Applied Problems of Nonlinear Oscillation and Wave Theory

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Solving a nonlinear problem for a one-sided dynamically loaded sliding thrust bearing[1]

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Abstract. The *purpose* of this study is to propose an efficient numerical method for solving the inverse nonlinear problem of the movement of the compressor rotor collar in a fluid film thrust bearing. *Methods.* A periodic thermoelastohydrodynamic (PTEHD) mathematical model of hydrodynamic and thermal processes in a bearing is constructed under the condition of the rotor collar motion. Within the framework of the model, an inverse nonlinear problem of determining the position of the collar under a given external load is formulated. An iterative solution method is proposed, which utilizes the solution of the direct problem. To reduce computational costs, a modified Dekker–Brent method is employed in conjunction with a modified Newton's method. *Results.* Numerical experiments have been conducted, demonstrating the effectiveness of the proposed approaches. The suggested methods significantly reduce the required computational resources by minimizing the number of calls to the target function in the optimization problem. A software suite has been developed that allows for the calculation of the nonlinear system of rotor motion under various physical and geometric parameters. *Conclusion.* An efficient set of numerical methods for solving the inverse nonlinear problem of the motion of the rotor collar in the compressor fluid film thrust bearing is proposed. The method's effectiveness lies in substantial savings of computational resources. The method's efficiency has been demonstrated in numerical experiments.

Keywords: fluid film thrust bearing, differential equations, inverse nonlinear problem, zeroin, Dekker–Brent method, Newton's method.

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Introduction

Traditionally, liquid friction bearings [1,2] are used as a support for the rotor of a compressor machine with a rotating rotor and force loads acting on it. They are widespread due to their excellent speed and durability under certain conditions: regular testing of the physical and chemical properties of the oil, cleanliness of oil filtration, temperature conditions, etc. One of the main aspects in calculating and designing the internal layout of the compressor casing is to ensure stable spatial movement of the rotor on the sliding bearings, which is checked at the stage of mechanical testing of the casing. At the same time, a distinctive feature of the occurrence of unacceptable vibrations in compressor rotor systems is frequent axial (longitudinal) oscillations. For a complete analysis and prediction of the vibration state of the rotor system, many authors [3, 4] point out the need to study its vibrodynamic state taking into account the hydrodynamic processes in a fluid film thrust bearing.

The investigated fluid film thrust bearing serves to reduce wear and friction between rotating and stationary parts of the structure, to perceive disturbing forces along the compressor rotor axis and to fix the rotor relative to the housing in the axial direction, including during transient compressor modes. Axial forces may arise from the total pressure drop at the working stages 2 (Fig. 1), from internal non-stationary gas-dynamic processes of the compressed gas in the flow part of the centrifugal compressor (CC), from the pressure drop of the compressed gas between the discharge and suction sides of the high-pressure screw compressor (SC) (with a final pressure of more 10 MPa) and/or from the axial component of the load of the helical gearing, for example, a multiplier centrifugal compressor [1, 2, 5, 6].

The study of nonlinear dynamic operating modes of the fluid film thrust bearing is not widely used. This is due to the complex joint solution of the equations of bearing hydrodynamics and rotor dynamics, despite the higher accuracy of calculating the dynamics of a turbomachine rotor. As a rule, a linear (linearized) formulation is solved with the derivation of the stiffness and damping coefficients of the bearing [7, 8] lubricant layer. When calculating a thrust bearing, the perceived axial force is usually taken as constant to simplify the calculation procedure. However, the axial load largely depends on the variable internal physical parameters of the compressed gas medium during operation of a centrifugal or screw compressor.

Of particular interest are transient (unsteady) modes of compressor operation. They can be observed when the compressor is connected to an external network containing inertia and capacity; when the compressor is disconnected from the network; when the characteristics of the compressor itself change, for example, when the frequency of rotation of the rotor of the

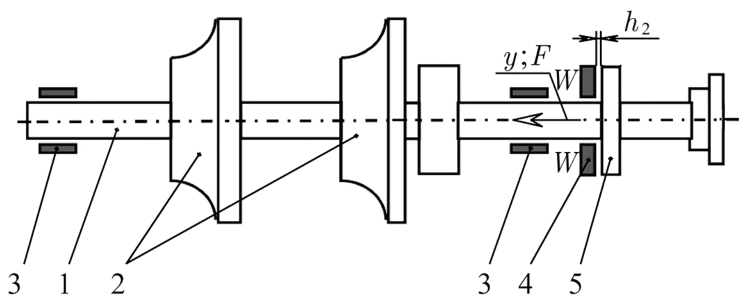

Fig 1. Scheme of the rotary part of a centrifugal compressor: 1 — shaft; 2 — working stage; 3, 4 — fluid film journal and thrust bearings; 5 — collar; h_2 — working clearance of the thrust bearing

centrifugal compressor [5,6,9] changes. It is necessary to separately consider the modes that are dangerous for the strength of the centrifugal compressor structure and are associated with non-stationary gas-dynamic processes directly in the flow part: stall, rotating stall and surge [6,9,10].

During transient conditions, an additional non-design dynamic component of the axial force acts on the thrust bearing. This leads to a change in the load in absolute value and, in some cases, direction. This may result in the failure of the thrust bearing due to possible contact between the rotor disk and the working surface of the bearing pads.

Theoretical analysis of possible axial displacements of the compressor rotor under the action of an external load allows, in the process of designing the housing, to evaluate the correctness of the choice of geometric and operating parameters of the rotor system and to select the optimal option for its layout taking into account the axial action of the thrust bearing. In this case, the basis of this analysis is the determination of the axial displacement of the rotor disk with a given layout and the operating conditions of the thrust bearing while maintaining the requirement for reliability and durability of its operation.

Thus, the dynamic operating conditions of the fluid film thrust bearing of the centrifugal compressor and screw compressor and ensuring the stability of the axial movement of the rotor disk determine the relevance of the research. A nonlinear problem is used to describe the dynamic behavior of the rotor. This article is devoted to describing the solution to this problem.

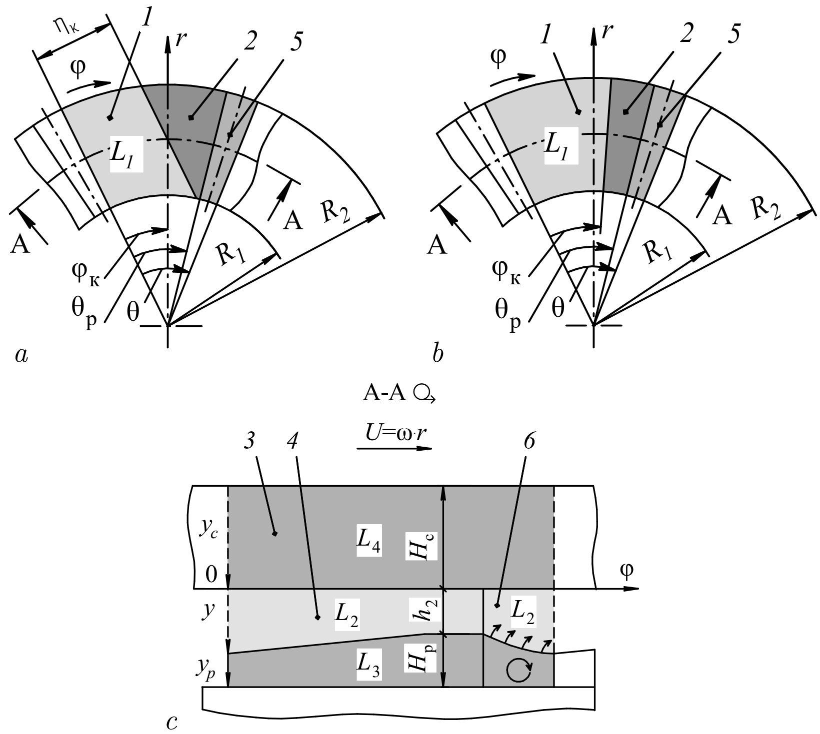

Fig 2. Calculation scheme of the fluid film thrust bearing: *a* — pad profile with a parallel taper land; *b* — pad profile with a taper land; *c* — section along A-A along the middle radius: 1, 2 — wedge and flat parts of the pad; 3 — collar; 4 — lubricating film; 5 — interpad groove; 6 — boundary layer

1. Statement of the problem

The fluid film thrust bearing under consideration consists of stationary (fixed) cushions 1, 2 of a solid annular thrust bearing and a rotating thrust disk 3, which are separated from each other by lubricating layers 4 above the surface of the cushions (Fig. 2). Such separation is ensured by the occurrence of hydrodynamic pressure in the lubricating layers due to the rotation of the disk 3, narrowing the confuser gap in the direction of rotation and the supply of lubricant of the required volume and viscosity. When the disk 3 rotates, the lubricant is drawn by the surface into the narrowing wedge-shaped gap, as a result of which the pressure in the lubricating layers 4 of the bearing increases and balances the externally applied load. The space between the cushions is filled with inter-cushion channels 5 (ICC), through which fresh lubricant is supplied with a constant pressure of 0.24...0.26 MPa (abs.) from the external lubrication system of the compressor unit. In the ICC, on the surface of the disk and under the condition of its rotation, a thin boundary layer of lubricant 6 is formed, participating in the complex heat exchange process of mixing the lubricant [11]. On the one hand, the boundary layer transfers, during disk rotation, the lubricant heated by dissipation with a temperature of $t|_{\varphi=\theta_p}$ from the previous cushion to the next one. On the other hand, it receives fresh lubricant with a lower temperature of t_0 coming from the ICC, thus forming an important initial temperature $t|_{\varphi=0}$ at the entrance to the next cushion. All bearing pads have a single and unique geometry of the working surface, called a profile. The paper considers two most frequently used in compressor technology profiles of the stationary surface of the pad: a bevel parallel to the inter-pad channel (IPC) (see Fig. 2, *a*) and a helical surface of the wedge-shaped spike (see Fig. 2, *b*). The profile, in turn, allows the occurrence of hydrodynamic pressure with a certain diagram, the integral of which over the working surface of the pad forms the bearing capacity P, which balances the external axial force F and provides a guaranteed gap h_2 to prevent the pads from touching the disk, including during the transient modes of the CC and VK.

The hydrodynamic theory of lubrication describes the operating principle, operating conditions, and mathematical foundations of plain bearings. The result of its consistent development is the *periodic thermoelastohydrodynamic (PETH) theory* [12]. This formulation most fully describes the jointly occurring hydrodynamic and thermal processes in the lubricant and boundary layers, thermal processes in the supporting cushion and the rotating element, taking into account the thermoelastic deformations of the cushion from the temperature difference across the thickness and direction of rotation of the element. A special feature of this formulation is that for the internal energy equation describing the accumulation and propagation of heat, periodic thermal boundary conditions are specified near the inlet edge of the cushion at $\varphi = 0$ [13–15]. This allows one to directly calculate the previously unknown temperature at the inlet to the cushion $t|_{\varphi=0}$ during the calculations and not to perform preliminary temperature calculations by successive approximations [12]. Such a formulation also allows one to take into account the temperature fields of the cushion and especially the thrust disk, taking into account heat exchange with the external environment. As a consequence, due to the combination of factors taken into account, the flow of lubricant and the spread of heat in the elements of the fluid film thrust bearing, the PTUGD theory has the greatest convergence in verifying the results of numerical calculations with physical experiments [16].

When deriving the PTUGD equations of the mathematical model of the operation of a fluid film thrust bearing with stationary compressor cushions, the following simplifications were adopted, which made it possible to significantly simplify the original Navier–Stokes equations.

1. The thickness of the lubricant layer is small compared to the radial r and circumferential φ directions. For a thrust bearing, this is the y direction, limited by the surface of the disk

at $y = 0$ and the cushion at $y = h$ (see Fig. 2).
2. The pressure gradient across the thickness of the lubricant layer is taken to be zero $\partial p/\partial y = 0$ compared to the radial and circumferential pressure gradients and is not taken into account in the analysis.
3. The inertial forces are insignificant based on the analysis of the order of magnitude of the Navier–Stokes equations, that is, the substantial derivatives of the velocities $DV_{r(\varphi,y)}/dt = 0$. This also means that the viscosity forces of the lubricant prevail over the inertial forces.
4. The flow in the lubricant layer is assumed to be laminar in the absence of vortex flow and turbulence inside the layer, which is consistent with negligible inertial forces.
5. The lubricant layer is not affected by mass forces (centrifugal, gravitational, etc.).
6. At the boundary with the surface of solid bearing elements, i.e. the disk at $y = 0$ and the cushion at $y = h$, the lubricant is stationary or moves with the velocity of this surface (the condition of lubricant adhesion).
7. An analysis of the order of magnitude of the velocities in the Navier–Stokes equation showed the dominant effect of the velocity gradients across the thickness of the lubricant layer $\partial V_r/\partial y$ and $\partial V_\varphi/\partial y$ over the others.
8. The lubricant is Newtonian and isotropic, i.e. the physical properties at each of its points are the same in all directions.

Further, the PTUGD mathematical model is presented in dimensionless form through relative (dimensionless) quantities, related to characteristic dimensions (the «–» sign above the quantity). The main defining equations with the corresponding boundary conditions are:

1. Reynolds equation describing the two-dimensional pressure distribution in the lubricating layer of the region $\bar{L}_1\left(-1 \leqslant \bar{r} \leqslant 1,\ 0 \leqslant \bar{\varphi} \leqslant \bar{\theta}_p,\ 0 \leqslant \bar{y} \leqslant 1\right)$ and is the fundamental equation of bearing hydrodynamics. The equation is derived with a minimum of restrictive assumptions, in which the density and viscosity of the lubricant are functions of all three coordinates. In the dimensionless non-stationary form, the equation takes the following form:

$$-\lambda^2 \frac{\partial}{\partial \bar{r}}\left[(\sigma\bar{r}+1)\bar{h}^3 \bar{f}_0 \frac{\partial \bar{p}}{\partial \bar{r}}\right] - \frac{\partial}{\partial \bar{\varphi}}\left[\frac{\bar{h}^3}{\sigma\bar{r}+1}\bar{f}_0 \frac{\partial \bar{p}}{\partial \bar{\varphi}}\right] =$$

$$= -\operatorname{Re}\psi\sigma\lambda^2 \frac{\partial(\bar{h}^3 \bar{f}_1)}{\partial \bar{r}} + \bar{\omega}(\sigma\bar{r}+1)\frac{\partial(\bar{h}\bar{f}_2)}{\partial \bar{\varphi}} + Sh(\sigma\bar{r}+1)\bar{A}, \quad (1)$$

where $\bar{r}, \bar{\varphi}, \bar{y}$ are dimensionless coordinates; $\bar{p} = ph_{20}^2/\left(\mu_0\omega_* R_{cp}^2\theta\right)$ is local dimensionless pressure; $\bar{A} = \frac{\partial}{\partial \bar{\tau}}\left(\bar{h}\int_0^1 \bar{\rho}\, d\bar{y}\right) - \bar{\rho}_{\bar{y}=1}\frac{\partial \bar{h}}{\partial \bar{\tau}}$ is non-stationary multiplier; $\bar{f}_0, \bar{f}_1, \bar{f}_2$ are functions that take into account the variability of lubricant viscosity across the layer thickness; λ, σ are relative length and width of the cushion; $\psi = h_{20}/(R_{cp}\theta)$ is relative thickness; θ is angular extent of the cushion with the MPC; $\bar{\omega}, \omega_*$ are current dimensionless and characteristic (usually maximum) angular velocities of the disk; R_{cp} is average radius of the cushion; h_{20} is characteristic thickness of the bearing lubricant layer; μ_0 is viscosity at the lubricant supply temperature t_0 in the MPC; $\bar{\tau} = \tau/\tau_*$ is dimensionless time.
2. The internal energy balance equation describing the three-dimensional temperature distribution in the lubricating and boundary layers of the region $\bar{L}_2(-1 \leqslant \bar{r} \leqslant 1,\ 0 \leqslant \bar{\varphi} \leqslant \bar{\theta}_p, \bar{\theta}_p \leqslant \bar{\varphi} \leqslant \bar{\theta},\ 0 \leqslant \bar{y} \leqslant 1)$. In the divergent dimensional non-stationary form, the energy equation

takes the following form:

$$c_p\left(\rho\frac{\partial t}{\partial\tau}+t\frac{\partial\rho}{\partial\tau}\right)+\frac{1}{r}\frac{\partial}{\partial r}(c_p\rho rV_rt)+\frac{\partial}{\partial\varphi}\left(\frac{c_p\rho}{r}V_\varphi t-\frac{\lambda_o}{r^2}\frac{\partial t}{\partial\varphi}\right)+$$
$$+\frac{\partial}{\partial y}\left(c_p\rho V_yt-\lambda_o\frac{\partial t}{\partial y}\right)=\mu\left[\left(\frac{\partial V_\varphi}{\partial y}\right)^2+\left(\frac{\partial V_r}{\partial y}\right)^2\right],\quad(2)$$

where t is local temperature, c_p, λ_o are isobaric heat capacity and thermal conductivity of the lubricant, ρ is local density of the lubricant. The transformation into a dimensionless form of the equation (2) is performed at the stage of numerical implementation using the dimensionless temperature $\bar{t}=c_{p0}\rho_0h_{20}^2(t-t_0)/\mu_0\omega_*R_{cp}^2\theta$ and preserving the divergent form. The right-hand side of the equation (2) describes the dissipation of mechanical energy and its conversion into thermal energy.

3. Three-dimensional temperature distribution in the cushion regions $\bar{L}_3(-1\leqslant\bar{r}\leqslant1,\ 0\leqslant\bar{\varphi}\leqslant\bar{\theta}_p, 0\leqslant\bar{y}_p\leqslant1)$ and thrust disk $\bar{L}_4(-1\leqslant\bar{r}\leqslant1,\ 0\leqslant\bar{\varphi}\leqslant1,\ 0\leqslant\bar{y}_p\leqslant1)$ is described by proper heat conduction equations with the corresponding boundary conditions: at the outer boundaries, heat exchange is taken into account by the Newton–Richmann boundary conditions; between the lubricating layer and the cushion, as well as between the lubricating and boundary layers and the disk, the conditions of continuity of temperatures and heat flows (conjugation condition) are set.

In the mathematical model developed by PTUGD, the boundary condition for the energy equation (2) at the entrance to the lubricating layer is expressed in a periodic form, which implies equality of temperatures and heat flows of the lubricant (due to convection and thermal conductivity). In dimensional form, in the absence of skew of the cushions and runout of the disk, the condition takes the following form:

$$t|_{\varphi=0}=t|_{\varphi=\theta},\quad\left.\left(\frac{c_p\rho}{r}V_\varphi t-\frac{\lambda_o}{r^2}\frac{\partial t}{\partial\varphi}\right)\right|_{\varphi=0}=\left.\left(\frac{c_p\rho}{r}V_\varphi t-\frac{\lambda_o}{r^2}\frac{\partial t}{\partial\varphi}\right)\right|_{\varphi=\theta}.$$

The radial and circumferential velocities of the lubricant $\bar{V}_r$ and $\bar{V}_\varphi$ of the mathematical model are derived from the truncated Navier–Stokes equations after estimating the dimensionless quantities using the method of N.A. Slezkin and taking into account the condition of equality to zero of the pressure gradient along the gap height $\bar{h}$. The transverse velocity $\bar{V}_y$ is obtained by solving the truncated Navier–Stokes equation along the [17] axis.

The shape of the gap of the flat-wedge surface of the stationary pad of the fluid film thrust bearing in the absence of distortions in dimensionless form looks like this:

- in the case of a parallel MPC bevel (see fig. 2, *a*)

$$\bar{h}=\frac{h}{h_{20}}=1-\bar{y}+\varepsilon_0\left[1-\frac{(\sigma\bar{r}+1)}{\bar{\eta}_k}\sin(\sigma\lambda\bar{\varphi})\right]\delta_k+$$
$$+\alpha_p^*\chi\frac{\Psi_p}{\psi}\int_0^1\left[\bar{T}_p(\bar{\theta}_p,\bar{y}_p)-\bar{T}_p(\bar{\varphi},\bar{y}_p)\right]d\bar{y}_p,\quad(3)$$

$$\delta_k=\begin{cases}1, & 0\leqslant\bar{\varphi}\leqslant\bar{\varphi}_k,\\0, & \bar{\varphi}_k<\bar{\varphi}\leqslant\bar{\theta}_k,\end{cases}\quad\text{where}\quad\bar{\varphi}_k=\frac{1}{\sigma\lambda}\arcsin\frac{\bar{\eta}_k}{\sigma\bar{r}+1};$$

- in the case of a helical surface of a wedge bevel (see fig. 2, *b*)

$$\bar{h} = \frac{h}{h_{20}} = 1 - \bar{y} + \varepsilon_0 \left[1 - \frac{\bar{\varphi}}{\bar{\theta}_k \bar{\theta}_p}\right] \delta_k + \alpha_p^* \chi \frac{\Psi_p}{\psi} \int_0^1 \left[\bar{T}_p(\bar{\theta}_p, \bar{y}_p) - \bar{T}_p(\bar{\varphi}, \bar{y}_p)\right] d\bar{y}_p, \qquad (4)$$

$$\delta_k = \begin{cases} 1, & 0 \leqslant \bar{\varphi} \leqslant \bar{\theta}_k \theta_p, \\ 0, & \bar{\theta}_k \theta_p < \bar{\varphi} \leqslant \bar{\theta}_p, \end{cases}$$

where $\bar{y} = y/h_{20}$ is dimensionless coordinate of disk position; $\varepsilon_0 = \Delta h/h_{20}$ is relative wedge shape of the cushion; $\bar{\eta}_k = \eta_k/R_{cp}$ is relative width of the wedge bevel; $\alpha_p^* = \alpha_p/\beta$; α_p is coefficient of linear expansion of the cushion material; β is temperature coefficient of lubricant viscosity; δ_k is unit function.

The fourth term on the right of the gap shape equations (3), (4) takes into account the thermal deformations of the fixed cushion due to free thermal expansion, described by the formula of A. I. Golubev for mechanical seals [18]. The $\bar{y}$ coordinate is determined either by direct substitution into the gap shape equation (statement of the direct problem), or is calculated based on the equation of compressor rotor dynamics (statement of the inverse nonlinear problem).

The dimensionless form of the main defining equations of the PTUGD mathematical model is necessary for the transformation of the considered dimensional curvilinear region L_2 ($R_1 \leqslant r \leqslant R_2$, $0 \leqslant \varphi \leqslant \theta_p$, $\theta_p \leqslant \varphi \leqslant \theta$, $0 \leqslant y \leqslant h$) into a rectangular form $\bar{L}_2$ ($-1 \leqslant \bar{r} \leqslant 1, 0 \leqslant \bar{\varphi} \leqslant \bar{\theta}_p$, $\bar{\theta}_p \leqslant \bar{\varphi} \leqslant \bar{\theta}$, $0 \leqslant \bar{y} \leqslant 1$) (see fig. 2) at the stage of numerical implementation with the obligatory condition of preserving the divergent form of the equations. The dimensionless form is necessary to reduce the time of numerical calculations and to reduce the total number of parameters under study when conducting subsequent parametric analysis of the thrust bearing characteristics.

The differential equations of the mathematical model in the PTUGD formulation and their boundary conditions are related through such physical properties of the working lubricant as viscosity, density, heat capacity and thermal conductivity, as well as through the shape of the gap, including the geometric profile of the working surface of the cushion, and some operating parameters. The mathematical model developed by the authors of PTUGD and its subsequent numerical implementation led to writing the calculation program $Sm2Px3Tx\tau$ [19]. The program, taking into account the modern trend of intensification of thermal processes and the three-dimensional distribution of heat in the bearing elements, allows study the static and dynamic operating modes of a one-way fluid film thrust bearing with fixed pads of a centrifugal or screw compressor. The static (integral) characteristics include the bearing capacity, friction power losses, lubricant consumption, heat flows, etc. Features of constructing a mathematical model, a description of the main defining Reynolds equations (area L_1), internal energy (area L_2) and the thermal conductivity of the pad (area L_3) and thrust disk (area L_4) and their boundary conditions (see Fig. 2), non-dimensionalization and the specifics of numerical implementation can be found in the articles [13–17, 20, 21].

Under the action of a variable external load during transient operating modes of a centrifugal or screw compressor, the thrust disk moves along the rotor axis. The deviation of the disk from an arbitrary predetermined point is determined by the formula

$$y = y_{st} + y_d,$$

where y_{st}, y_d are the static and dynamic components of the disk position. The initial position of the disk, characterized by a gap of h_{20}, is set before the bearing starts operating. The dynamic component y_d determines the displacement of the disk relative to the static position of the disk y_{st}.

The mathematical model developed by PTUGD assumes two formulations depending on the definition of the dynamic component y_d:

1) *direct*, in which the trajectory of the disk movement is specified and, based on it, the bearing characteristics are determined. The solution to the direct problem is described in article [22];
2) *inverse nonlinear*, in which the change in the external load is specified and the position of the disk and the bearing characteristics corresponding to this position are determined.

This paper considers an inverse nonlinear problem, which is based on the joint numerical integration of the equations of rotor dynamics and thrust bearing hydrodynamics. The shape of the thrust disk displacement near the point of the static position on the curve of the bearing's dynamic equilibrium is the geometric locus of points defining at a specific moment in time the position of the disk center moving under the action of an external disturbing force and the reaction of the thrust bearing. This formulation allows us to study the influence of the nonlinear reaction of the thrust bearing and to simulate the real dynamic behavior of the rotor at any axial eccentricities e_p [23, 24]. The stability of the rotor's axial motion under the action of an external force can be judged by analyzing the amplitude and frequency of the rotor disk displacement and the possibility of contact between the moving and fixed parts of the bearing.

The rotor is represented as a concentrated mass, which is acted upon by an external force from the compressor F and the reaction of the thrust bearing P. The relationship between the disk position, the load-bearing capacity, and the external axial force is expressed through the rotor dynamics equation

$$m_p \frac{d^2 y}{d\tau^2} = P - F, \tag{5}$$

where $m_p = \text{const}$ is rotor mass, y is disk displacement coordinate along the rotor axis, τ is time, $P = P(\tau, y)$ is one-way bearing load-bearing capacity, $F = F(\tau, y)$ is axial force. The external axial force is represented as the sum of the static and dynamic components $F = F_{st} + F_d$. The dynamic component of the force can be specified according to the harmonic law, describing soft surge of the CC, or in the form of piecewise continuous functions in the case of hard surge of the CC [6, 13]. To solve the equation (5), the initial condition the disk position is determined based on the solution of the stationary problem: at $\tau = 0$ the position $y = y_{st}$.

Using the following dependencies: coordinate $y = h_{20}\bar{y}$, time $\tau = \tau_0\bar{\tau}$, bearing capacity $P(\bar{\tau}, \bar{y}) = C_{pf}\bar{P}(\bar{\tau}, \bar{y})$, force $F(\bar{\tau}, \bar{y}) = C_{pf}\bar{F}(\bar{\tau}, \bar{y})$, $C_{pf} = \frac{\mu_0 \omega_* R_{cp}^3 \theta^2 (R_2 - R_1)}{h_{20}^2}$. The equation (5) takes the following dimensionless form:

$$\Lambda \frac{\partial^2 \bar{y}}{\partial \bar{\tau}^2} = \bar{P} - \bar{F},$$

where $\Lambda = \frac{m_p h_{20}^3}{\tau_0^2 \mu_0 \omega_* R_{cp}^3 \theta^2 (R_2 - R_1)} = \frac{m_p \psi^3 Sh}{\tau_0 \mu_0 (R_2 - R_1)}$ is reduced mass characterizing the inertial properties of the rotor; $\tau_0 = 2\pi/\omega$ is dimensionless time (the characteristic time is taken to be the time of one disk revolution); $Sh = \theta/(\omega_* \tau_0)$ is strouhal number; $\psi = h_{20}/(R_{cp}\theta)$ is relative thickness; h_{20} is characteristic thickness of the bearing layer; μ_0 is viscosity at the lubricant feed temperature t_0 in the MPC; R_1, R_2, R_{cp} are inner, outer and average radii; θ_p, θ are angular extents of the cushion and the periodicity element; ω_* is characteristic (usually maximum) angular velocity of the disk.

The hydrodynamics of the thrust bearing, in turn, is represented by the load-bearing capacity of the i-th bearing cushion. In the absence of thrust disc runout and skew of the common surface of the thrust bearing cushions, the load-bearing capacity is $P = zP_i$, where z

— the number of cushions. The bearing capacity of the i–th cushion is calculated by integrating the pressure fields [15]:

$$P_i = \int_0^{\theta_p} \int_{R_1}^{R_2} p_i r d\varphi dr, \quad (6)$$

where $p_i = p_i(\tau, y)$ is pressure distribution in the lubricating layer of the i–th cushion. It is determined on the basis of the solution of the Reynolds equation of the periodicity element (see equation (1)) on the surface of the cushion at $R_1 \leqslant r \leqslant R_2, 0 \leqslant \varphi \leqslant \theta_p$ (see Fig. 2) taking into account the distribution of temperatures and the viscosity formed in the volume of lubricant [12, 13].

The dimensionless form of the main governing equations of the PTUGD mathematical model is necessary to transform the considered dimensional curvilinear region $L_2(R_1 \leqslant r \leqslant R_2, 0 \leqslant \varphi \leqslant \theta_p, \theta_p \leqslant \varphi \leqslant \theta, 0 \leqslant y \leqslant h)$ into a rectangular form $\bar{L}_2(-1 \leqslant \bar{r} \leqslant 1, 0 \leqslant \bar{\varphi} \leqslant \bar{\theta}_p, \theta_p \leqslant \bar{\varphi} \leqslant 1, 0 \leqslant \bar{y} \leqslant 1)$ (see Fig. 2) at the stage of numerical implementation with the mandatory condition of preserving the divergent form equations. The dimensionless form is necessary for the correct evaluation of the values of the variables and reducing the total number of parameters under study when conducting subsequent parametric analysis of the thrust bearing characteristics.

2. Methodology

Considering the type of equation (5), the problem can be divided into two parts: stationary, associated with the static force F_{st}, and non-stationary, taking into account the dynamic change of F_d.

In the case of a stationary problem, the equation (5) will have the following form:

$$0 = P - F_{st}. \quad (7)$$

Therefore, it is necessary to balance the stationary force F_{st} by finding a certain position of the disk y and calculating the bearing capacity P. The above can be reduced to an optimization problem

$$y : \min |P - F_{st}|. \quad (8)$$

The position of the disk naturally affects the thickness of the lubricating layer, on which the system parameters depend. If the thickness of the lubricating layer decreases to zero, this will mean that the disk and the pads touch, which, as was said, will lead to failure of the thrust bearing. Therefore, the condition $\max_{x \in L_2} h(x, y) > 0$ must be imposed on the parameter y, where h is the thickness of the lubricating layer.

The bearing capacity P is an integral characteristic, the calculation of which requires knowledge of the temperature and pressure distribution in the thrust bearing. In turn, temperature and pressure depend on each other through such physical properties of the liquid as viscosity and density of the lubricant, and are determined through a system of three-dimensional nonlinear differential equations of the second order. It is worth noting that the energy equation (2) for calculating the temperature in the lubricant has dominant convective terms [12,13]. This imposes special requirements on the methods used, the accuracy of the calculation and, therefore, requires computational resources. To solve the optimization problem (8), it is necessary to choose an iterative method. Since the optimal position of the disk y is sought, at each iteration of the method, some y^{k+1} will be selected, for which it will be necessary to calculate the bearing capacity P and, therefore, to perform a complete calculation of the direct stationary problem. This is an expensive process.

After solving the stationary problem (7), the system reaches equilibrium, and the thrust bearing perceives the magnitude of the stationary force F_{st}. Based on this solution, further calculations related to the dynamics can be performed. To solve the equation (5) we construct a three-layer time iterative process.

$$m_p \frac{\hat{y}_h - 2y_h + \check{y}_h}{h_\tau^2} = P(\hat{y}_h) - F, \tag{9}$$

here y_h is grid approximation of disk displacement y, $\check{y}_h$ is disk displacement on the previous time layer, $\hat{y}_h$ is on the next one, h_τ is time step.

The entry $P(\hat{y}_h)$ additionally emphasizes the dependence of the bearing capacity P on the disk position, namely on the value of the disk displacement on the next time layer.

The equation (9) can be reduced to a similar optimization problem (8)

$$\hat{y}_h : \min \left| \left(m \frac{\hat{y}_h}{h_\tau^2} - P(\hat{y}_h) \right) - \left(\frac{2y_h - \check{y}_h}{h_\tau^2} - F \right) \right|. \tag{10}$$

As in the case of (8), with some iterative method, at each iteration at a fixed disk position, it is necessary to solve the direct problem, which is also an extremely costly process. Accordingly, the implicit scheme is preferable.

For comparison, in works [23,24] for plain bearings, the authors solved the derived system of differential equations of rotor motion by the explicit Adams–Bashforth method, which has the fourth order of accuracy. Due to the impossibility of self-starting, the first three or four points of the motion curve were calculated by the Euler method.

3. Results

To solve the presented optimization problems, due to their computational complexity, it is necessary to select a minimization method that will require recalculating the coefficients of the equation (8) and (10) as few times as possible. If we consider the target functions of the given optimization problems, it turns out that their behavior is affected by the physical and geometric parameters of the bearing. The only thing that can be said with certainty is the monotonicity of the target function. Therefore, the method must take into account nonlinearity and be able to find a solution for functions of various types. Below is a table of the number of calls to the target function depending on its type.

The table compares several optimization methods, including those from the MatLab package for finding the zero of a function. Let us describe the parameters of the experiments: zeroin implements the Dekker–Brent algorithm [25] (the accuracy of 10^{-15} is selected); fzero — the Dekker algorithm (accuracy of 10^{-15}); fminsearch — the Nelder–Mead algorithm (accuracy of 10^{-4}); fminbnd — the golden section algorithm with parabolic interpolation (accuracy of 10^{-4}); newton — the Newton algorithm with derivatives calculated by the difference method (accuracy of 10^{-4}). The zeroin method is presented modified; additional checks have been added, which in some cases reduce the number of calls to the objective function. The main method recommended by MatLab is the fzero method. Since some methods required a large number of calls to the target function, the accuracy for them was reduced.

The results of the experiments showed that the modified zeroin algorithm significantly outperforms the methods proposed by MatLab. However, zeroin requires knowing the localization

Table. Number of target function calls depending on its form. → — the solution is located at the right boundary of the localization region, ≫ — the solution is shifted to the left from the boundary

Objective function	zeroin	fzero	fminsearch	fminbnd	newton
$\mathrm{const} \neq 0$	1	1	29	22	200
$x - \mathrm{const}$	3	27	36	16	4
$x^2 - \mathrm{const} \rightarrow$	2	32	52	20	18
$x^2 - \mathrm{const} \leftarrow$	1	1	22	20	2
$x^2 - \mathrm{const} \ll\rightarrow$	10	32	52	15	138
$x^2 - \mathrm{const} \gg\leftarrow$	10	24	30	15	18
$\cos(x), [0, \pi]$	4	33	52	12	8
$\cos(x), [-\pi, 0]$	4	27	34	12	8

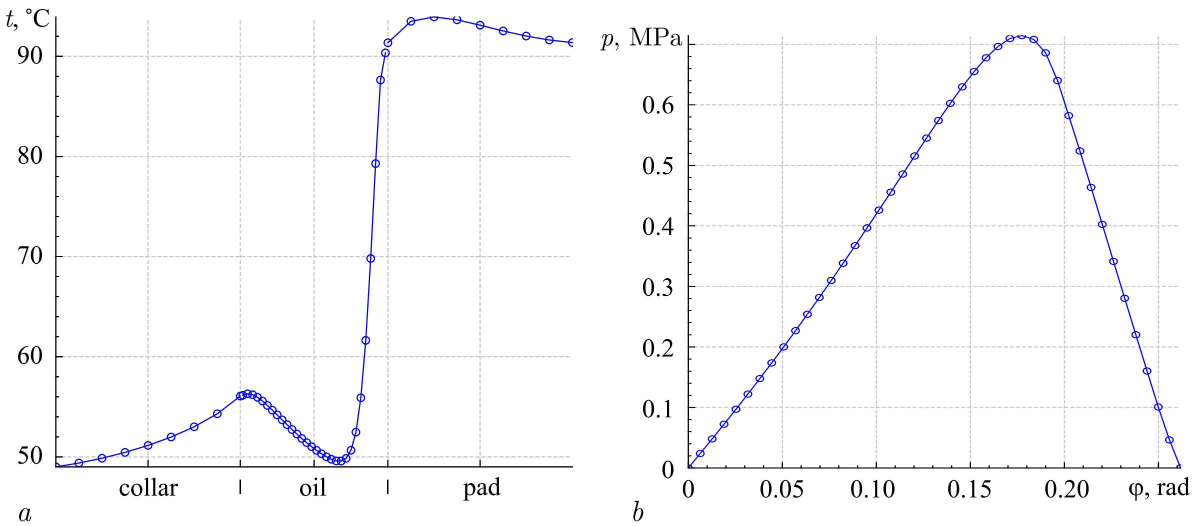

Fig 3. Distribution at the optimal collar position: a — temperature along the y axis, b — pressure along the angular coordinate φ

domain of the function zero. A method based on Newton's method was created to find the localization domain. It is formulated as follows. Let $[a, b]$ be a certain segment in the domain of definition of the function f. If $f(a) * f(b) < 0$, then $[a, b]$ is the localization domain and does not require clarification. Otherwise, it is necessary to perform a shift, the localization domain, in this case, will be $[\alpha, \beta]$, where

$$\alpha = (a + \Delta c)\sigma + \left(\frac{b + a}{2} + \Delta c\right)(1 - \sigma),$$

$$\beta = (b + \Delta c)\sigma + \left(\frac{b + a}{2} + \Delta c\right)(1 - \sigma).$$

Here $\Delta c = \frac{\Delta b - \Delta a}{2}$, $\Delta a = -f(a)/f'$, $\Delta b = -f(b)/f'$, $f' = (f(b) - f(a))/(b - a)$, σ is compression parameter that determines the correspondence between the sizes of the initial and final regions.

Below are graphs of the results of calculations using this technique.

To calculate the temperature and pressure, the authors of the article PTUGD used the bearing model and the $Sm2Px3Tx\tau$ software package. The geometric dimensions of the thrust plain bearing of a centrifugal compressor, located in the laboratory of the Department of «Compressor Machines and Installations» of the Federal State Budgetary Educational Institution of Higher

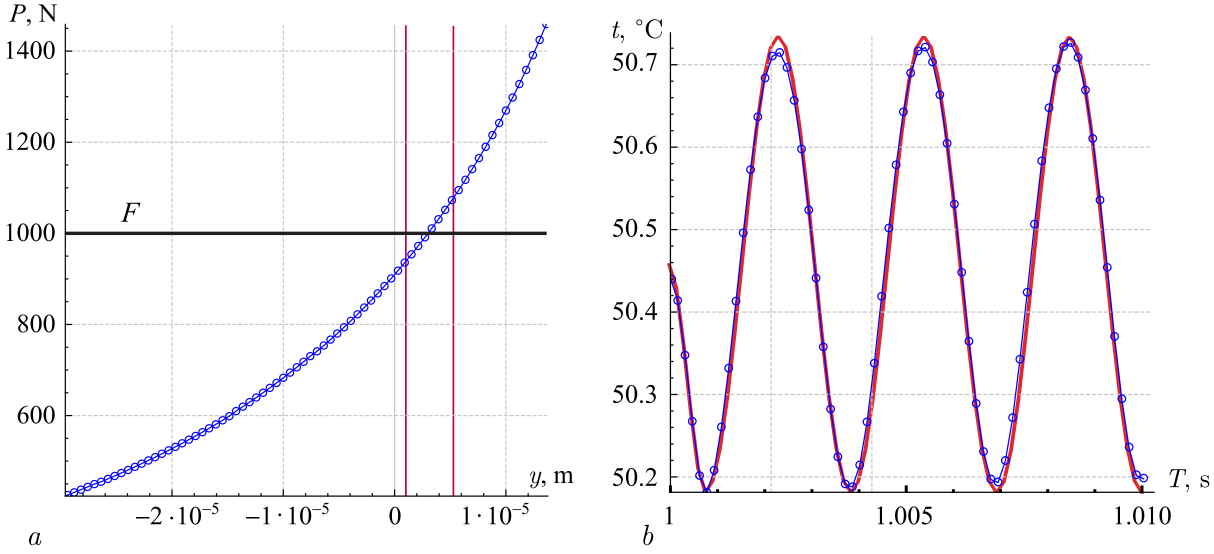

Fig 4. raphs demonstrating the relationship of the force F acting on the rotor. *a* — The region of equilibrium localization between the load-bearing capacity P and the force F, *b* — the relationship between temperature (blue line) and the trajectory of the collar movement (red line) (color online)

Education «KNITU» [13–15, 17, 20, 21], were adopted as the initial data for the program in numerical experiments.

Fig. 3 shows the graphs of the distribution of temperature along the y axis in the center of the computational domain and pressure p along the angular coordinate φ at the optimal position of the disk. As can be seen, the temperature reaches its maximum value near the cushion surface due to braking by a stationary surface, but does not exceed the limit value of 110°C [1,2]. Some increase in temperature t is observed near the disk surface due to the transfer of heated layers of grease from the previous cushion due to viscous shear. In the body of the disk and cushion, the temperature decreases slightly due to heat exchange with the environment. The previously selected flat-wedge profile of the working surface of the cushion (see Fig. 2, *a*, *b*) determines the shape of the pressure distribution curve p and its value. In this case, the maximum of the distribution is shifted towards disk rotation. The integral over the working surface allows us to calculate the bearing capacity of the thrust bearing P, balancing the external axial force F (see equation (6)).

Fig. 4 shows two graphs demonstrating the operation of the algorithms proposed above. The graph in Fig. 4, *a* shows the behavior of the load-bearing capacity depending on the disk position y in blue. The graph also shows the force F acting on the rotor at a fixed point in time. The vertical lines show the region of localization of the equilibrium state between the load-bearing capacity P and the force F. The graph in Fig. 4, *b* shows the relationship between the temperature in the center of the calculation region (blue) and the trajectory of the disk movement in time (red, in dimensionless form on the graph). For demonstration, soft surge was chosen as the simplest. As can be seen from the graph, the curves practically coincide with a single cyclic frequency without time inertia. In future articles, the issues of changing the integral characteristics of the compressor thrust bearing when solving direct and inverse nonlinear problems, surge intensity and disk rotation frequency will be considered separately.

Conclusion

Based on the proposed methods, a software package $Sm2Px3Tx\tau$ has been developed, allowing to solve direct and inverse nonlinear problems of a one-sided dynamically loaded fluid film thrust bearing with fixed pads under various geometric, physical and operating parameters. The program can be used to study the rotor stability taking into account the impact of the thrust bearing under various loading conditions, which is of great practical importance in the design of various compressor machines. The advantages of the mathematical model applied by PTUGD are the ability to fully take into account the temperature field in the thrust bearing, including heat transfer between adjacent pads and fresh lubricant supply. With a nonlinear formulation, the equations of rotor motion and thrust bearing hydrodynamics are integrated together, which allows for the nonlinear behavior of the bearing reaction to be taken into account and the actual dynamic behavior of the rotor to be modeled. ПIn the numerical implementation, an iterative solution method is proposed using the solution of the direct problem. To reduce computational costs, a modified Dekker-Brent method is used together with a modified Newton method. The solution of the inverse problem will optimize the geometric dimensions of the thrust bearing to achieve increased load-bearing capacity and stability of the rotor motion. However, it should be noted that solving the inverse problem using the PTUGD mathematical model can be complex and require significant computational resources. Thus, the development of effective numerical methods and algorithms for solving this problem remains a relevant area of research in engineering science.

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