings occur during an observation time. However, it does not require instantaneous phase locking in course of

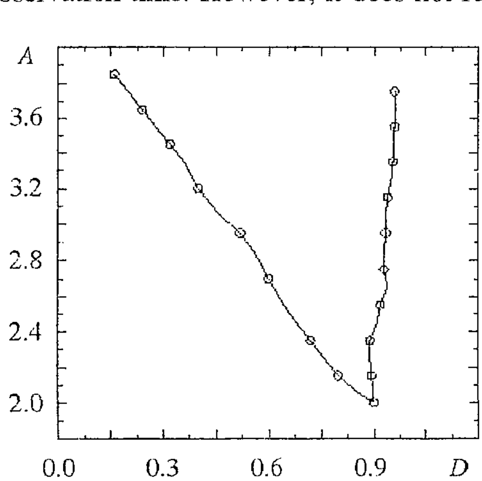

Fig. 6. The region of effective synchronization. Other parameters are the same as in the previous figure

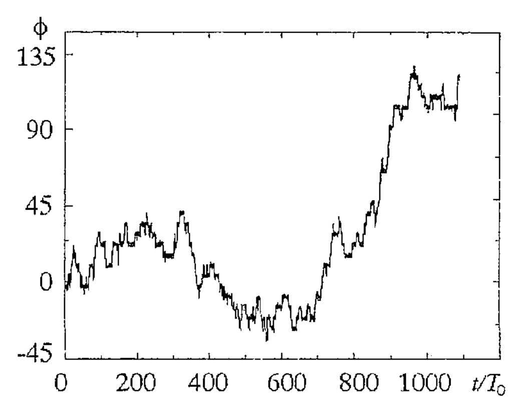

Fig. 7. The phase difference versus time (in units of driving period) for A=1, D=1.0215. Other parameters are the same as in previous figures

long times. Therefore, the residence time distribution recover an average phase preference of the system. That is why the measures based on the residence time distribution reflect synchronization nature of SR even for weak signals.

Our investigations have shown that for comparatively small signals the phase dynamics cannot be viewed as synchronized, although the residence time distribution displays synchronization-like behavior [30]. Even in the case when the mean frequency is equal to the driving frequency $\langle\omega\rangle=\Omega_0$, the phase difference performs Brownian-like motion with a zero slope (see Fig. 7).

The measure of synchronization proposed by Gammaitoni et.al. [30] is a strength of the peak in the residence times distribution centered at the half driving period, e.g., the area under the peak. The dependence of this quantity versus noise intensity has a bell shaped form. For a small amplitude of the periodic force it achieves a maximal value at an optimal *single* noise level. With the increase of the driving amplitude the maximal value of the strength of first peak tends to 1 and this maximal level keeps nearly constant in a finite region of noise intensity. We can, therefore, define a synchronization region at which this quantity is nearly 1. In this region the residence time distribution is represented by a single narrow peak at the half driving period and instantaneous phase locking can be observed again. Therefore, for a large enough driving amplitude (but still subthreshold) both definitions of synchronization match.

IV. Phase synchronization of switchings in periodically driven chaotic bistable system

As known, one of the typical properties of chaotic dynamical systems with quasiattractors is the coexistence of a number of different attractors in phase space [45,46]. The effect of external perturbations on such systems or the variation of their control parameters causes the interaction of attractors and can generate a set of interesting phenomena one of which is stochastic resonance [45]. Really, if a system has few attractors in the phase space separated by separatrix hyperplanes then the effect of noise or parameter variation can lead to the destruction of these hyperplanes and cause the intermittency of different types [47]. The low-dimensional dynamical systems with symmetry which have two chaotic attractors in the phase space separated by a separatrix plane form an important class of *bistable chaotic systems* [17]. Chaotic attractors play the role of the metastable states in this case. The external perturbations or parameter variation destroy the separatrix surface separating the basins of attraction of chaotic attractors that leads to the noise-induced or dynamical intermittency of «chaos-chaos» type, respectively [16,17]. In this case we can stay the tasks about SR and phase synchronization of switchings as well as in the case of stochastic bistable systems.

SR in chaotic bistable systems with continiuos and discrete time was studied in works [16,17,20,33,43,19]. It was shown, that SR in such systems can be realized via both parameter variation (in the absence of noise when the hoppings caused by the internal chaotic dynamics) and variation of the noise intensity with fixed values of other parameters. The results of investigations of synchronization-like phenomena in chaotic bistable systems are reported in work [48].

In this section we will consider the purely deterministic task about *phase synchronization* of switchings in a chaotic bistable system with discrete time. To describe this effect we use the same approach as in our previuos study.

Let us consider a discrete system

$$x_{n+1} = (ax_n - x_n^3)\exp(-x_n^2/b). \qquad (16)$$

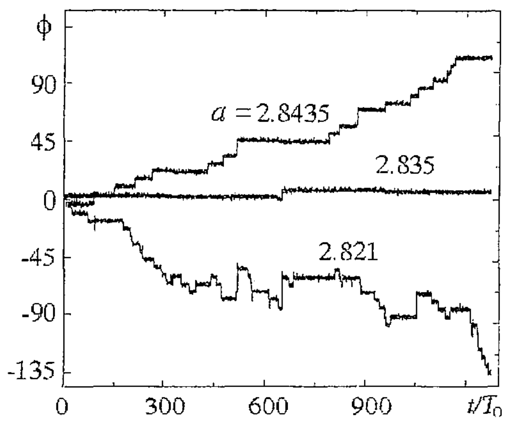

Fig. 8. The instantaneous phase difference in the units of driving period for indicated values of control parameter. Other parameters are A=0.025, Ω_0=0.0153

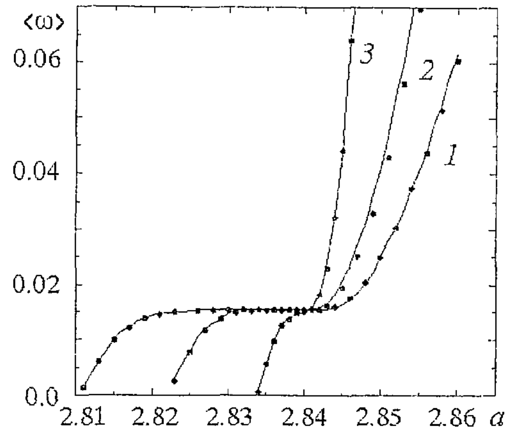

Fig. 9. Mean frequency (8) (solid line) and the mean switching frequency (13) (symbols) versus control parameter for different values of driving amplitude: A=0.025 (*1*), 0.015 (*2*) and 0.005 (*3*), Ω_0=0.0153

This system has the only stable fixed point (x_1=0) for 0<a<2 (parameter b=10). A pitchfork bifurcation takes place at a=2.39. The cascade of period-doubling is realized in the interval 2.4<a<2.5 and map (16) demonstrates chaos for a>2.5. For values of the control parameter from the interval 2.5<a≤2.84 there are two symmetric chaotic attractors whose basins of attraction are separated by the saddle point x_1=0. At a≅2.84 the crisis of attractors takes place. The merging of attractors is followed by a phenomenon of dynamical intermittency of the «chaos-chaos» type when the phase trajectory resides on the partial attractors for a long time and makes random switchings from one region to another. It is important to underline that random switchings in the considered system are caused by the internal dynamics only and parameter a controls by the «frequency of switchings» playing the role that is similar to the role of noise intensity in our previuos consideration.

Let us add periodic modulation to the system (16):

$$x_{n+1} = (ax_n - x_n^3)\exp(-x_n^2/b) + A\cos(\Omega_0 n), \qquad (17)$$

where A and Ω_0 are the amplitude and frequency of the periodic force. As in the previos section, we define the instantaneuos phase of the input signal and of chaotic signal through the Hilbert transfomation. The results of our phase calculations are pictured in Fig. 8. As clearly seen, for an optimal value of the control parameter the instanteneuos phases of the chaotic system and the input signal are locked. This fact means that in purely deterministic case we can observe exactly the same effect of phase synchronization of switchings as in periodically driven stochastic bistable systems. The effect of mean frequency locking also takes place and, as seen from Fig. 9, there is an interval of control parameter values where the mean frequency of chaotic oscillations coincides with Ω_0. The effective diffusion constant (14) (see Fig. 10) also demonstrates the behavior which is similar to the previuos one. It's very small in a certain interval of control parameter values, that means a high degree of phase the coherence between the input signal and switchings in the system (17). For a weak signal, the spreading of the distribution of the phase difference is sufficiently large and the system (17) cannot be treated as effectively synchronized, that is to be in good agreement with our previuos results. By analogy with the study in Sec. III, we constructed the region of phase synchronization of switchings, that is presented in Fig. 11. We have considered the system (17) being effectively synchronized by a periodic force if, as earlier, its instantaneuos phase is locked in course of at least 100 periods of the signal. The region

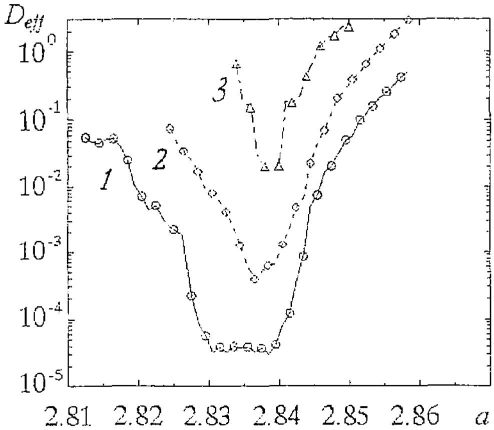

Fig. 10. The dependence of effective diffusion constant versus control parameter in purely deterministic case for different values of driving amplitude: A=0.025 (*1*), 0.015 (*2*) and 0.005 (*3*)

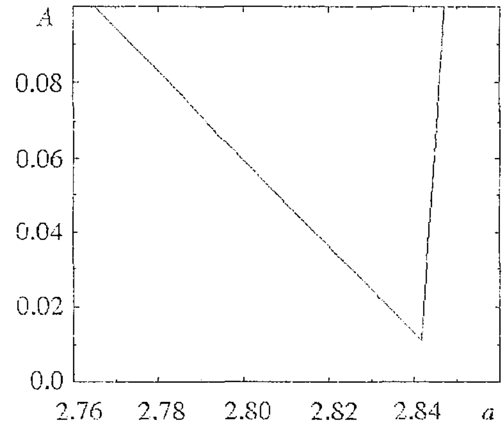

Fig. 11. The region of synchronization of switchings in deterministic case. The value of driving frequency is Ω_0=0.0153

has a tongue-like shape. Synchronization of switchings has a threshold character, although the threshold is very small in this case.

V. Mutual synchronization of switching processes

As it should follow from the above results, the effect of phase synchronization of switchings can be observed in periodically driven chaotic or stochastic bistable systems. It is natural to make the next step in study of synchronization in the systems which possessed of statistical time scales and to ask the following questions: Is it possible to observe mutual synchronization of switchings when the deterministic time scales are absence in the system? How long the switching processes in subsystems may be coherent? Can we generalize the notion of *phase synchronization* for this case? The positive answer to the first question has been already given in [32] and [50], where the phenomena of mutual synchronization of switching processes in symmetrically coupled stochastic and chaotic bistable systems have been reported, respectively. To answer the other questions, in this section we will consider phenomena of synchronization of switching processes in symmetrically coupled stochastic and chaotic bistable systems from the above developed approach point of view.

At it first, let us consider two symmetrically coupled stochastic bistable systems, which governed by the following SDE:

$$\dot{x}_1 = \alpha_1 x_1 - x_1^3 + \xi_1(t) + \gamma(x_2 - x_1),$$
$$\dot{x}_2 = \alpha_2 x_2 - x_2^3 + \xi_2(t) + \gamma(x_1 - x_2), \qquad (18)$$

where γ is the coupling parameter and $\xi_{1,2}$ are statistically independent white noise sources, e.g., $\langle\xi_i(t)\xi_j(t+s)\rangle=2D\delta_{ij}\delta(s)$. Parameter α (α>0) of a subsystem characterizes the deepness of potential wells and can be used as a control parameter changing of which we can detune subsystems. Bifurcations occuring in this system were analyzed in detail in work [32]. It was shown, that the growth of the coupling parameter causes the reconstruction of the two-dimensional stationary probability density and leads to the synchronization of switchings in subsystems. As mentioned in Sec. II, the stochastic bistable systems such as (2) and (19) have a statistical time scale which deals with the

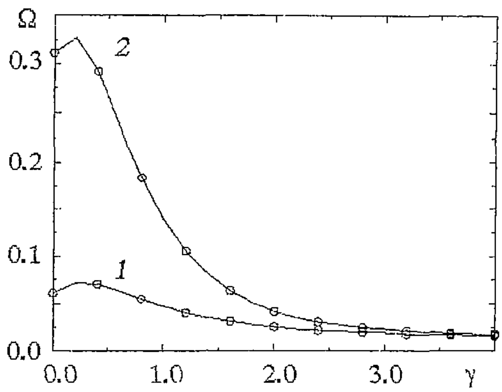

Fig. 12. The dependencies of mean switching frequencies of subsystems (*1*) and (*2*) versus coupling parameter. Other parameters are D=1.6, α_1=5.0, α_2=3.5

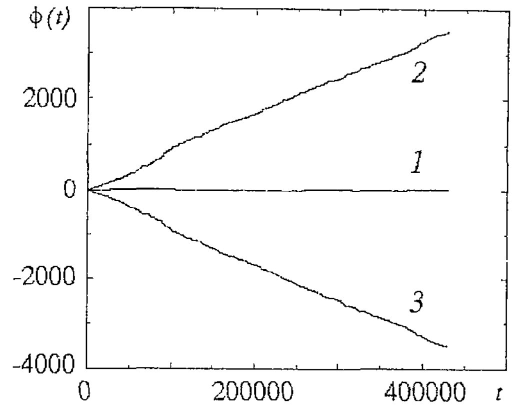

Fig. 13. The instantaneous phase difference versus time for different values of detuning α_1/α_2=0.98 (*1*), α_1/α_2=0.7 (*2*), α_1/α_2=1.3 (*3*). Other parameters are D=1.6, γ=2.5

mean escape rate from a potential well and is represented in the frequency domain by the mean frequency of switchings. As seen from Fig. 12, the increase of the coupling parameter leads to the approach of mean switching frequencies, that means the synchronization of switching processes.

As it was mentioned above, the description of synchronization of stochastic bistable systems by means of mean switching frequency only is not complete. It provides no information about matching of switchings in subsystems. By analogy with our previous consideration, let us consider this effect in terms of the theory of phase synchronization. The instantaneos phase of stochastic oscillations can be introduced by the above described ways, e.g., through the HT (6) and through the switchings times (12). In this case, we introduced the phase of stochastic bistable systems through the switching times. The dependence of the instantaneos phase difference versus time is presented in Fig. 13. As seen, for an optimal value of coupling and detuning between subsystems the phases of subsystems are *instantaneuosly locked*. When the values of parameters differ from the optimum the increase of the phase difference should be observed. Such a behavior of the phase difference is typical for the phenomenon of mutual phase synchronization of classical coupled oscillatory systems. The dependencies of the difference of mean frequencies versus detuning for different values of the coupling parameter are presented in Fig. 14. They also demonstrate the behavior which is similar to the classical one. In some range of detuning values they coincide that means the *mutual locking of mean frequencies*.

We have used the effective diffusion constant (14) again (see Fig. 15), as the measure of the coherence degree between switching processes in subsystems. Its behavior clearly demonstrates that for an optimal value of coupling the spreading of the initial distribution of the phase difference is very small and switchings in subsystems are highly coherent.

Now, let us consider the purely deterministic case when the switchings in subsystems are caused by their internal dynamics only and there are no any sources of random forces in the system. A similar task was considered in work [50]. A dynamical system considered in this work is described by the following ordinary differental equations:

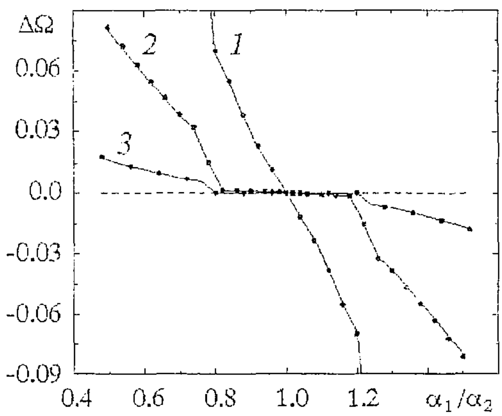

Fig. 14. The dependence of the difference of mean frequencies versus parameters detuning value for different value of coupling parameter: γ=0.5 (*1*), γ=1.5 (*2*), γ=2.5 (*3*)

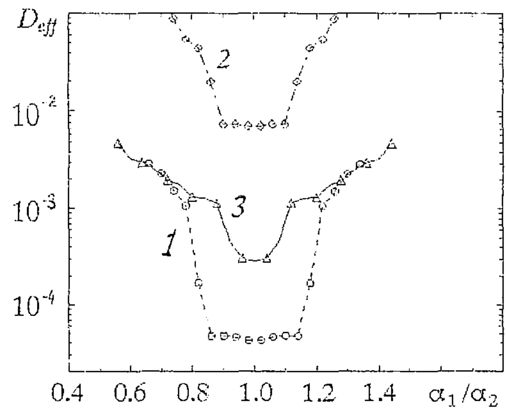

Fig. 15. The effective diffusion constant versus detuning value for different values of coupling parameter γ=2.5 (*1*), γ=0.5 (*2*), γ=1.5 (*3*). Noise intensity equals D=1.6

$$\dot{x}_{1,2} = 10(y_{1,2}-x_{1,2}) + \gamma(x_{2,1} - x_{1,2}),$$
$$\dot{y}_{1,2} = r_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2}, \qquad (19)$$
$$\dot{z}_{1,2} = x_{1,2}y_{1,2} - 8/3\,z_{1,2}.$$

It is clearly seen that these equations describe the dynamics of two symmetrically coupled Lorenz systems, each of them is considered as a chaotic bistable system [49]. As it was found in [50], the effect of synchronization of switching processes in subsystems takes place for some value of coupling parameter γ. This effect was described in terms of the mean switching frequency and residence time distributions, which, as mentioned above do not provide any information about the matching of switchings in subsystems. It is reasonable to try to generalize the above approach based on the notion of *instantaneous phase* of oscillations for this purely deterministic case, especially as the Lorenz system can be considered as a bistable system driven by some effective noise [49]. We defined the instantaneous phases of the subsystems through the switching times as in the previous case of the interaction of stochastic bistable systems. As seen from Fig. 16, we have got exactly the same results. The phases of switchings in the subsystems are instantaneuosly

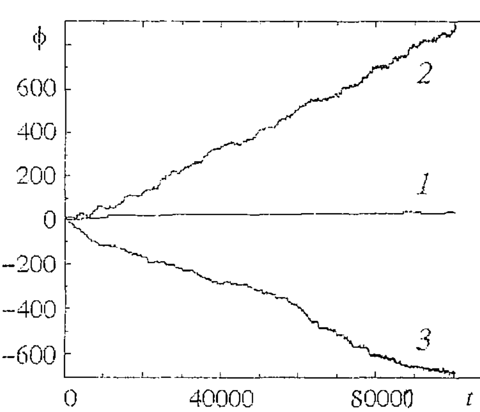

Fig. 16. The dependence of phase deference of couled Lorenz systems versus time for different values of detuning: r_1/r_2=1.00357 (*1*), 1.042857 (*2*), 0.9589 (*3*). The coupling parameter is γ=6.0

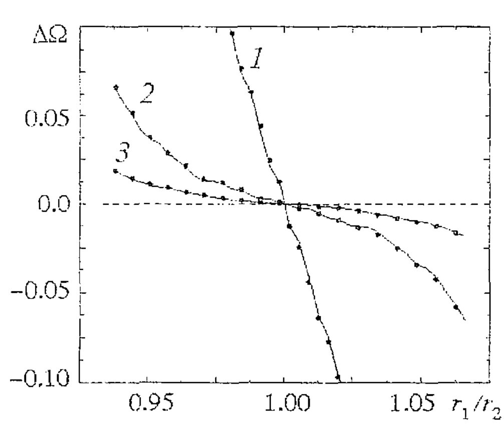

Fig. 17. The dependence of the deference of mean frequencies versus detuning value for different value of coupling parameter: γ=1.0 (*1*), 4.0 (*2*), 6.0 (*3*)

locked for a certain value of coupling and detuning between the subsystems. The mean frequencies of switchings coincide with the mean frequency of chaotic oscillations and demonstrate the behavior which is similar to the previous one. The increase of the coupling parameter leads to their approaching to each other, e.g., the effect of *mutual locking of mean frequencies of chaotic oscillations* takes place. Moreover, the mean frequencies of the subsystems coincide in some range of detuning values for sufficiently large values of γ (see Fig. 17), that means the existence of some region on the parameter plane «coupling - detuning» in which the switchings are synchronized and occured at exactly the same times.

Thus, as seen from the results presented in this section the notion of *phase synchronization* can be successfully generalized for the case of the interaction of subsystems which have only statistical time scales.

VI. Conclusion

We have studied the phenomena of synchronization of switching processes in stochastic and chaotic bistable systems in classical terms of phase synchronization. We have used two definitions of the instantaneous phase, basing on the analytic signal concept and on the switching times sequences. The first definition appreciates both switchings between wells and intrawell motion, whereas the second definition takes into consideration the process of switchings only. Both phase definitions provide the same results for averaged quantities. The effect of phase synchronization of switching process is shown to occur in a finite region of noise intensity or in a finite range of a control parameter in purely deterministic case. However, this effect is restricted by comparatively large amplitudes of external signal. The above results of our study allow to understand deeper the essence of the phenomenon of synchronization in SR systems, which was previously described in terms of residence times distributions [30]. As it was mentioned above, synchronization of switchings in the frame of this approach deals with the presence of single gaussian-like peak in the residence time distribution at the half period of external force [30]. For small signals the measure which was proposed by authors in [30] demonstrates the resonance-like behavior and achieves its maximal value at an optimal single level of noise intensity. However, according to the classical definition, the phenomenon of synchronization is characterized by the presence of a finite region of control parameter values in which the input signal and response are synchronized. The appearance of such a region, in our case, is possible for large enough amplitude of external force only and it deals with the effect of noise enhanced phase locking which was described in details in the present work. We have used the effective diffusion constant as the measure of the phase coherence between the input signal and response. This characteristic passes through a minimum being plotted versus noise intensity or control parameter that means the increase of «phase order» in a system. This ordering is also reflected in a non-monotonous behavior of the source entropy which takes its minimal value (see Fig. 18) in the case of synchronization [51]. Moreover, the effect of phase synchroniza-

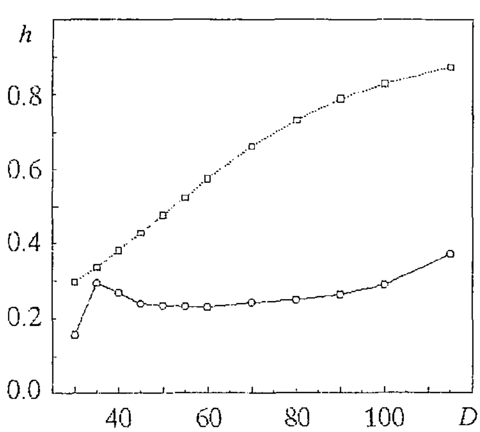

Fig. 18. The dependence of source entropy calculated by the residence times distribution for Shmitt trigger - symbols O. Symbols □ indicate the source entropy of input signal. See [51] for details

tion can be observed in a more complex case of the interaction of symmetrically coupled stochastic or chaotic bistable systems when there are no deterministic time scales in the system and the interaction of statistical time scales takes place.

We acknowledge fruitful discussions with L. Schimansky-Geier, Yu.Klimontovich, A.Malakhov, F.Moss, I.Khovanov, M.Rosenblum, A.Pikovsky, and J.Kurths. A.Neiman is a recipient of a Fetzer Institute post-doctoral fellowship.

This work has been supported in part by RFBR (grant № 98-02-16531), by the State Committee on Higher Education of the Russian Federation (grant № 97-0-8.3-47) and by the Royal Society of London.

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Department of Physics, Saratov State University
Center for Neurodynamics, University of Missouri at St. Louis, USA

Received 19.06.98

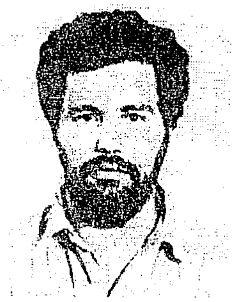
УДК 621.373

ФАЗОВАЯ СИНХРОНИЗАЦИЯ ПЕРЕКЛЮЧЕНИЙ В СТОХАСТИЧЕСКИХ И ХАОТИЧЕСКИХ БИСТАБИЛЬНЫХ СИСТЕМАХ

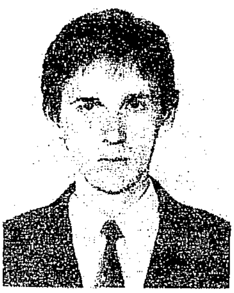
В.С. Анищенко, А.Б.Нейман, А.Н. Сильченко, И.А. Хованов

В данной работе эффекты синхронизации переключений в неавтономных и связанных стохастических и хаотических бистабильных системах рассмотрены с позиций классической теории колебаний. Мгновенная фаза колебаний в системах демонстрирующих бистабильную динамику вводилась как с помощью концепции аналитического сигнала, так и непосредственно через времена переключений между метастабильными состояниями. Показано, что действие шума (изменение управляющего параметра в детерминистском случае) обуславливает фазовую синхронизацию переключений. В качестве величины, характеризующей степень когерентности переключений и внешнего воздействия (или взаимной когерентности в случае связанных систем) использовался коэффициент эффективной диффузии мгновенной разности фаз, традиционно применяемый в теории колебаний.

Anishchenko Vadim Semenovich was born in Saratov in 1943. He obtained his Candidate of Physical and Mathematical Sciences (Ph.D.) (1970) and a degree of Doctor of Science (1986) from the Saratov State University. In 1987 he read the lectures at Humboldt University (Berlin). From 1988 he is a chief of Radiophysics Chair at Saratov State University. He is the author more 200 scientific papers and 4 books. His works are devoted to the problems of statistical radiophysics, theory of oscillations and to experimental research of dynamical chaos. He is Soros's professor (1994, 1998), Honoured scientist of Russia (1995), corresponding member of Russian Academy of Natural Science (1998). E-mail: wadim@chaos.ssu.runnet.ru



Neiman Alexander Borisovich was born in Saratov in 1962. He has from graduated Saratov State University in 1993 and obtained his Candidate of Physical and Mathematical Sciences (Ph.D.) in 1991. He is author more than 40 articles. In 1998 he got a degree of Doctor of Science. He is currently a recipient of a Fetzer Institute post-doctoral fellowship and works at the Center of Neurodynamics (St. Louis, USA). Scientific interests: nonlinear dynamics, stochastic dynamics of nonlinear systems, theory of information. E-mail: neiman@chaos.ssu.runnet.ru



Silchenko Alexander Nikolaevich was born in Saratov in 1972. He was educated in Saratov, gaining his undergraduate degree in physics from Saratov State University in 1994. He is currently Ph.D. student in Saratov University. Current research interests are nonlinear dynamics, synchronization in chaotic and stochastic systems, controlling chaos. E-mail: sendy@chaos.ssu.runnet.ru

Khovanov Igor Alexandrovich was born in Saratov in 1971. He has graduated from Saratov State University in 1993 and obtained his Candidate of Physical and Mathematical Sciences (Ph.D.) in 1996. Current research interests are nonlinear dynamics, statistical physics, biophysics, time-series analysis. E-mail: igor@chaos.ssu.runnet.ru.