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Synchronization of oscillators with hard excitation coupled with delay Part 1. Phase approximation

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Abstract. *Aim* of this work is to develop the theory of mutual synchronization of two oscillators with hard excitation associated with a delay. Taking into account the delay of a coupling signal is necessary, in particular, when analyzing synchronization at microwave frequencies, when the distance between the oscillators is large compared to the wavelength. *Methods.* Theoretical analysis is carried out under the assumption that the delay time is small compared to the characteristic time for the oscillations. The phase approximation is used when the frequency mismatch and the coupling parameter are considered small. *Results.* Taking into account the change in oscillation amplitudes up to first-order terms in the coupling parameter, a generalized Adler equation for the phase difference of the oscillators is obtained, which takes into account the combined type of the coupling (dissipative and conservative coupling) and non-isochronism. The conditions for saddle-node bifurcations are found and the stability of various fixed points of the system is analyzed. The boundaries of the domains of in-phase and anti-phase synchronization are plotted on the plane of the parameters "frequency mismatch–coupling parameter". *Conclusion.* It is shown that, depending on the control parameters (non-isochronism parameter, excitation parameter, phase advance of the coupling signal), the system exhibits behavior typical of either dissipative or conservative coupling. The obtained formulas allow for trace the transition from one type of coupling to another when varying the control parameters.

Keywords: coupled generators, self-oscillating systems with hard excitation, synchronization, delay, phase approximation, generalized Adler equation.

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Introduction

The study of mutual synchronization in ensembles of coupled oscillators is a fundamental problem of radiophysics and is of great importance for many applications [1–5]. In particular, for modern ultrahigh frequency electronics, the addition of the capacities of several coupled generators [6] is of great interest. Synchronization in systems of coupled magnetrons and vircators has been most widely studied (see, for example, [7–10]). In particular, mutual synchronization was experimentally observed in ensembles of several relativistic magnetrons with different coupling topology [11]. Recently, the problem of mutual synchronization of powerful gyrotrons [12,13] has also attracted attention, since complexes consisting of several gyrotrons are used to heat plasma in controlled thermonuclear fusion plants [14].

Since at ultrahigh frequencies the distance between the coupled generators can significantly exceed the wavelength, it is necessary to take into account the delay of the signal propagating between them. There is no doubt that the synchronization pattern in delayed systems has a number of specific features compared to systems with a small number of degrees of freedom. As is known, in the theory of synchronization, two types of communication are usually distinguished: dissipative (diffusive) and conservative (inertial, reactive) [2–5, 15, 16]. These cases differ in the arrangement of synchronization languages, and with conservative communication, the synchronization mode becomes bistable: synchronization is possible on both in-phase and antiphase modes. In the works [17–19], where a simple model of two self-oscillating systems with a limit cycle associated with delay was studied, it was shown that, depending on the phase of the signal, either dissipative or conservative coupling dominates in the communication channel. Since the phase advance is determined by the time of propagation of the signal between the generators, when the distance between the generators changes by an order of magnitude of the wavelength, the nature of the connection and, accordingly, the device of the synchronization languages can change significantly.

It should be noted that in high-power gyrotrons, the maximum efficiency is achieved in the hard excitation mode [20]. Therefore, the study of the synchronization features of generators operating in the hard excitation mode is of considerable interest. In particular, in [21], the theory of synchronization of a generator with rigid excitation, which is affected by an external harmonic signal, was developed. A number of important differences were found from the well-known pattern of synchronization of a generator with mild self-excitation. These differences are mainly due to the unstable nature of the self-oscillating system with rigid excitation [1, 3].

This work is devoted to the study of the mutual synchronization of two generators with rigid excitation associated with delay. The work consists of two parts. In the first part, the problem is solved within the framework of the phase approximation, which is valid in the case of two weakly coupled systems whose natural frequencies differ slightly. At the same time, it can be assumed that the coupling practically does not change the oscillation amplitude of the interacting subsystems, so that we can limit ourselves exclusively to analyzing the dynamics of the phase difference, which greatly simplifies the consideration. At the same time, from a practical point of view, this situation is of the greatest interest.

A more rigorous bifurcation analysis of synchronization, which is not limited to the phase approximation, will be presented in the second part of this paper.

1. Basic equations and classification of fixed points

In the works [17–19], where a system of two generators with soft self-excitation associated with delay was considered, a system of quasi-linear differential equations for slowly varying complex oscillation amplitudes was formulated, which was further generalized to the case of two coupled gyrotrons [12, 13]. A model of coupled generators with rigid excitation can be constructed in a similar way. It is only necessary to modify the function that defines the nonlinear characteristic of the generator so that it describes the effects of hard excitation. The generators are considered identical except for a slight natural frequency disorder $\omega_1 \neq \omega_2$, and $|\omega_1 - \omega_2| \gg \omega_{1,2}$. As a result, we can write the following system of equations:

$$\begin{aligned} \frac{dA_1}{dt} + \frac{i\Delta}{2}A_1 &= \left(\sigma + (1+ib)\left|A_1\right|^2 - \left|A_1\right|^4\right) A_1 + \rho e^{-i\psi} A_2\left(t-\tau\right), \\ \frac{dA_2}{dt} - \frac{i\Delta}{2}A_2 &= \left(\sigma + (1+ib)\left|A_2\right|^2 - \left|A_2\right|^4\right) A_2 + \rho e^{-i\psi} A_1\left(t-\tau\right). \end{aligned} \tag{1}$$

Here $A_{1,2}$ is slowly changing (compared to $\exp\left(i\omega_{1,2}t\right)$) complex oscillation amplitudes of the first and second generators, σ is excitation parameter, b is nonisochronity parameter, Δ is normalized natural frequency detuning, τ is delay time (for more information, see [12,13,17–19]). The coupling coefficient ρ is determined in such a way that the value ρ^2 characterizes the proportion of power coming from the output of one generator to the input of another, while it obviously takes the values $0 < \rho < 1$ [7–13, 17–19]. The parameter ψ represents the phase shift of the signal propagating in the communication channel. All values in (1) are considered dimensionless, and in the accepted normalization, the hard excitation mode is realized at $-1/4 < \sigma < 0$ (see [3,21]).

We will assume that the delay time is small compared to the characteristic time for the establishment of oscillations, that is, $\tau \ll 1$. In this case, the equations (1) turn into a system of ordinary differential equations

$$\begin{aligned} \frac{dA_1}{dt} + \frac{i\Delta}{2}A_1 &= \left(\sigma + (1+ib)\left|A_1\right|^2 - \left|A_1\right|^4\right) A_1 + \rho e^{-i\psi} A_2, \\ \frac{dA_2}{dt} - \frac{i\Delta}{2}A_2 &= \left(\sigma + (1+ib)\left|A_2\right|^2 - \left|A_2\right|^4\right) A_2 + \rho e^{-i\psi} A_1. \end{aligned} \tag{2}$$

Assuming $A_{1,2} = R_{1,2}\exp\left(i\varphi_{1,2}\right)$, where $R_{1,2}$ and $\varphi_{1,2}$ are real amplitudes and phases of oscillations, respectively, we obtain from (2) a system of equations of the third order

$$\begin{aligned} \dot{R}_1 &= \left(\sigma + R_1^2 - R_1^4\right) R_1 + \rho R_2 \cos\left(\psi + \varphi\right), \\ \dot{R}_1 &= \left(\sigma + R_2^2 - R_2^4\right) R_2 + \rho R_1 \cos\left(\psi - \varphi\right), \\ \dot{\varphi} &= -\Delta + b\left(R_2^2 - R_1^2\right) + \rho\left[\frac{R_1}{R_2}\sin\left(\psi - \varphi\right) - \frac{R_2}{R_1}\sin\left(\psi + \varphi\right)\right], \end{aligned} \tag{3}$$

where $\varphi = \varphi_1 - \varphi_2$ is the phase difference, the dot on top means time differentiation t.

To analyze synchronization modes, it is first necessary to consider the fixed points of the system (3). Note that in a system of coupled generators with hard excitation the situation becomes more complicated compared to that considered in [17–19], since the number of fixed points increases. Indeed, let us first consider isolated generators ($\rho = 0$). In this case, from the equations (3) we obtain

$$\sigma + R_{1,2}^2 - R_{1,2}^4 = 0. \tag{4}$$

The solutions to this equation have the form

$$R_{1,2}^2 = R_\pm^2 + \frac{1 \pm \sqrt{1+4\sigma}}{2}. \tag{5}$$

The solution R_+ is stable, but R_- is unstable [3]. In addition, for $\sigma < 0$ the zero solution $R_{1,2} = 0$ is also stable.

Now let us consider the case of coupled generators, assuming that the coupling is weak, that is, $\rho \ll 1$, and for simplicity we will also set $\Delta = 0$. In this case, to determine the fixed points we will have the equations

$$\begin{aligned}
&\left(\sigma + R_1^2 - R_1^4\right) R_1 + \rho R_2 \cos\left(\psi + \varphi\right) = 0,\\
&\left(\sigma + R_2^2 - R_2^4\right) R_2 + \rho R_1 \cos\left(\psi - \varphi\right) = 0,\\
&b\left(R_2^2 - R_1^2\right) + \rho\left[\frac{R_1}{R_2}\sin\left(\psi - \varphi\right) - \frac{R_2}{R_1}\sin\left(\psi + \varphi\right)\right] = 0.
\end{aligned} \tag{6}$$

The solutions of the equations (6) can be divided into symmetric ones, for which the oscillation amplitudes of the first and second generators are the same, and asymmetric ones. Obviously, there are two types of symmetric solutions. First, these are solutions for which $R_{1,2} = R_+ + O\left(\rho\right)$. They correspond to the modes of in-phase and out-of-phase synchronization; we will denote them as I and A, respectively. There are also solutions for which $R_{1,2} = R_- + O\left(\rho\right)$, they will be denoted as I_- and A_-. At the same time, it is obvious that since R_- corresponds to an unstable oscillation mode of an isolated generator, then the solutions I_-, A_- will be unstable.

Next, we will discuss asymmetric solutions. The equations (6) obviously have solutions for which one of the amplitudes is close to R_+, and the other to R_-. Indeed, with weak coupling, in addition to the two limit cycles mentioned above, for which $R_{1,2} \approx R_+$ or $R_{1,2} \approx R_-$, two more limit cycles appear in the phase space of the system (3), for which $R_1 \approx R_+$, and $R_2 \approx R_-$ or vice versa. Both of these cycles are obviously unstable. With an increase in the coupling parameter, as a result of saddle nodal bifurcations, pairs of saddle (unstable node points) are born on these cycles. We will denote them as $S_k, k = 1, 2, 3, 4$.

In addition, there are four more asymmetric solutions, for which one of the amplitudes is close to zero, and the other is to R_+ or to R_-. However, the bifurcations of these points are not described by the phase approximation, so we will postpone their consideration until the second part of this paper.

Finally, the equations (6) have a zero solution $R_1 = R_2 = 0$, which, at least with weak coupling, is stable.

As is known, with weak communication, the transition to synchronous mode is carried out as a result of mutual frequency capture [1–4]. Such a mechanism corresponds to a saddle nodal bifurcation. Since the number of possible fixed points increases in the case of coupled systems with rigid excitation, the situation becomes more complicated. The bifurcations of symmetric and non-symmetric fixed points should be considered separately.

2. Synchronization analysis in the phase approximation

2.1. Saddle-node bifurcations of symmetric solutions. As noted above, in the case of weak coupling and small detuning, the bifurcation of the fixed points of the system (3) can be analyzed within the framework of a phase approximation. By introducing a weak coupling between generators, $\rho \ll 1$, stable solutions $R_{1,2} = R_+$ turn into a stable limit cycle. If the frequency disorder is small, a saddle node bifurcation occurs on this cycle, as a result of which the points I and A are born (see section 1). Since the coupling is considered weak, the oscillation amplitudes vary slightly compared to isolated generators, $R_{1,2} \approx R_+$. In this case, the system (3) is reduced to a first-order equation for the phase difference φ, which is often called the Adler equation [3, 5, 22]. However, as shown in [17–19], in the case of delay-related generators, the Adler equation in its traditional form does not adequately describe the synchronization pattern, in particular, to trace the transition from dissipative to conservative coupling. It is necessary to use a more accurate approximation, finding the oscillation amplitudes up to terms of the order of ρ. So, we will look for solutions in the form of $R_{1,2} = R_+ + r_{1,2}$, where $r_{1,2} \sim \rho$ are small additives. Substituting these relations into the first two equations of the system (6) and limiting them to terms of the order ρ, we obtain

$$\begin{aligned} \left(\sigma + R_+^2 - R_+^4\right) r_1 + \left(2R_+ - 4R_+^3\right) R_+ r_1 + \rho R_+ \cos\left(\psi + \varphi\right) = 0, \\ \left(\sigma + R_+^2 - R_+^4\right) r_2 + \left(2R_+ - 4R_+^3\right) R_+ r_2 + \rho R_+ \cos\left(\psi - \varphi\right) = 0. \end{aligned} \tag{7}$$

From here, taking into account (4), we find that

$$\begin{aligned} r_1 &= -\frac{\rho}{2R_+\left(1 - 2R_+^2\right)} \cos\left(\psi + \varphi\right) = 0, \\ r_2 &= -\frac{\rho}{2R_+\left(1 - 2R_+^2\right)} \cos\left(\psi - \varphi\right) = 0. \end{aligned} \tag{8}$$

Now it is possible to approximate the ratio of the oscillation amplitudes included in the equation for the phase in the system (3). After a number of transformations, we get

$$\begin{aligned} \frac{R_1}{R_2} &\approx 1 - \frac{2\rho}{\sqrt{1+4\sigma}\left(1 + \sqrt{1+4\sigma}\right)} \sin\psi \sin\varphi, \\ \frac{R_2}{R_1} &\approx 1 + \frac{2\rho}{\sqrt{1+4\sigma}\left(1 + \sqrt{1+4\sigma}\right)} \sin\psi \sin\varphi. \end{aligned} \tag{9}$$

These relations should be substituted into the third equation of the system (3). Also included in this equation is the term $b\left(R_2^2 - R_1^2\right)$, which is decomposed up to terms of the order ρ^2:

$$\begin{aligned} b\left(R_2^2 - R_1^2\right) \approx \frac{b\rho}{1 - 2R_+^2}\left(\cos\left(\psi + \varphi\right) - \cos\left(\psi - \varphi\right)\right) + \\ + \frac{b\rho^2}{4R_+^2\left(1 - 2R_+^2\right)}\left(\cos^2\left(\psi - \varphi\right) - \cos^2\left(\psi + \varphi\right)\right). \end{aligned} \tag{10}$$

As a result, substituting (9) and (10) into the third equation of the system (3), we obtain the generalized Adler equation:

$$\dot{\varphi} + \Delta = -2\rho\sin\varphi\left(\cos\psi - \frac{b}{\sqrt{1+4\sigma}}\sin\psi\right) - \frac{\rho^2}{\lambda}\sin 2\varphi\left(\sin^2\psi - \frac{b}{4\sqrt{1+4\sigma}}\sin 2\psi\right), \tag{11}$$

where the designation is entered

$$\lambda = \frac{1}{2}\sqrt{1+4\sigma}\left(1 + \sqrt{1+4\sigma}\right). \tag{12}$$

A member of the order ρ proportional to $\sin\varphi$, on the right side of the equation (11) is responsible for the dissipative coupling, a member of the order ρ^2 proportional to $\sin 2\varphi$ for the conservative [5, 16].

In Fig. 1 shows the dependence of the parameter λ on the excitation parameter σ, constructed according to (12). At $\sigma \to -0.25$, the parameter λ turns to zero, and with the growth of σ increases monotonously and at $\sigma = 0$ becomes equal to one.

In synchronization mode, when $\varphi = \text{const}$, the equation (11) is rewritten as

$$\Delta = -2\rho\sin\varphi\left(\cos\psi - \frac{b}{\sqrt{1+4\sigma}}\sin\psi\right) - \frac{\rho^2}{\lambda}\sin 2\varphi\left(\sin^2\psi - \frac{b}{4\sqrt{1+4\sigma}}\sin 2\psi\right). \tag{13}$$

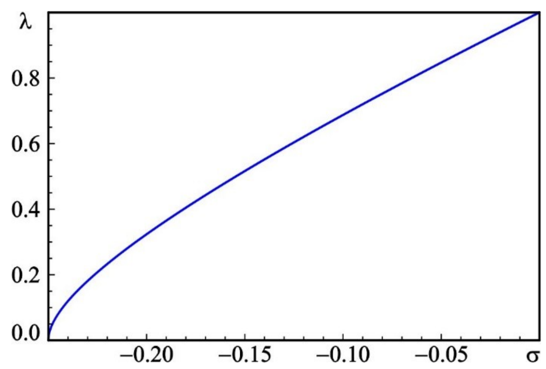

Fig 1. Dependence of the parameter λ on the excitation parameter σ

It is not difficult to show that the stability boundary (that is, the condition of saddle-node bifurcation) is determined from the following relation, which is the condition for merging the two roots of the equation (13):

$$\frac{d\Delta}{d\varphi} = -2\rho\cos\varphi\left(\cos\psi - \frac{b}{\sqrt{1+4\sigma}}\sin\psi\right) - \frac{2\rho^2}{\lambda}\cos 2\varphi\left(\sin^2\psi - \frac{b}{4\sqrt{1+4\sigma}}\sin 2\psi\right) = 0. \quad (14)$$

From here we find

$$\rho = -\frac{4\lambda\cos\varphi\left(\cos\psi\sqrt{1+4\sigma} - b\sin\psi\right)}{\cos 2\varphi\left(4\sin^2\psi\sqrt{1+4\sigma} - b\sin 2\psi\right)}. \quad (15)$$

The relations (13) and (15) set parametrically the boundaries of the synchronization language on the plane Δ, ρ. Note that with

$$\operatorname{ctg}\psi = \frac{b}{\sqrt{1+4\sigma}} \quad (16)$$

the connection is purely conservative. When $\sin\psi = 0$, as well as when

$$\operatorname{tg}\psi = \frac{b}{2\sqrt{1+4\sigma}} \quad (17)$$

the connection is purely dissipative. In the case of isochronous oscillators ($b = 0$) (16) and (17) turn into $\cos\psi = 0$ and $\sin\psi = 0$, respectively.

In Fig. 2 the corresponding dependencies are constructed $\psi = \psi(b)$. Note that in the isochronous case, the bond is purely dissipative when $\psi = \pi n$, and purely conservative, when $\psi = \pi n + \pi/2$, $n \in \mathbb{Z}$ (cf. [17]). With an increase in b, the value of the phase raid, in which the dissipative bond dominates, increases, while the value of ψ, in which the bond is conservative, decreases. At the point where the graphs of the functions in Fig. 2 intersect, that is, when $b = \sqrt{2(1+4\sigma)}$, both terms on the right side (13) vanish, that is, the situation becomes degenerate, and the equation (13) is no longer applicable. To correctly describe the synchronization process, it is necessary to look for solutions for $r_{1,2}$ up to terms of the order ρ^2, and terms of the order ρ^3 will appear in the generalized Adler equation. Note that degeneration occurs when $\psi = \arctan\left(1/\sqrt{2}\right) \approx \pi n + 0.2\pi$.

In Fig. 3 on the plane of the parameters Δ, ρ, the regions of in-phase (I) and antiphase (A) synchronization are constructed for different values of the phase raid in the communication channel ψ at $b = 0.2$. For for certainty, let's choose $\sigma = -0.16$, then $R_+ = \sqrt{0.8} \approx 0.894$,

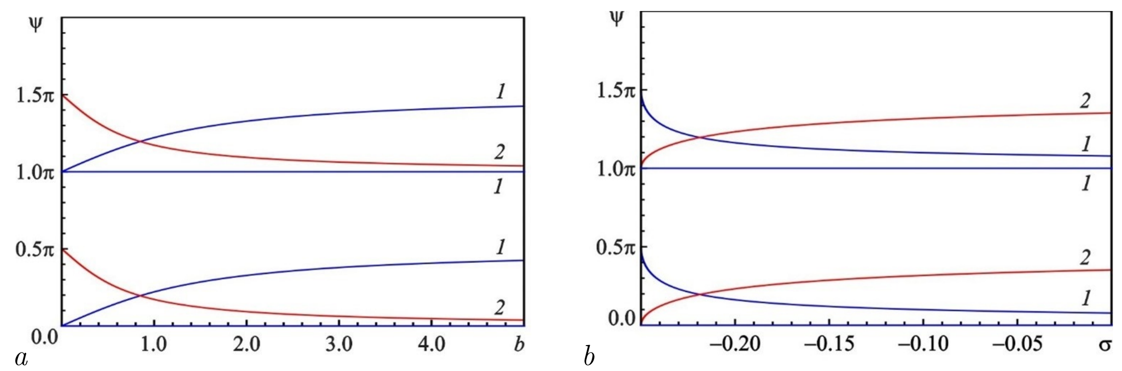

Fig 2. Dependences of the phase shift ψ, at which the coupling is purely dissipative (curves *1*) and purely conservative (curves *2*), on the nonisochronism parameter b for $\sigma = -0.16$ (a) and on the parameter σ for $b = 0.5$ (b)

$R_- = \sqrt{0.2} \approx 0.447$. At $\psi = 0$, when dissipative coupling dominates (Fig. 3,a), synchronization is possible only in common mode, and the boundaries of the language are straight lines. At $\psi \neq 0$, conservative coupling begins to affect (Fig. 3, b, c) and synchronization regions appear in the antiphase mode. The multistability of synchronous modes is observed within this region.

With an increase in ψ, the size of the antiphase synchronization region increases, and at $\psi \approx 0.4\pi$, when the coupling becomes purely conservative, the synchronization boundaries on the in-phase and antiphase modes are degenerated (Fig. 3, d). In this case, phase bistability is observed throughout the synchronization region. With a further increase in the phase inrush, the synchronization boundaries in the common-mode and antiphase modes change places. From Fig. 3, e it can be seen that now, with weak communication, synchronization is possible only in the antiphase mode. At $\psi = \pi$, the dissipative coupling dominates again, but the generators synchronize in the opposite phase (see Fig. 3, f). It is clear that there is a fundamental difference between Fig. 3, a and 3, f none: in the second case, during the passage through the communication channel, the signal acquires an additional phase shift $\psi = \pi$ and enters the oscillatory system of another generator exactly in phase with by his own hesitation. A similar behavior occurs for coupled generators with mild excitation [17, 18].

Now let's consider the transformation of synchronization languages when changing the excitation parameter. In Fig. 4 common-mode and antiphase synchronization regions are constructed at $\psi = 0.3\pi$ and $b = 0.5$. According to the formulas (16) and (17), a purely dissipative bond is realized at $\sigma \approx -0.24$, and a purely conservative one at $\sigma \approx -0.13$. Indeed, in Fig. 4, a only common-mode synchronization is observed. When the σ parameter is increased, the influence of a conservative connection begins to manifest itself. This leads to the appearance of areas of antiphase synchronization, which increase in size, as shown in Fig. 4, b, c. In Fig. 4, d the boundaries of in-phase and out-of-phase synchronization are degenerating. With a further increase in σ, the in-phase synchronization boundary breaks away from the horizontal axis and swaps places with the antiphase synchronization boundary (Fig. 4, e, f). Now, with small values of the coupling parameter and frequency detuning, only antiphase synchronization takes place.

Similarly, it is possible to find the boundaries of a saddle-node bifurcation on an unstable limit cycle, as a result of which the points I_- and A_- are born. To do this, it is enough in the above formulas (7), (8) to replace R_+ with R_-. As a result, the boundaries of the saddle-node

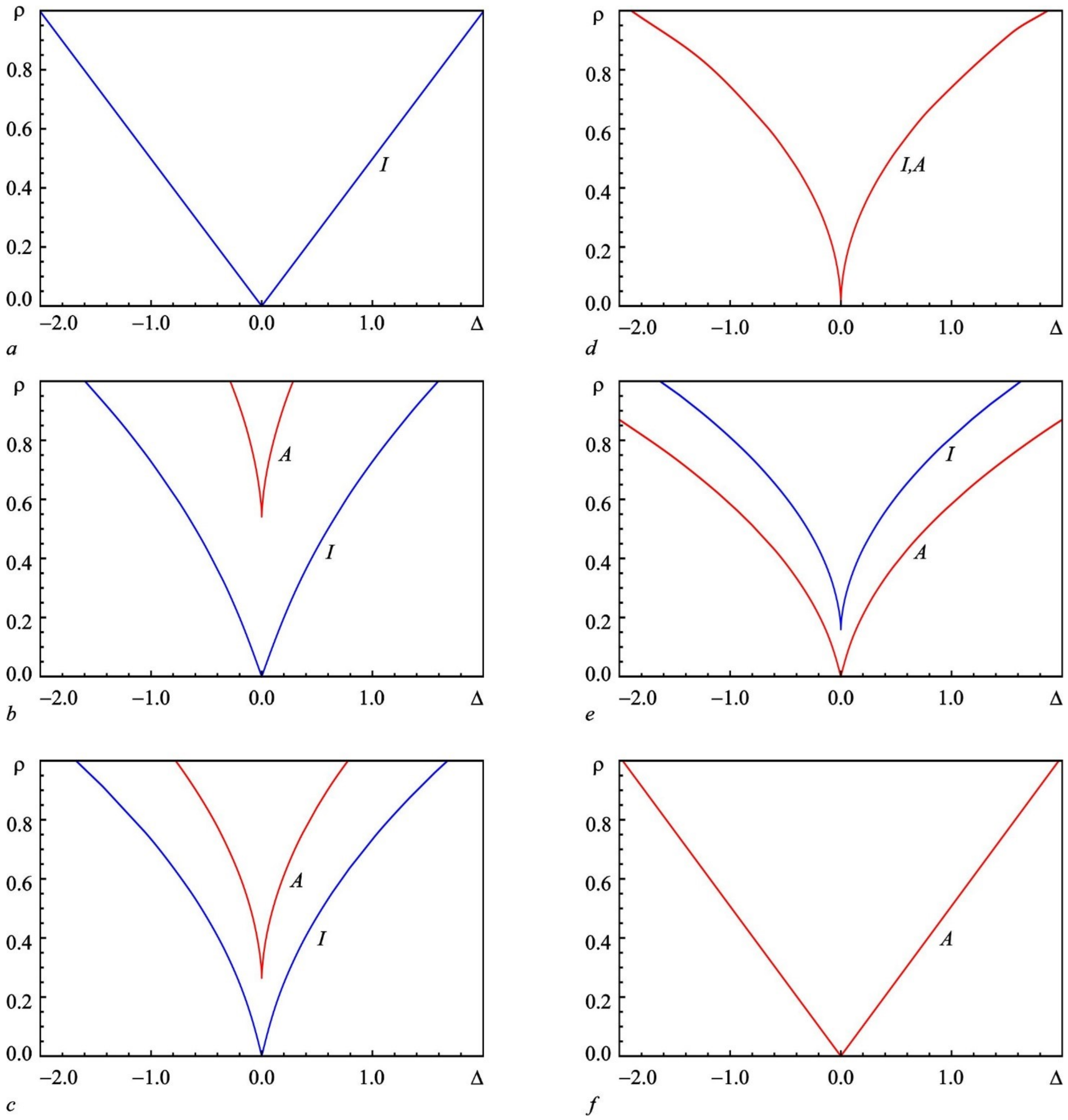

Fig 3. Domains of in-phase and anti-phase synchronization on the parameter plane Δ, ρ for $\sigma = -0.16$, $b = 0.2$ and for different values of the phase shift $\psi = 0$ (a), 0.25π (b), 0.3π (c), 0.4π (d), 0.5π (e), 1.0π (f)

bifurcation instead of (13), (15) will be determined by the following formulas:

$$\Delta = -2\rho \sin\varphi \left(\cos\psi + \frac{b}{\sqrt{1+4\sigma}}\sin\psi\right) - \frac{\rho^2}{\lambda}\sin 2\varphi \left(\sin^2\psi + \frac{b}{4\sqrt{1+4\sigma}}\sin 2\psi\right), \quad (18)$$

$$\rho = -\frac{4\lambda\cos\varphi\left(\cos\psi\sqrt{1+4\sigma} + b\sin\psi\right)}{\cos 2\varphi\left(4\sin^2\psi\sqrt{1+4\sigma} + b\sin 2\psi\right)}, \quad (19)$$

and the expression (12) will take the form

$$\lambda = -\frac{1}{2}\sqrt{1+4\sigma}\left(1 - \sqrt{1+4\sigma}\right). \quad (20)$$

It makes sense to talk about this bifurcation only at $-1/4 < \sigma < 0$, that is, when $R_-^2 > 0$ (see

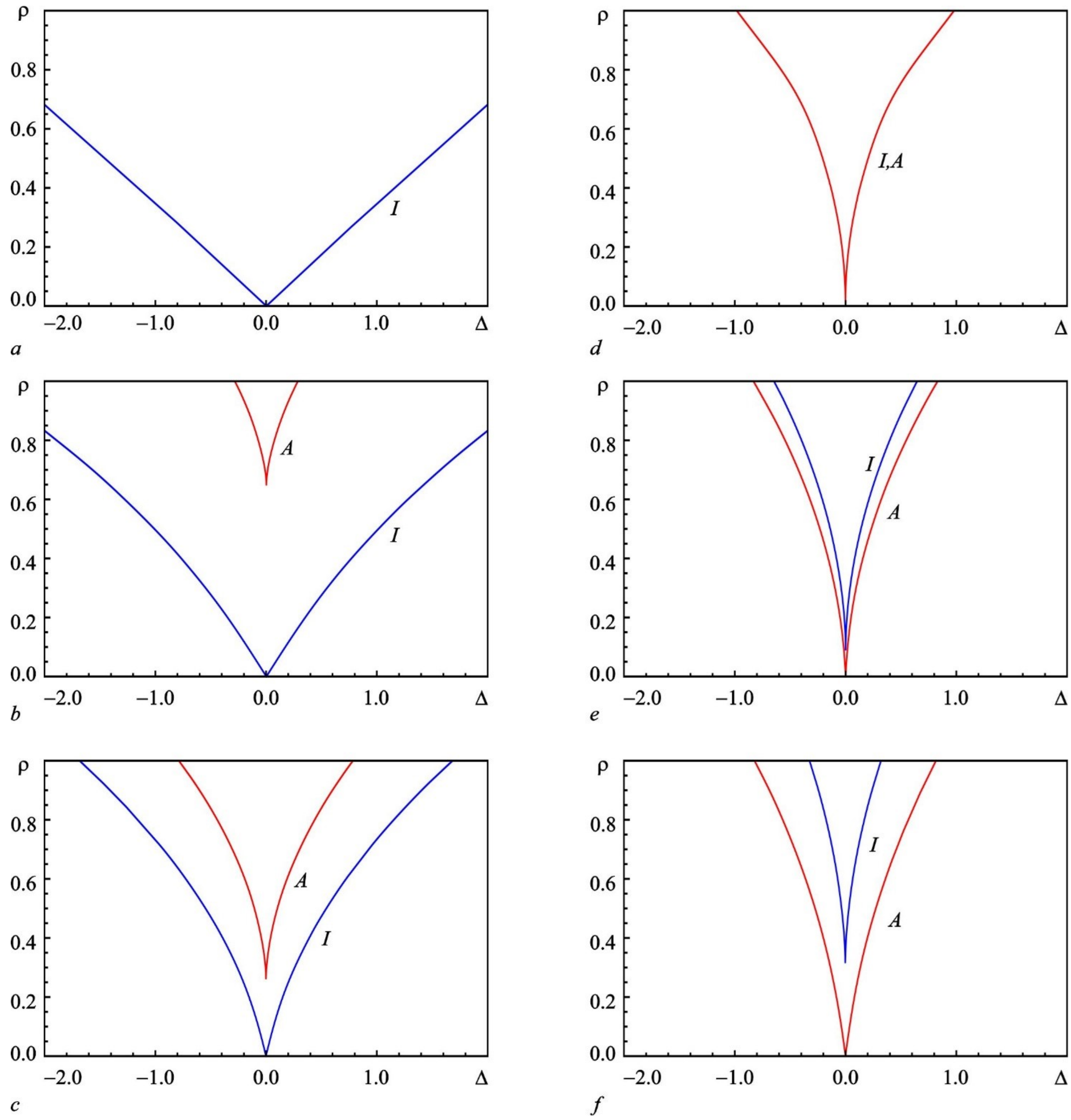

Fig 4. Domains of in-phase and anti-phase synchronization on the parameter plane Δ, ρ for $b = 0.5$, $\psi = 0.3\pi$ and for different values of excitation parameter $\sigma = -0.24$ (a), -0.23 (b), -0.2 (c), -0.13 (d), -0.1 (e), -0.01 (f)

(5)). Since this bifurcation results in the birth of a saddle and an unstable node, the lines (18), (19) are not the boundaries of the synchronization region.

2.2. Saddle-node bifurcations of asymmetric solutions. Within the framework of the phase approximation, it is also possible to analyze saddle node bifurcations of another type, as a result of which asymmetric fixed points $S_{1,2,3,4}$ appear. To do this, we will look for solutions in the form of $R_1 = R_+ + r_1, R_2 = R_- + r_2$, where $r_{1,2} \sim \rho$ are small additives. Then instead of the ratios (8) we get

$$\begin{aligned} r_1 &= -\frac{\rho R_-}{2R_+^2\left(1 - 2R_+^2\right)} \cos\left(\psi + \varphi\right) = 0, \\ r_2 &= -\frac{\rho R_+}{2R_-^2\left(1 - 2R_-^2\right)} \cos\left(\psi - \varphi\right) = 0. \end{aligned} \tag{21}$$

Using (21), taking into account the expressions (5) for $R_\pm$, after a series of calculations we find that

$$\frac{R_1}{R_2} \approx \frac{1+\sqrt{1+4\sigma}}{1-\sqrt{1+4\sigma}} + \frac{\rho\left(1+\sqrt{1+4\sigma}\right)}{4\sigma^2\sqrt{1+4\sigma}}\left((1+2\sigma)\cos\psi\cos\varphi + \sqrt{1+4\sigma}\sin\psi\sin\varphi\right), \quad (22)$$

$$\frac{R_2}{R_1} \approx \frac{1-\sqrt{1+4\sigma}}{1+\sqrt{1+4\sigma}} - \frac{\rho\left(1-\sqrt{1+4\sigma}\right)}{4\sigma^2\sqrt{1+4\sigma}}\left((1+2\sigma)\cos\psi\cos\varphi + \sqrt{1+4\sigma}\sin\psi\sin\varphi\right). \quad (23)$$

Substituting (22) and (23) into the third equation of the system (3), we obtain a generalized Adler equation, which, after a series of transformations, can be reduced to a relatively compact form:

$$\begin{aligned}\dot\varphi + \Delta = b\sqrt{1+4\sigma} + \frac{\rho}{\sqrt{-\sigma}}\left(\sqrt{1+4\sigma}\sin\psi\cos\varphi - \cos\psi\sin\varphi\right) \times \\ \times\left[1 + \frac{\rho}{2\sigma^2\sqrt{1+4\sigma}}\left((1+2\sigma)\cos\psi\cos\varphi + \sqrt{1+4\sigma}\sin\psi\sin\varphi\right)\right].\end{aligned} \quad (24)$$

Note that for $\rho \ll 1$, the second term in the square bracket can be neglected (except for the degenerate cases $\sigma \to 0$ and $\sigma \to -1/4$). As a result (24) is greatly simplified:

$$\dot\varphi + \Delta = b\sqrt{1+4\sigma} + \frac{\rho}{\sqrt{-\sigma}}\left(\sqrt{1+4\sigma}\sin\psi\cos\varphi - \cos\psi\sin\varphi\right). \quad (25)$$

Unlike the equation (11), the coefficient for a term of the order of ρ in (25) does not identically vanish at any particular value of the phase raid parameter ψ.

Assuming in (25) $\dot\varphi = 0$, we obtain an equation for determining fixed points

$$\Delta = b\sqrt{1+4\sigma} + \frac{\rho}{\sqrt{-\sigma}}\left(\sqrt{1+4\sigma}\sin\psi\cos\varphi - \cos\psi\sin\varphi\right). \quad (26)$$

Large industrial enterprises are distinguished from the association of two key enterprises (26):

$$\frac{\partial\Delta}{\partial\varphi} = \frac{\rho}{\sqrt{-\sigma}}\left(\sqrt{1+4\sigma}\sin\psi\sin\varphi + \cos\psi\cos\varphi\right) = 0,$$

how can we find that

$$\operatorname{tg}\varphi = -\frac{\operatorname{ctg}\psi}{\sqrt{1+4\sigma}}.$$

Substituting this ratio into (26), after a series of calculations we find

$$\Delta = b\sqrt{1+4\sigma} \pm \frac{\rho}{\sqrt{-\sigma}}\sqrt{1+4\sigma\sin^2\psi}. \quad (27)$$

This expression defines the bifurcation lines on the plane Δ, ρ.

Obviously, there is another asymmetric solution for which $R_1 = R_- + r_1$, $R_2 = R_+ + r_2$. For him, the conditions of saddle node bifurcation coincide with (27) up to the sign:

$$\Delta = -b\sqrt{1+4\sigma} \pm \frac{\rho}{\sqrt{-\sigma}}\sqrt{1+4\sigma\sin^2\psi}. \quad (28)$$

Based on the above relations, it is possible to construct all lines of saddle-node bifurcations on the plane of the parameters Δ, ρ. An example for the case of $\sigma = -0.16, b = 0.2, \psi = 0.35\pi$ is shown in Fig. 5. The common-mode synchronization area is shaded in blue, and the antiphase one is purple. They are bounded by the lines SN_1 saddle of nodal bifurcations of symmetric solutions.

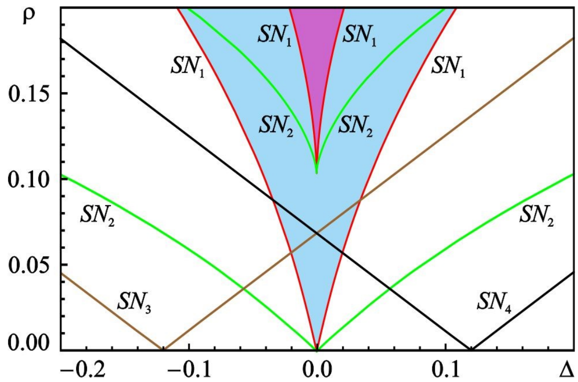

Fig 5. Lines of saddle-node bifurcations SN_{1-4}, plotted on the parameter plane Δ, ρ for $\sigma = -0.16$, $b = 0.2$ and $\psi = 0.35\pi$. Synchronization domains are shaded (color online)

$R_{1,2} = R_+ + O(\rho)$, which are set by the ratios (13), (15). This figure also shows the lines SN_2 of saddle-node bifurcations of unstable solutions $R_{1,2} = R_- + O(\rho)$, constructed according to the relations (18), (19). They lie below the lines SN_1.

The lines $SN_{3,4}$ correspond to saddle-node bifurcations of asymmetric solutions. Since in the equations (27), (28) there is a term responsible for non-isochronism on the plane Δ, ρ these lines are based on the horizontal axis not at the origin, but at points $\Delta = \pm b\sqrt{1+4\sigma} = \pm 0.12$.

Using the above relations, it is also useful to construct dependences of the oscillation amplitudes of various modes on the parameter ρ. An example of such dependencies is shown in Fig. 6 (for simplicity, consider the case of $\Delta = 0$). The figure shows the dependencies only for the amplitude R_1. It is not necessary to give values for the oscillation amplitude of the second oscillator, since at zero detuning for in-phase and antiphase modes $R_1 = R_2$, and the solutions $S_{1,2}$ and $S_{3,4}$ are pairwise symmetric to each other with respect to substitution

$$(R_1, R_2, \varphi) \to (R_2, R_1, -\varphi)\,.$$

In the case of a non-zero disorder, the degeneracy is removed and the specified symmetry is violated.

From Fig. 6 it can be seen that with increasing coupling, the amplitude of the in-phase mode I increases, and the amplitude of the antiphase mode A decreases. For solutions corresponding to saddle node bifurcation on an unstable cycle, the situation is the opposite. Indeed, the formulas (8) include the value $1 - 2R_+^2 = -\sqrt{1+4\sigma}$. When we consider the solutions I_-, A_-, in (8) it is

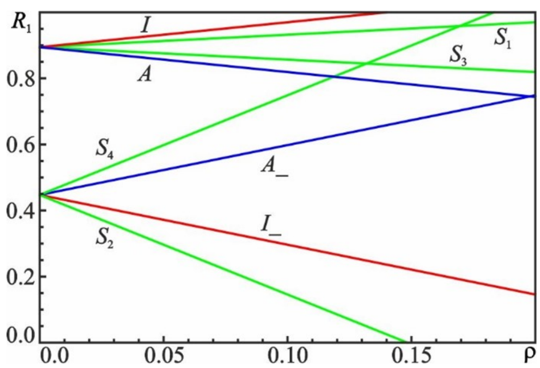

Fig 6. Dependences of the oscillation amplitudes of various modes on the coupling parameter ρ for $\sigma = -0.16$, $\psi = 0.2\pi$, and $b = 0$ (color online)

necessary to replace R_+ with R_-, while $1 - 2R_-^2 = \sqrt{1 + 4\sigma}$ and the right parts (8) change the sign.

As for the asymmetric solutions, for two of them, with increasing coupling, the oscillation amplitudes of the first oscillator increase (at a given value of ψ it is S_1 and S_4), and the second one decrease, for the other two the opposite situation takes place.

Conclusion

In this paper, based on the phase approximation, an analysis of the synchronization of a system of two generators with rigid excitation associated with a delay is carried out in an approximation when the delay time is considered small compared with the characteristic oscillation time. It is shown that the dynamics in such a system is significantly more complicated compared to coupled systems with mild self-excitation. In the phase space, in addition to the fixed points corresponding to the modes of in-phase and antiphase synchronization, there are a couple more unstable fixed points for which the oscillation amplitudes of the first and second oscillators are close to the amplitude of the unstable state R_-, as well as asymmetric fixed points for which the oscillation amplitudes of the first and second oscillators differ significantly from each other.

Generalized Adler equations for various situations are obtained, from which simple analytical formulas follow for conditions of saddle-node bifurcations, as a result of which fixed points arise. Conditions have been found under which the bond is purely conservative or purely dissipative (ratios (16) and (17)). With an increase in the nonisochronous parameter b, the value of the phase incursion ψ, at which the dissipative coupling dominates, increases, and the value at which the coupling is purely conservative decreases. When dissipative coupling dominates, synchronization is possible only in common mode. When conservative coupling dominates, phase bistability appears in the system, that is, areas of antiphase synchronization appear. The obtained formulas made it possible to trace the transition from one type of connection to another when changing control parameters.

However, it should be noted that the phase approximation is valid only for weak coupling and for small detunements. A more rigorous analysis within the framework of the so-called amplitude-phase approximation will be presented in the second part of this paper.

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