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Discrete traveling waves in a relay system of differential-difference equations modeling a fully connected network of synaptically connected neurons

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Abstract. *Purpose.* Consider a system of differential equations with delay, which models a fully connected chain of $m+1$ neurons with delayed synaptic communication. For this fully connected system, construct periodic solutions in the form of discrete traveling waves. This means that all components are represented by the same periodic function $u(t)$ with a shift that is a multiple of some parameter Δ (to be found). *Methods.* To search for the described solutions, in this work we move from the original system to an equation for an unknown function $u(t)$, containing m ordered delays, differing with step Δ. It performs an exponential substitution (typical of equations of the Volterra type) in order to obtain a relay equation of a special form. *Results.* For the resulting equation, a parameter range is found in which it is possible to construct a periodic solution with period T depending on the parameter Δ. For the found period formula $T = T(\Delta)$, it is possible to prove the solvability of the period equation, that is, to prove the existence of non-zero parameters — integer p and real Δ — satisfying the equation $(m + 1)\Delta = pT(\Delta)$. The constructed function $u(t)$ has a bursting effect. This means that $u(t)$ has a period of n high spikes, followed by a period of low values. *Conclusion.* The existence of a suitable parameter Δ ensures the existence of a periodic solution in the form of a discrete traveling wave for the original system. Due to the choice of permutation, the coexistence of $(m + 1)!$ periodic solutions is ensured.

Keywords: differential-difference equations, fully coupled system, discrete traveling waves, bursting effect, periodic solutions, neuron modeling.

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Introduction

In this paper, we consider a model of a fully coupled neuron network. It is based on the equation [1–3]

$$\dot{u} = \lambda F(u(t-1))u, \tag{1}$$

which is used to describe the behavior of a single neuron. Here $u(t) > 0$ is the normalized membrane potential of the neuron, $\lambda > 0$ characterizes the rate of electrical processes, F is a piecewise constant function,

$$F(u) = \begin{cases} -a, & u \in (0,1], \\ 1, & u > 1, \end{cases}$$

$a = \mathrm{const} > 0$. Equation (1) is in some sense the limit version of the generalized Hutchinson equation [4, 5]

$$\dot{u} = \lambda f(u(t-1))u, \tag{2}$$

where f is a smooth function,

$$u = u(t) > 0; \quad \lambda \gg 1; \quad f(0) = 1; \quad \lim_{u\to\infty} f(u) = -a \ (a > 0); \quad f^{'}(u), uf^{''}(u) = O(u^{-2}), \ u \to \infty.$$

In [4] it was proved that equation (2) admits a stable relaxation cycle $u(t) = e^{\lambda(x_0(t)+O(1/\lambda))}$ for $\lambda \to +\infty$, where

$$x_0(t) = \begin{cases} t, \ t \in [0,1], \\ -a(t-t_0), \ t \in [1, t_0+1], \\ t - T_0, \ t \in [t_0+1, T_0], \end{cases} \tag{3}$$

$$x_0(t+T_0) = x_0(t), \quad t_0 = (a+1)/a, \quad T_0 = (a+1)^2/a.$$

The function $x_0(t)$ is shown in Fig. 1. Returning to equation (1), we note that it has an orbitally stable T_0-periodic solution $u(t) = e^{\lambda x_0(t)}$.

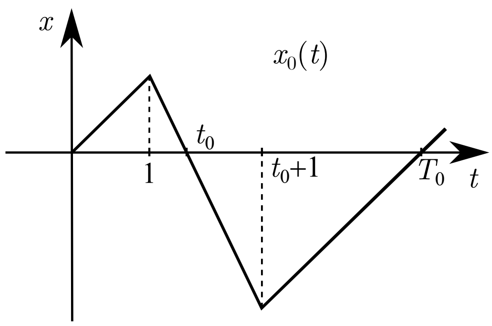

Fig 1. The function $x_0(t)$

In this paper, neurons, each of which is individually modeled by equation (1), are connected into a fully coupled system with synaptic one-way interaction.

The method for modeling synaptic connection is based on the idea of fast threshold modulation, described, for example, in [6–8]. It is chosen as a certain limiting version of the connection used, for example, in the works [9, 10] when modeling ring chains of neurons. To describe the complete connection, a function that is symmetric with respect to the permutation of its arguments is used

$$G(u_1, \dots, u_m) = \begin{cases} 0, & \text{if for } \forall i \ u_i < 1, \\ b, & \text{if } \exists k \ u_k > 1, \end{cases}$$

$b = \mathrm{const} > 0$. The synaptic connection is assumed to be retarded, so each argument of the function G has a delay $h > 0$.

As a model of a fully coupled chain of neurons with a retarded synaptic connection, a system of differential equations with delay is proposed

$$\dot{u}_j = \Big(\lambda F(u_j(t-1)) + G(u_0(t-h), \ldots, u_{j-1}(t-h), u_{j+1}(t-h), \ldots, u_m(t-h)) \cdot \ln \frac{u_*}{u_j}\Big) u_j, \quad (4)$$

$j = 0, 1, \ldots, m$. Here $u_j(t) > 0$ — normalized membrane potentials of neurons, $\lambda > 0$ characterizes the rate of electrical processes, $h > 0$ — delay in the communication chain, the terms $G(u_0(t-h), \ldots, u_{j-1}(t-h), u_{j+1}(t-h), \ldots, u_m(t-h)) \cdot \ln(u_*/u_j) u_j$ model synaptic interaction with a time delay. When describing the jth neuron, the function G is multiplied by the logarithm $\ln(u_* u_j)$, which changes sign when the values of the function u_j pass the threshold value $u_* = \exp(c\lambda)$, $c = \mathrm{const} \in \mathbb{R}$.

Similar (4) systems of equations that describe networks of neurons and are based on the solitary neuron model (1) or (2) are considered in a number of works [1, 2, 11]. In [11], the so-called impulse-refractory mode was constructed for a ring chain of neurons with one-way synaptic interaction. It is understood as a periodic regime in which components with exponentially high spikes alternate with components with exponentially small values. The works [1, 2] consider the interaction of two neurons. Results on the multistability of coexisting periodic regimes with a fixed total number of spikes are proven.

In contrast to the above-mentioned works, this paper considers a fully coupled system, i.e. we assume that each neuron is connected to each other. In [12], a fully coupled system of neuro-oscillators with electrical synaptic connection is introduced, each of the oscillators is described by equation (2). For the case of a ring system with unidirectional connection, results on buffering are proved, and for a fully coupled network, where all connections are equal and identical, the issue of two-cluster synchronization is studied. In [13], the issue of two-cluster synchronization is also studied, but using van der Pol oscillators as an example. In [14], a fully connected system of nonlinear oscillators is considered, and the dynamic properties of chimeric solutions arising during two-cluster synchronization are studied.

Section 1 is devoted to the formulation of the problem. It describes the mechanism for searching for discrete traveling waves, the transition to an auxiliary relay equation with m delays, and introduces a set of initial functions for it. Section 2 presents the main result in the form of three theorems. Sections 3 and 4 prove the formulated theorems.

1. Statement of the problem

1.1. Discrete traveling waves. We are interested in the existence of a periodic solution of the system (4) in the form of a discrete traveling wave. The technique for constructing solutions of this type for a fully coupled system is the same as, for example, in [15], and is similar to the technique for a ring chain of generators (see [16]): we assume that all functions u_k are represented by the same periodic function u with shifts multiple of some parameter Δ:

$$u_k(t) = u(t + j_k \Delta), \quad k = 0, 1, \ldots, m, \quad (5)$$

where $j_0, j_1, \ldots, j_m$ denote some permutation of numbers $0, 1, \ldots, m$, and the parameter Δ is to be determined.

We fix k. After substituting (5) into the k-th equation (4) and renormalizing time $t+j_k\Delta \mapsto t$, we obtain

$$\dot{u} = \Big(\lambda F\big(u(t-1)\big) + G\big(u(t+(j_0-j_k)\Delta-h),\ldots,u(t+(j_{k-1}-j_k)\Delta-h), \\ u(t+(j_{k+1}-j_k)\Delta-h),\ldots,u(t+(j_m-j_k)\Delta-h)\big)\cdot\ln\frac{u_*}{u}\Big)u. \quad (6)$$

Note that the differences j_l-j_k $(l=0,1,\ldots,k-1,k+1,\ldots,m)$ take all values from the ordered set $\{-j_k,\ldots,-1,1,\ldots,m-j_k\}$. Let $T=T(\Delta)$ denote the period of $u(t)$. Given the symmetry of G with respect to the permutation of its arguments, equation (6) is equivalent to the following:

$$\dot{u} = \Big(\lambda F\big(u(t-1)\big) + G\big(u(t-\Delta-h),\ldots,u(t-j_k\Delta-h), \\ u(t+(m-j_k)\Delta-T-h),\ldots,u(t+\Delta-T-h)\big)\cdot\ln\frac{u_*}{u}\Big)u. \quad (7)$$

Since for each $k=0,1,\ldots,m$ the same equation for the function $u(t)$ must be obtained, the requirement

$$u\left(t-(j_k+1)\Delta-h\right) \equiv u\left(t+(m-j_k)\Delta-T-h\right),$$

is natural, that is, the value $(j_k+1)\Delta+(m-j_k)\Delta-T$ must be a multiple of T. From this we obtain that for the periodicity of the solution $(u_0,u_1,\ldots,u_m)$ of the system (4) it is necessary that the parameter Δ and the period $T=T(\Delta)$ of the function $u(t)$ satisfy the period equation

$$(m+1)\Delta = pT(\Delta), \quad p\in\mathbb{Z}\setminus\{0\}, \quad \Delta\in\mathbb{R}\setminus\{0\}. \quad (8)$$

In this case, all equations of the system (7) are transformed into

$$\dot{u} = \Big(\lambda F\big(u(t-1)\big) + G\big(u(t+m\Delta-h),\ldots,u(t+\Delta-h)\big)\cdot\ln\frac{u_*}{u}\Big)u. \quad (9)$$

Thus, the problem of finding periodic solutions of system (4) in the form of discrete traveling waves (5) has been reduced to finding a periodic function $u(t)$ satisfying (9) and a parameter Δ such that the period $T=T(\Delta)$ of the function $u(t)$ satisfies the period equation (5) for some integer p.

Note that in the case of the existence of the specified function $u(t)$ and the parameter Δ, the choice of permutation ensures the coexistence of $(m+1)!$ periodic solutions of the system (4).

1.2. Relay equation with m delays. Instead of equation (9), consider the equation

$$\dot{u} = \Big(\lambda F\big(u(t-1)\big) + G\big(u(t-h_1),\ldots,u(t-h_m)\big)\cdot\ln\frac{u_*}{u}\Big)u \quad (10)$$

with ordered delays

$$h_1 < h_2 < \ldots < h_m, \quad h_{s+1} = h_s + |\Delta|, \quad s=1,\ldots,m-1. \quad (11)$$

1.3. Transition to logarithmic scale. We make an exponential change of variables in (10): $u=e^{\lambda x}$, assuming that $c=\text{const}$ such that $u_*=e^{\lambda c}$:

$$\dot{x} = F\left(e^{\lambda x(t-1)}\right) + (c-x)\,G\left(e^{\lambda x(t-h_1)},\ldots,e^{\lambda x(t-h_m)}\right).$$

We introduce notations for functions with exponential arguments

$$R(x) = F(e^{\lambda x}) = \begin{cases} -a, & x < 0, \\ 1, & x > 0, \end{cases}$$

$$H(x_1, \ldots, x_m) = G(e^{\lambda x_1}, \ldots, e^{\lambda x_m}) = \begin{cases} 0, & \text{if for } \forall i \; x_i < 0, \\ b, & \text{if } \exists k \; x_k > 0. \end{cases}$$

Thus, the new unknown function $x(t)$ must satisfy the equation

$$\dot{x} = R(x(t-1)) + (c - x)H(x(t-h_1), \ldots, x(t-h_m)). \tag{12}$$

1.4. Initial set of functions. As a set of initial functions for equation (12), we choose the following:

$$S = \{\varphi \in C[-h_m, 0] : \\ \varphi(t) < 0 \text{ for } t \in [-h_m, 0), \; \varphi(0) = 0\}. \tag{13}$$

These are negative functions on the interval of the length of the largest delay h_m, taking the value zero at zero (see schematic fig. 2).

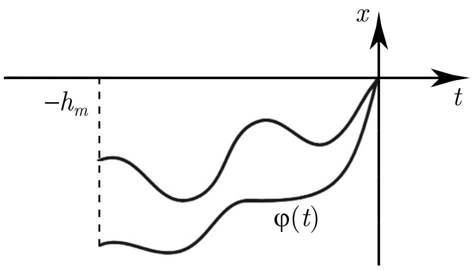

Fig 2. Representatives of the set of initial functions

2. Result

2.1. Solution of the relay equation with respect to the variable x. We introduce the notation

$$y_0(\tilde{x}, t) = \begin{cases} \left(\tilde{x} - \frac{1}{b} - c\right) e^{-bt} + \frac{1}{b} + c, & t \in [0, t_0], \\ t - t_0 + y_0(\tilde{x}, t_0) & t \in [t_0, T_0], \end{cases} \qquad y_0(\tilde{x}, t + T_0) = y_0(y_0(\tilde{x}, T_0), t). \tag{14}$$

This function is shown in Fig. 3. It consists of continuously "glued" alternating parts of the exponential and linear functions with slope 1. The quantity $\tilde{x}$ denotes the value of the function y_0 at $t = 0$. Below, in lemma 2, it will be proved that the condition

$$c < -a - \frac{1}{b} - \frac{a+1}{1 - e^{-bt_0}} \tag{15}$$

guarantees that $y_0((k+1)T_0) < y_0(kT_0)$. We also introduce the quantities

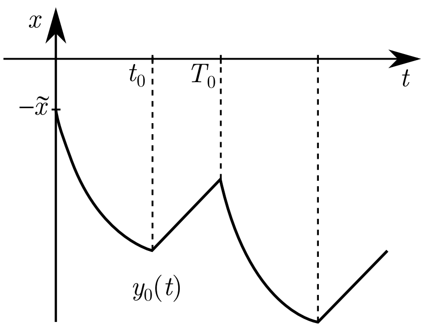

Fig 3. The function $y_0(t)$

$$x_0^{(1)} = x_0(h_1),$$
$$x_0^{(s+1)} = h_{s+1} - h_s - t_0 - (n-1)T_0 + y_0(x_0^{(s)}, t_0 + (n-1)T_0), \; s = 1, \ldots, m-1,$$

Theorem 1. *Let us fix a natural number n. Let*

1) *parameters $a > 0$, $b > 0$ and c satisfy the constraint* (15)*;*
2) *delays h_s satisfy the inequalities*

$$(n-1)T_0 + t_0 + 1 < h_1 < nT_0, \tag{16}$$

$$h_{s+1} - h_s > t_0 + (n-1)T_0, \quad s = 1, \dots, m-1; \tag{17}$$

$$h_{s+1} - h_s < nT_0 + a(1 - e^{-nbt_0}), \quad s = 1, \dots, m-1. \tag{18}$$

Then equation (12) *with any initial function from the set* (13) *has a T-periodic solution*

$$x(t) = \begin{cases} x_0(t), & t \in [0, h_1] \\ y_0(x_0^{(s)}, t - h_s), & t \in [h_s, h_s + t_0 + (n-1)T_0], \\ & s = 1, \dots, m, \\ t - h_s - t_0 - (n-1)T_0 + y_0(x_0^{(s)}, t_0 + (n-1)T_0), & t \in [h_s + t_0 + (n-1)T_0, h_{s+1}], \\ & s = 1, \dots, m-1, \\ t - h_m - t_0 - (n-1)T_0 + y_0(x_0^{(m)}, t_0 + (n-1)T_0), & t \in [h_m + t_0 + (n-1)T_0, T], \end{cases} \tag{19}$$

$$T = h_m + t_0 + (n-1)T_0 - \left(x_{n-1}^{(m)} - \frac{1}{b} - c\right) e^{-bt_0} - \frac{1}{b} - c. \tag{20}$$

The quantities $x_0^{(s)}$ denote the values of the solution $x(t)$ at the points h_s. The schematic graph of the function $x(t)$ is shown in Fig. 4.

A detailed proof of the theorem 1 is presented in section 3. Here we restrict ourselves to describing the meaning of the conditions on the parameters given in the theorem.

The double inequality (16) means that the time moment $t = h_1$ falls on the n-th period of the function $x_0(t)$, and on the segment where $x_0(t)$ increases and is negative (see Fig. 1).

The constraint (17) means that the lengths of the segments $[h_s, h_{s+1}]$ are greater than the length of the interval on which the solution $x(t)$ coincides with the function $x_0(t)$ and changes sign (up to the last point of positivity on the period).

Inequalities (18) ensure that the quantities $x_0^{(s)}$ are negative for $s = 1, \dots, m$. This will be proved in lemma 6.

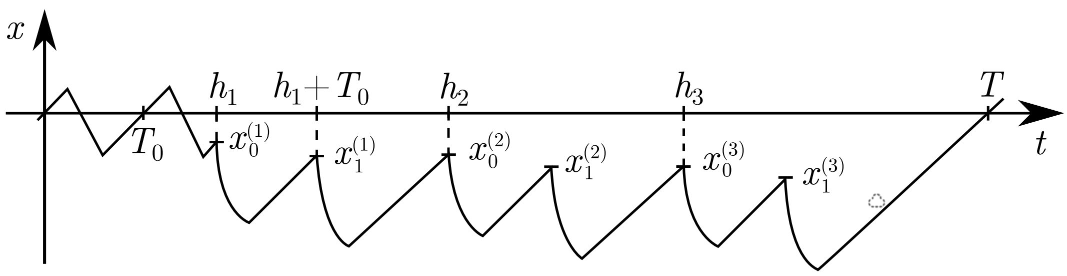

Fig 4. The function $x(t)$ for $n = 2$, $m = 3$

2.2. Solvability of the period equation. For the period T described by the formula (20), it is possible to prove the statement on the solvability of the period equation (8).

Theorem 2. *We fix natural numbers $m \geqslant 2$ and n. Let*

1) *parameters $a > 0$, $b > 0$;*
2) *the value of c satisfies the constraints (15) and*

$$c > -2a - \frac{1}{b} - \frac{a+1}{1 - e^{-bt_0}}; \tag{21}$$

3) *the delays h_s satisfy (16) and (11).*

Then there exist $\Delta \in \mathbb{R} \setminus \{0\}$ and $p \in \mathbb{Z} \setminus \{0\}$ satisfying equation (8), and

$$t_0 + (n-1)T_0 < |\Delta| < nT_0 + a(1 - e^{-nbt_0}). \tag{22}$$

The inequality (22) means that for $h_{s+1} - h_s = |\Delta|$ the conditions are satisfied (17) and (18) of the theorem 1. The proof of the theorem is given in section 4.

2.3. Periodic solution of the original system. The presence of a suitable parameter Δ ensures the existence of a periodic solution of the original system in the form of a discrete traveling wave. Thus, the following result follows from theorems 1 and 2.

Theorem 3. *Let*

1) *n is a fixed natural number,*
2) *parameters a, b, c, h_s $(s = 1, \ldots, m)$ satisfy the conditions of the theorem 1,*
3) *$\lambda > 0$,*
4) *$\Delta \neq 0$ satisfies the period equation (8) for some integer $p \neq 0$, then there exists h such that the system (4) has $(m+1)!$ coexisting solutions of the form*

$$u_k = e^{\lambda x(t + j_k \Delta)}, \tag{23}$$

where $x(t)$ is described by the formula (19), $j_0, j_1, \ldots, j_m$ represent some permutation of the numbers $0, 1, \ldots, m$. In this case, each component has «high-amplitude» (of the order of e^{λ}) bursts on the period n, after which there follows an interval with «small» (of the order of $e^{-\lambda}$) values of the function $u_k(t)$. The value $(m+1)\Delta$ is a multiple of the period of this solution.

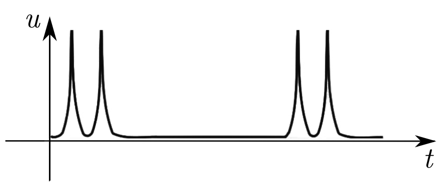

Fig 5. A function $u(t)$ with a bursting effect, containing $n = 2$ bursts per period

The behavior of the solution components described in the theorem can naturally be called the bursting effect (see Fig. 5).

Taking into account (5) and the exponential replacement $u = e^{\lambda x}$, the validity of the theorem 3 follows from the theorem 1 and the theorem 2. Note the method of choosing the parameter h. If $h_1 > \Delta$, then $h = h_1 - \Delta$. If $h_1 < \Delta$, then we can take $h = \Delta - h_1$.

3. Proof of the theorem 1

On the interval $[0, h_1]$ the function $H(x(t-h_1), \dots, x(t-h_m))$ is equal to 0, therefore, on the current interval $x(t)$ is found from the initial Cauchy problem

$$\begin{cases} \dot{x} = R(x(t-1)), \\ x(t)|_{t\in[-h_m,0]} = \varphi(t). \end{cases} \tag{24}$$

The solution to problem (24) is a periodic function $x_0(t)$ described by formulas (3). This function is shown in Fig. 1. It vanishes at the points

$$kT_0,\ t_0 + kT_0,\ k = 0, 1, \dots \tag{25}$$

and breaks at the points

$$1 + kT_0,\ t_0 + 1 + kT_0,\ k = 0, 1, \dots. \tag{26}$$

By the condition of the theorem 1, the value h_1 falls on the n-th period of the function $x_0(t)$. More precisely: the constraint (16) guarantees that h_1 falls on the segment of the period of the function $x_0(t)$ where it increases and is negative.

Equation (12) on the interval $[h_1, h_2]$ takes the form

$$\dot{x} = 1 + (c - x)H(x_0(t-h_1), \varphi(t-h_2) \dots, \varphi(t-h_m)).$$

The first argument of the function H changes sign, and the arguments from the 2nd to the m-th are negative. This means that the value of the function H here is determined by the sign of its first argument $x_0(t-h_1)$. On the segments $[h_1 + kT_0, h_1 + t_0 + kT_0],\ k = 0, 1, \dots, n-1$ the function $x_0(t-h_1)$ is positive, and on the segments $[h_1+t_0+kT_0, h_1+t_0+(k+1)T_0]$ it is negative. We denote the value of the function x at the points $h_1 + kT_0$ by $x_k^{(1)}$; these quantities are to be determined. Thus, depending on the sign of $x_0(t-h_1)$ on the interval $[h_1, h_1 + t_0 + (n-1)T_0]$, we obtain one of two possible Cauchy problems.

1. For $t \in [h_1 + kT_0, h_1 + t_0 + kT_0]$, $k = 0, 1, \dots, n-1$,

$$\begin{cases} \dot{x} = 1 + (c - x)b, \\ x|_{t=h_1+kT_0} = x_k^{(1)}, \end{cases} \tag{27}$$

from where

$$x(t) = \left(x_k^{(1)} - \frac{1}{b} - c\right) e^{-b(t-h_1-kT_0)} + \frac{1}{b} + c. \tag{28}$$

2. For $t \in [h_1 + t_0 + kT_0, h_1 + (k+1)T_0]$, $k = 0, 1, \dots, n-2$,

$$\begin{cases} \dot{x} = 1, \\ x|_{h_1+t_0+kT_0} = \left(x_k^{(1)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c. \end{cases} \tag{29}$$

Here the initial value is determined by formula (28). The solution of the Cauchy problem (29) has the form

$$x(t) = t - h_1 - t_0 - kT_0 + \left(x_k^{(1)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c \tag{30}$$

Thus, it is proved that on the interval $[h_,h_1 + t_0 + kT_0]$ the solution $x(t)$ of equation (12) coincides with the function $y_0(x_0^{(1)}, t - h_1)$ described by the formula (14), $x_0^{(1)} = x_0(h_1)$. In this case,

$$x_k^{(1)} = y_0(x_0^{(1)}, kT_0).$$

Next, we prove two lemmas describing the behavior of the sequence $x_k^{(1)}$.

Lemma 1. *The sequence $x_k^{(1)}$ is given by the formula*

$$x_k^{(1)} = (a+1)\frac{1 - e^{-kbt_0}}{1 - e^{-bt_0}} + \left(x_0^{(1)} - \frac{1}{b} - c\right) e^{-kbt_0} + \frac{1}{b} + c. \tag{31}$$

Proof. Given the recurrent form of y_0 and the fact that $T_0 - t_0 = a + 1$, we obtain a recurrent sequence for $x_k^{(1)}$:

$$x_{k+1}^{(1)} = a + 1 + \left(x_k^{(1)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c. \tag{32}$$

From here, summing the corresponding geometric progression, accumulating in the coefficient at $(a + 1)$, we find an explicit formula for calculating $x_k^{(1)}$:

$$x_k^{(1)} = (a+1)\frac{1 - e^{-kbt_0}}{1 - e^{bt_0}} + \left(x_0^{(1)} - \frac{1}{b} - c\right) e^{-kbt_0} + \frac{1}{b} + c.$$

The lemma 1 is proved.

Let us prove the lemma on the monotonicity of the sequence $x_k^{(1)}$.

Lemma 2. *Let the parameters a, b, c satisfy the inequality (15). Then the sequence $x_k^{(1)}$ described by the formula (31) is decreasing.*

Proof. We will prove that

$$x_{k+1}^{(1)} < x_k^{(1)}.$$

This inequality, taking into account (31), is equivalent to the following inequality:

$$(a+1)\frac{1 - e^{-(k+1)bt_0}}{1 - e^{-bt_0}} + \left(x_0^{(1)} - \frac{1}{b} - c\right) e^{-(k+1)bt_0} < (a+1)\frac{1 - e^{-kbt_0}}{1 - e^{-bt_0}} + \left(x_0^{(1)} - \frac{1}{b} - c\right) e^{-kbt_0},$$

whence follows

$$c < x_0^{(1)} - \frac{1}{b} - \frac{a+1}{1 - e^{-bt_0}}.$$

The last inequality is true, since $x_0^{(1)} = x_0(h_1) > x_0(t_0 + 1 + (n-1)T_0) = -a$, and by condition (15) is satisfied.

Under the assumption (16), $x_0^{(1)} = x_0(h_1) = h_1 - nT_0 < 0$ is valid, therefore, lemma 2 and the decrease of the exponent in the function y_0 ensure the negativeness of the solution on the interval $[h_1, h_1 + t_0 + (n-1)T_0]$.

The next interval of construction starts at the point $h_1 + t_0 + (n-1)T_0$ and will last either to the point h_2 or to the next root increased by 1. Here the problem takes the form

$$\begin{cases} \dot{x} = 1, \\ x|_{t=h_1+t_0+(n-1)T_0} = \left(x_{n-1}^{(1)} - \dfrac{1}{b} - c\right) e^{-bt_0} + \dfrac{1}{b} + c, \end{cases}$$

from where

$$x(t) = t - h_1 - t_0 - (n-1)T_0 + \left(x_{n-1}^{(1)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c. \tag{33}$$

Substituting h_2 into formula (33), using (31), inequalities (18), (15) and the negativeness of $x_0^{(1)}$, we see that $x(h_2) < 0$. This means that h_2 is less than the next root of function $x(t)$. Thus, the current construction interval is $[h_1 + t_0 + (n-1)T_0, h_2]$, and the formula (33) is valid on it.

Then the solution is constructed in the same way as it was done on the interval $[h_1, h_2]$.

Lemma 3. *For $t \in [h_s, h_s + t_0 + (n-1)T_0]$, $s = 1, \ldots, m$, the function H has one argument with alternating sign (this is $x(t - h_s)$), and the other arguments are negative. On the intervals $[h_s + t_0 + (n-1)T_0, h_{s+1}]$, $s = 1, \ldots, m-1$, all arguments of the function H are negative.*

The proof of the lemma is carried out by the method of mathematical induction and includes several lemmas on the behavior of the quantities $x_k^{(s)}$.

On each of the intervals $[h_s, h_s+t_0+(n-1)T_0]$, when solving the equation, Cauchy problems arise similar to problems (27) and (29).

1. For $t \in [h_s + kT_0, h_s + t_0 + kT_0]$, $k = 0, 1, \ldots, n-1$, $s = 1, \ldots, m$

$$\begin{cases} \dot{x} = 1 + (c - x)b, \\ x|_{t=h_s+kT_0} = x_k^{(s)}. \end{cases} \tag{34}$$

The sequence $x_k^{(s)}$ is equal to the values of the function $x(t)$ at the points $h_s + kT_0$. They will be described by recurrence relations below. From (34) we find

$$x(t) = \left(x_k^{(s)} - \frac{1}{b} - c\right) e^{-b(t-h_s-kT_0)} + \frac{1}{b} + c. \tag{35}$$

Substituting $t = h_s + t_0 + kT_0$ into (35), we obtain the initial value for the next Cauchy problem by number.

2. For $t \in [h_s + t_0 + kT_0, h_s + (k+1)T_0]$, $k = 0, 1, \ldots, n-2$, $s = 1, \ldots, m$

$$\begin{cases} \dot{x} = 1, \\ x|_{h_s+t_0+kT_0} = \left(x_k^{(s)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c, \end{cases}$$

from

$$x(t) = t - h_s - t_0 - kT_0 + \left(x_k^{(s)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c. \tag{36}$$

3. For $t \in [h_s + t_0 + (n-1)T_0, h_{s+1}]$, $s = 1, \ldots, m-1$

$$\begin{cases} \dot{x} = 1, \\ x|_{h_s+t_0+(n-1)T_0} = \left(x_{n-1}^{(s)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c, \end{cases}$$

from

$$x(t) = t - h_s - t_0 - (n-1)T_0 + \left(x_{n-1}^{(s)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c. \tag{37}$$

To complete the proof, we formulate 3 more lemmas.

Lemma 4. *The sequences $x_k^{(s)}$ for $s = 1, \dots, m$, $k = 0, \dots, n-1$ are defined by the formulas*

$$x_0^{(1)} = x_0(h_1), \quad x_k^{(s+1)} = (|\Delta| - nT_0)\frac{1 - e^{-snbt_0}}{1 - e^{-nbt_0}}e^{-kbt_0} + (a+1)\frac{1 - e^{-(sn+k)bt_0}}{1 - e^{-bt_0}} + \\ + \left(x_0^{(1)} - \frac{1}{b} - c\right)e^{-(sn+k)bt_0} + \frac{1}{b} + c. \quad (38)$$

Proof. Similarly to how formula (31) was obtained, we find

$$x_k^{(s)} = (a+1)\frac{1 - e^{-kbt_0}}{1 - e^{-bt_0}} + \left(x_0^{(s)} - \frac{1}{b} - c\right)e^{-kbt_0} + \frac{1}{b} + c. \quad (39)$$

Formula (37) for $t = h_{s+1}$ gives the value for $x_0^{(s+1)}$:

$$x_0^{(s+1)} = h_{s+1} - h_s - t_0 - (n-1)T_0 + \left(x_{n-1}^{(s)} - \frac{1}{b} - c\right)e^{-bt_0} + \frac{1}{b} + c, \quad (40)$$

Substituting formula (39) into (40) for $k = n-1$ and taking into account the equality $T_0 - t_0 = = a + 1$, we obtain the recurrent dependence of $x_0^{(s+1)}$ on $x_0^{(s)}$:

$$x_0^{(s+1)} = h_{s+1} - h_s - nT_0 + (a+1)\frac{1 - e^{-nbt_0}}{1 - e^{-bt_0}} + \left(x_0^{(s)} - \frac{1}{b} - c\right)e^{-nbt_0} + \frac{1}{b} + c. \quad (41)$$

Hence, summing up the geometric progression and taking into account that $h_{s+1} - h_s = |\Delta|$, we obtain

$$x_0^{(s+1)} = (|\Delta| - nT_0)\frac{1 - e^{-snbt_0}}{1 - e^{-nbt_0}} + (a+1)\frac{1 - e^{-snbt_0}}{1 - e^{-bt_0}} + \left(x_0^{(1)} - \frac{1}{b} - c\right)e^{-snbt_0} + \frac{1}{b} + c. \quad (42)$$

From formulas (39) and (42) follows (38).

Lemma 5. *Let parameters a, b, c satisfy inequality (15). Then sequences $x_k^{(s)}$, $s = 1, \dots, m$, described by formulas (38), decrease in k for fixed s.*

This lemma is proved similarly to lemma 2.

The following lemma guarantees that the values of the function x at the points h_s are negative.

Lemma 6. *Let the parameters a, b, c, h_s ($s = 1, \dots, m$) satisfy conditions (18) and (15), then the values $x_0^{(s)} < 0$ for $s = 1, \dots, m$.*

Proof. We prove the statement by mathematical induction. It follows from condition (16) that $x_0^{(1)} = x_0(h_1) = h_1 - nT_0 < 0$. This means that the induction base is satisfied. Suppose that $x_0^{(s)} < 0$, then, applying to (41) the inequalities (18) and $c + \frac{1}{b} < -a - \frac{a+1}{1-e^{-bt_0}}$, which follows from (15), we obtain

$$x_0^{(s+1)} = h_{s+1} - h_s - nT_0 + (a+1)\frac{1 - e^{-nbt_0}}{1 - e^{-bt_0}} + x_0^{(s)}e^{-nbt_0} + \left(\frac{1}{b} + c\right)(1 - e^{-nbt_0}) < x_0^{(s)}e^{-nbt_0} < 0.$$

The lemma 6 is proved.

The lemmas 2 and 6 guarantee the negative values of the solution $x(t)$ for $t \in [h_s, h_{s+1}]$, thus justifying the induction step of the proof of the lemma 3.

At the final stage of constructing the solution, we obtain the Cauchy problem

$$\begin{cases} \dot{x} = 1, \\ x|_{t=h_m+t_0+(n-1)T_0} = \left(x_{n-1}^{(m)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c, \end{cases}$$

from where

$$x(t) = t - h_m - t_0 - (n-1)T_0 + \left(x_{n-1}^{(m)} - \frac{1}{b} - c\right) e^{-bt_0} + \frac{1}{b} + c. \tag{43}$$

The right-hand side of equation (12) turns out to be equal to 1, since all arguments of H are negative after $t = h_m + t_0 + (n-1)T_0$. This situation will persist on the interval $[h_m + t_0 + (n-1)T_0, T]$, where T denotes the first zero of the function $x(t)$ greater than $(n-1)T_0 + t_0$ (the positive root with number $2n$). From formula (43) follows (20).

Lemma 7. *Let $x(t)$ be a solution of equation (12) constructed on the interval $[0, T]$ and described by the formulas (3) for $t \in [0, h_1]$, (35), (36) and (37) for $t \in [h_1, h_m + t_0 + (n-1)T_0]$, (43) for $t \in [h_m + t_0 + (n-1)T_0, T]$. Then the function $x(t+T)$ for $t \in [-h_m, 0]$ belongs to the set (13).*

Proof. Let us prove that the interval $((n-1)T_0 + t_0, T)$, on which $x(t) < 0$, has a length greater than h_m. Indeed,

$$T - (n-1)T_0 - t_0 = h_m \underbrace{- \left(x_{n-1}^{(m)} - \frac{1}{b} - c\right) e^{-bt_0} - \frac{1}{b} - c}_{=-x(h_s)>0} > h_s.$$

Then the interval $[T - h_m, T)$ is embedded in $((n-1)T_0 + t_0, T)$, and $x(T) = 0$.

Lemma 7 ensures the T-periodicity of the constructed solution $x(t)$.

Theorem 1 is completely proved.

4. Proof of Theorem 2

Note that in the period equation (8), due to the positivity of T and $m+1$, the quantities Δ and p have the same sign, so we can write

$$T = \frac{(m+1)|\Delta|}{|p|}. \tag{44}$$

We give an explicit formula for calculating the period T. Substituting (38) into (20), and also taking into account that $h_m = h_1 + (m-1)|\Delta|$ and $x_0^{(1)} = h_1 - nT_0$, we get

$$T = |\Delta| \Big(m - \frac{1 - e^{-mnbt_0}}{1 - e^{-nbt_0}}\Big) + h_1(1 - e^{-mnbt_0}) + nT_0 \frac{1 - e^{-(m+1)nbt_0}}{1 - e^{-nbt_0}} - \\ - (a+1)\frac{1 - e^{-mnbt_0}}{1 - e^{-bt_0}} - \frac{1}{b} - c. \tag{45}$$

From (44) and (45) it follows

$$\Big(\frac{m+1}{|p|}-m+\frac{1-e^{-mnbt_0}}{1-e^{-nbt_0}}\Big)|\Delta| =$$
$$= h_1(1-e^{-mnbt_0})+nT_0\frac{1-e^{-(m+1)nbt_0}}{1-e^{-nbt_0}}-(a+1)\frac{1-e^{-mnbt_0}}{1-e^{-bt_0}}-\frac{1}{b}-c. \quad (46)$$

Here the coefficient of $|\Delta|$ is positive, since $\frac{m+1}{|p|}>m$ and the exponent with a negative exponent is less than 1. The right-hand side is positive due to the inequality (15), which implies

$$-(a+1)\frac{1-e^{-mnbt_0}}{1-e^{-bt_0}}-\frac{1}{b}-c>-(a+1)\frac{1-e^{-mnbt_0}}{1-e^{-bt_0}}+a+\frac{a+1}{1-e^{-bt_0}}=\frac{(a+1)e^{-mnbt_0}}{1-e^{-bt_0}}+a>0.$$

Thus, we have verified the correctness of finding Δ from equation (46) and can write

$$|\Delta| = \frac{h_1(1-e^{-mnbt_0})+nT_0\dfrac{1-e^{-(m+1)nbt_0}}{1-e^{-nbt_0}}-(a+1)\dfrac{1-e^{-mnbt_0}}{1-e^{-bt_0}}-\dfrac{1}{b}-c}{\dfrac{m+1}{|p|}-m+\dfrac{1-e^{-mnbt_0}}{1-e^{-nbt_0}}}.$$

Now it remains to check the validity of the constraints (22). Let $|p|=1$, then $\frac{m+1}{|p|}-m=1$. Applying inequalities (15), (21) and (16), we obtain the required.

Theorem 2 is proved.

Conclusion

We have introduced a system (4) that models a fully connected chain of $m+1$ neurons with a synaptic delay connection. For it, the theorem 3 on the coexistence of $(m+1)!$ periodic solutions in the form of discrete traveling waves with a bursting effect is proved. To this end, for the auxiliary equation (12) with m ordered delays $h_1,\dots,h_m$, the theorem 1 on the existence of a special periodic regime is proved. After that, theorem 2 on the solvability of the period equation is proved.

Note that the constraint (16) in the theorem 1 is artificial. The value h_1 falls on a certain period of the function $x_0(t)$, the number of which is designated by n. In this case, four cases of mutual arrangement of points (25), (26) and h_1 are possible:

(I) $(n-1)T_0<h_1<(n-1)T_0+1$;
(II) $(n-1)T_0+1<h_1<(n-1)T_0+t_0$;
(III) $(n-1)T_0+t_0<h_1<(n-1)T_0+t_0+1$;
(IV) $(n-1)T_0+t_0+1<h_1<nT_0$.

For definiteness, we have considered in detail the case (IV), for which the theorem 1 is proved. In other cases, similar statements can be proved, in which the form of the solution is slightly different from (19), but the general idea of alternating intervals with positive and negative values is preserved, which ensures the presence of a bursting effect in the function $u(t)$. Thus, the bursting effect can be preserved when h_1 varies over a wider range.

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