Nonlinear Waves. Solitons. Autowaves. Self-Organization

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Solitary deformation waves in two coaxial shells made of material with combined nonlinearity and forming the walls of annular and circular cross-section channels filled with viscous fluid

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Abstract. The *aim* of the paper is to obtain a system of nonlinear evolution equations for two coaxial cylindrical shells containing viscous fluid between them and in the inner shell, as well as numerical modeling of the propagation processes for nonlinear solitary longitudinal strain waves in these shells. The case when the stress-strain coupling law for the shell material has a hardening combined nonlinearity in the form of a function with fractional exponent and a quadratic function is considered. *Methods.* To formulate the problem of shell hydroelasticity, the Lagrangian–Eulerian approach for recording the equations of dynamics and boundary conditions is used. The multiscale perturbation method is applied to analyze the formulated problem. As a result of asymptotic analysis, a system of two evolution equations, which are generalized Schamel–Korteweg–de Vries equations, is obtained, and it is shown that, in general, the system requires numerical investigation. The new difference scheme obtained using the Gröbner basis technique is proposed to discretize the system of evolution equations. *Results.* The exact solution of the system of evolution equations for the special case of no fluid in the inner shell is found. Numerical modeling has shown that in the absence of fluid in the inner shell, the solitary deformation waves have supersonic velocity. In addition, for the above case, it was found that the strain waves in the shells retain their velocity and amplitude after interaction, i.e., they are solitons. On the other hand, calculations have shown that in the presence of a viscous fluid in the inner shell, attenuation of strain solitons is observed, and their propagation velocity becomes subsonic.

Keywords: solitary deformation waves, coaxial shells, viscous fluid, combined nonlinearity, computational experiment.

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Introduction

The studies of the deformation waves' propagation in elastic structures involve the formulation and solution of wave dynamics problems for such models of elastic elements as a rod, a plate, and a shell. Now, such problems in the linear formulation are well enough studied [1]. However, modern elastic structures can be made of materials with nonlinear physical properties, and work beyond the linear theory of elasticity, as well. Therefore, the problems of studying the evolution of deformation waves for nonlinear computational models of elastic elements and, in particular, the possibility of the formation of solitary nonlinear waves (solitons) in them that retain their velocity and shape are relevant [2]. In [3], the Korteweg–de Vries equation for a nonlinear-elastic rod was derived and the possibility of a solitary longitudinal strain wave (a strain soliton) in it was first substantiated. Later, the Korteweg–de Vries–Burgers equation for a viscoelastic rod and a plate describing the evolution of strain solitons in these elastic elements was obtained in [4]. The reviews of the main papers on theoretical and experimental studies of the evolution of solitary strain waves in nonlinear rods are given in [5, 6], and in the review [7], where, in addition, the studies of solitary strain waves in plates and shells are presented. Note that the studies on nonlinear wave dynamics of shells are much less than for rods and plates, and we present below a number of studies for shells that were not included in these reviews. The evolution equation describing the propagation of nonlinear longitudinal strain waves in a cylindrical Kirchhoff–Love type shell made of a linear viscoelastic material and operating under the condition of neglecting rotational inertia was obtained in [8]. It is shown that that equation is the Kadomtsev–Petviashvili–Burgers equation. In [9], the results of full-scale experiments on excitation, detection, and the study of the propagation of a volumetric strain soliton in a shell made of polymethyl methacrylate are presented. The authors also proposed an axisymmetric model of the evolution of volume longitudinal waves in a nonlinear-elastic cylindrical shell made of Murnaghan material. As a simplification in that model, torsion and bending are neglected. The numerical simulation of the strain soliton evolution in a nonlinearly elastic cylindrical shell with varying cross-section and physical properties of the material on the bases of the development of this model was carried out in [10]. In [11, 12] the evolution of axisymmetric longitudinal strain waves in a Kirchhoff–Love cylindrical shell surrounded by a generalized nonlinear-elastic medium, which in particular cases reduces to the Winkler, Pasternak, and Hetenyi models, is investigated. A nonlinear evolution equation of the sixth order modeling the propagation of these waves is obtained, and the physical realizability of its partial exact solutions in the form of periodic and solitary waves is discussed. The study of solitary strain waves in ribbed cylindrical shells made of incompressible material with physical softening nonlinearity, when the relationship between stress intensity and strain intensity is given in the form of a power law with a fractional exponent, is presented in [13, 14]. In [13], the authors use the structural anisotropy method to describe the presence of a system of orthogonal stiffeners rids at the ribreinforced shell, while in [14] they consider a shell with stiffeners ribs in the form of internal stringers, which are described as beams in contact with its skin. In the first of these papers [13], the evolutionary equation of longitudinal deformation waves was obtained for the shell, which is the generalized Schamel equation, and in the second one [14], an evolutionary equation generalizing the Schamel–Ostrovsky equation was derived. It is shown that these equations have partial exact solutions in the form of nonlinear solitary longitudinal deformation waves.

Cylindrical shells are used in various engineering structures, and in particular, in pipeline systems for transportation of fluids. Therefore, the problems of studying the interaction of shells with the liquid filling them are relevant and associated with the consideration of hydroelasticity problems. The first studies of wave processes in elastic shell-fluid systems were carried out in a linear formulation. The papers [15,16] should be noted among such investigations. Reference [15], the axisymmetric wave motion of incompressible fluid in a thin elastic cylindrical tube was investigated and the propagation velocity of the fluid pressure wave was determined for the case of considering the inertia of wall motion and the forces of viscous friction of the fluid. In [16] the wave pulsating motion of viscous incompressible fluid in a thin-walled elastic tube of circular cross-section was investigated in relation to the study of blood motion in vessels. The current state of the research in the field of hydroelasticity of cylindrical shells is given in [17–19]. It should be noted that most of the studies consider an ideal fluid and discuss problems of linear wave processes in shells. Below we present a number of works in which the nonlinear questions have been studied. In [20, 21] the propagation of nonlinear solitary waves in a geometrically nonlinear cylindrical shell filled with an ideal incompressible fluid was studied. The axisymmetric problem of hydroelasticity was formulated, and using its asymptotic analysis the Korteweg–de Vries equation for the shell deflection was obtained. The numerical examples of calculations of the evolution of solitary waves in a shell are presented for the following cases: absence of fluid in it, its complete filling with fluid, and the motion in it of a stationary fluid flow with constant velocity. In [22], an axisymmetric hydroelastic problem is formulated for a geometrically nonlinear cylindrical shell with structural damping, filled with a viscous incompressible fluid and surrounded by a generalized Vlasov–Leontiev medium, in which linear and cubic reactions to longitudinal displacement are presented. Using the perturbation method and considering the creeping motion of the fluid in the framework of the hydrodynamic theory of lubrication, an evolutionary integro-differential equation for longitudinal deformation waves in the shell generalizing the Korteweg–de Vries equation is obtained. The numerical solution of this equation was carried out, which allowed to evaluate the effects of fluid, structural damping and surrounding elastic medium on the evolution of nonlinear solitary strain waves. In [23, 24], longitudinal solitary strain waves in two coaxial shells with a viscous fluid between them are investigated. In [23], the inertia of viscous fluid motion in an annular gap is taken into account, and a system of two generalized modified Korteweg–de Vries–Burgers equations is obtained and numerically investigated for the shells with structural damping, whose material has a physical law with cubic nonlinearity and is surrounded by a Winkler elastic medium. In [24], the motion of a viscous fluid in an annular gap is considered as creeping. The authors obtained and numerically investigated the system of two generalized Schamel equations for cylindrical shells, the material of which has a law of dependence of stress on strain and strain intensity with a nonlinear term having a fractional exponent. This study is further developed in [25] by the case of filling the inner shell with viscous fluid and considering the inertia of fluid motion in the annular gap between the shells and the circular channel formed by the inner shell.

The proposed study is aimed at studying the evolution of solitary strain waves in two coaxial cylindrical shells, which material has a combined nonlinear physical law of stress-strain coupling, forming annular and circular channels filled with viscous incompressible fluid.

1. Derivation of the shell dynamics equations, the hydroelasticity problem formulation

Let us consider two cylindrical shells made of the same material and having a common longitudinal axis of symmetry. It is assumed that the entire space in the inner shell and between the shells is filled with viscous fluid (see Fig. 1). While studying the wave process in the shells, we will accept that they are infinitely long, i. e., exclude from consideration the reflection of waves from the shells' ends.

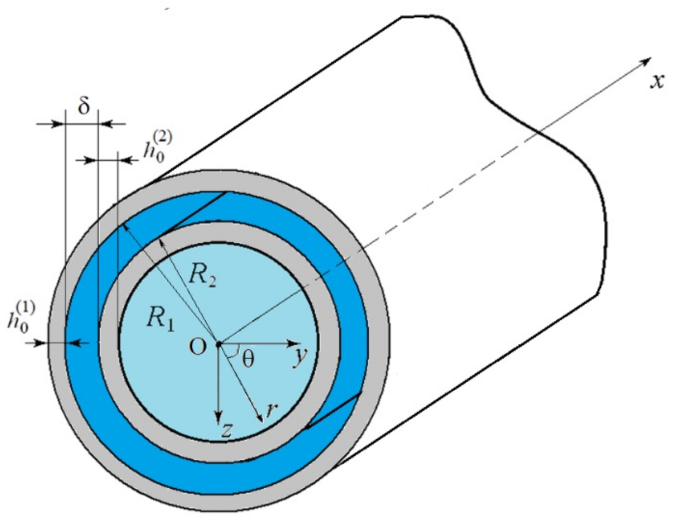

Fig. 1

Along the symmetry axis, we direct the x-axis of the Cartesian coordinate system xyz, the center of which is located at the point O of an arbitrary (initial) cross-section. The center of the cylindrical coordinate system $r\theta x$ is at the same point. Let us study an axisymmetric wave process the evolution of which occurs in the positive direction of the x-axis. The radius of the inner surface of the outer shell is R_1, and R_2 is the radius of the outer surface of the inner shell, then in the unperturbed state the gap between the shells is $\delta = R_1 - R_2$. The notation of thickness for the i-th shell as $h_0^{(i)}$, and $R^{(i)}$ for the radius of its middle surface is introduced, where $i = 1$ corresponds to the outer and $i = 2$ to the inner shell. Next, we will designate as the upper index i the parameters corresponding to the i-th shell.

Let us assume that the shells satisfy the Kirghoff–Love hypotheses and write the equations of their dynamics according to [26] considering the load on the shells from the side of the viscous fluid

$$\frac{\partial N_x^{(i)}}{\partial x} = \rho_0 h_0^{(i)} \frac{\partial^2 U^{(i)}}{\partial t^2} - \left(q_x^{(i)} + U^{(i)} \frac{\partial q_x^{(i)}}{\partial x} - W^{(i)} \frac{\partial q_x^{(i)}}{\partial r} + (i-1) \left(q_x^{cir} + U^{(i)} \frac{\partial q_x^{cir}}{\partial x} - W^{(i)} \frac{\partial q_x^{cir}}{\partial r} \right) \right) \Bigg|_{R^{(i)}},$$

$$\frac{\partial^2 M_x^{(i)}}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial W^{(i)}}{\partial x} N_x^{(i)} \right) + \frac{1}{R^{(i)}} N_\theta^{(i)} = \rho_0 h_0^{(i)} \frac{\partial^2 W^{(i)}}{\partial t^2} - \tag{1}$$

$$- \left((-1)^{i-1} \left(q_n + U^{(i)} \frac{\partial q_n}{\partial x} - W^{(i)} \frac{\partial q_n}{\partial r} \right) + (i-1) \left(q_n^{cir} + U^{(i)} \frac{\partial q_n^{cir}}{\partial x} - W^{(i)} \frac{\partial q_n^{cir}}{\partial r} \right) \right) \Bigg|_{R^{(i)}}, \quad i = 1, 2.$$

The following notations are adopted in equations (1): $M_x^{(i)}$ is the bending moment in the middle surface element of the i-th shell, $N_x^{(i)}$, $N_\theta^{(i)}$ are the normal forces, along the corresponding axes x and θ, of the middle surface element of the i-th shell, $W^{(i)}$ is the displacement of the middle surface element along the normal (deflection) of the i-th shell, the positive direction of which is taken to the center of curvature of the shell, $U^{(i)}$ — longitudinal displacement of the middle surface element of the i-th shell, $q_x^{(i)}$, q_n are the tangential and normal stresses of the viscous fluid between the shells, q_x^{cir}, q_n^{cir} are the tangential and normal stresses of the viscous fluid filling the inner shell, t is the time, ρ_0 is the density of the shell material.

Moment $M_x^{(i)}$ and normal forces $N_x^{(i)}$, $N_x^{(\theta)}$ are defined as [26]

$$M_x^{(i)} = \int_{-h_0^{(i)}/2}^{h_0^{(i)}/2} \sigma_x^{(i)} z dz, N_x^{(i)} = \int_{-h_0^{(i)}/2}^{h_0^{(i)}/2} \sigma_x^{(i)} dz, N_\theta^{(i)} = \int_{-h_0^{(i)}/2}^{h_0^{(i)}/2} \sigma_\theta^{(i)} dz, \tag{2}$$

where $\sigma_x^{(i)}$, $\sigma_\theta^{(i)}$ are the normal stresses, along the corresponding x and θ axes, in the element of the i-th shell, z is the local coordinate normal to the middle surface of the i-th shell.

To represent equations (1) in displacements, it is necessary to specify the physical law relating stresses and deformations in the shell material. In the case of physically nonlinear material, to approximate experimentally determined diagrams of its deformation, nonlinear power dependences are used: quadratic, cubic or fraction exponents, as well as their combinations [27]. For example, the case of the physical law with softening fractional exponent or combined softening quadratic and fractional exponent nonlinearity was studied in [13,14] for synthetic incompressible materials based on epoxy resins. Such an approximation allows to reflect the fact of limitation of stress growth with strain growth. In the proposed study, we use an approximation of a physical law with the hardening combined nonlinearity in the form of sum quadratic and fractional exponent function. This allows us to reflect the effect of material hardening, i.e. the presence of nonlinear stress growth with strain growth in stress-strain diagrams. For example, such behavior is characteristic of biotissues such as skin and blood vessels of animals and human circulatory systems [28]. According to the above, let us write down the relationship between the components of the stress tensor σ_x, σ_θ and the components of the strain tensor ε_x, ε_θ and strain intensity ε_u according to [29] as follows

$$\begin{aligned}
&\sigma_x^{(i)} = \frac{E}{1-\mu_0^2}\left[\left(\varepsilon_x^{(i)} + \mu_0 \varepsilon_\theta^{(i)}\right)\left\{1 + \frac{m}{E}\varepsilon_u^{(i)1/2} + \frac{m_2}{E}\varepsilon_u^{(i)}\right\}\right],\\
&\sigma_\theta^{(i)} = \frac{E}{1-\mu_0^2}\left[\left(\mu_0\varepsilon_x^{(i)} + \varepsilon_\theta^{(i)}\right)\left\{1 + \frac{m}{E}\varepsilon_u^{(i)1/2} + \frac{m_2}{E}\varepsilon_u^{(i)}\right\}\right],\\
&\varepsilon_u^{(i)} = \frac{\sqrt{3}}{1+\mu_0}\left[\mu_1\left(\varepsilon_x^{(i)^2} + \varepsilon_\theta^{(i)^2}\right) - \mu_2 \varepsilon_x^{(i)}\varepsilon_\theta^{(i)}\right]^{1/2},\\
&\mu_1 = \frac{1}{3}\left[1 + \frac{\mu_0}{(1-\mu_0)^2}\right], \quad \mu_2 = \frac{1}{3}\left[1 - \frac{2\mu_0}{(1-\mu_0)^2}\right].
\end{aligned} \tag{3}$$

In expressions (3) E is Young's modulus, μ_0 is Poisson's ratio of the shell material; m, m_2 are considered as positive material constants having the dimension of stresses and determined from tensile-compression experiments of nonlinear-elastic shell material [30]. The coefficients μ_1, μ_2 reflect the fact of compressibility of the material, which is characteristic for biotissues [28]. If we consider incompressible material, for example, synthetic materials based on epoxy resins, we can put $\mu_0 = 1/2$ (in this case $\mu_1 = -\mu_2 = 1$) and $m < 0$, $m_2 = 0$ or $m < 0$, $m_2 < 0$ similarly [13, 14].

We assume that the deformations and elastic displacements of the i-th shell are related to each other as [25]

$$\varepsilon_x^{(i)} = \frac{\partial U^{(i)}}{\partial x} - z\frac{\partial^2 W^{(i)}}{\partial x^2}, \quad \varepsilon_\theta^{(i)} = -\frac{W^{(i)}}{R^{(i)}} - z\frac{W^{(i)}}{R^{(i)2}}, \quad -h_0^{(i)}/2 \leqslant z \leqslant h_0^{(i)}/2. \tag{4}$$

Let us substitute (3), (4) into (2), and after that into (1), bearing in mind [25], where the validity of considering the intensity of deformations on the shell's middle surface, i. e. $z = 0$, was shown. As a result, we obtain the equations of dynamics of the considered coaxial shells, which material has a physical law with combined fractional-quadratic nonlinearity, written in displacements

$$\frac{Eh_0^{(i)}}{1-\mu_0^2}\frac{\partial}{\partial x}\left\langle\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}+\left[\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}\right]\left\{\frac{m}{E}\left(\frac{\sqrt{3}}{1+\mu_0}\right)^{1/2}\left[\mu_1\left(\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right)+\right.\right.\right.$$
$$\left.\left.\left.+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{1/4}+\frac{m_2}{E}\frac{\sqrt{3}}{1+\mu_0}\left[\mu_1\left[\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right]+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{1/2}\right\}\right\rangle=$$
$$=\rho_0h_0^{(i)}\frac{\partial^2U^{(i)}}{\partial t^2}-\left(q_x^{(i)}+U^{(i)}\frac{\partial q_x^{(i)}}{\partial x}-W^{(i)}\frac{\partial q_x^{(i)}}{\partial r}+(i-1)\left(q_x^{cir}+U^{(i)}\frac{\partial q_x^{cir}}{\partial x}-W^{(i)}\frac{\partial q_x^{cir}}{\partial r}\right)\right)\Bigg|_{R^{(i)}},$$
$$\frac{Eh_0^{(i)}}{12\left(1-\mu_0^2\right)}\frac{\partial^2}{\partial x^2}\left\langle-\frac{h_0^{(i)2}}{12}\left(\frac{\partial^2W^{(i)}}{\partial x^2}+\mu_0\frac{W^{(i)}}{R^{(i)2}}\right)\right\rangle+\frac{Eh_0}{1-\mu_0^2}\frac{\partial}{\partial x}\left\langle\frac{\partial W^{(i)}}{\partial x}\left[\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}+\right.\right.$$
$$\left.\left.+\left[\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}\right]\left\{\frac{m}{E}\left(\frac{\sqrt{3}}{1+\mu_0}\right)^{1/2}\left[\mu_1\left(\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right)+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{1/4}+\right.\right.\right.$$
$$\left.\left.\left.+\frac{m_2}{E}\frac{\sqrt{3}}{1+\mu_0}\left[\mu_1\left[\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right]+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{1/2}\right\}\right]\right\rangle+\frac{Eh_0^{(i)}}{1-\mu_0^2}\frac{1}{R^{(i)}}\left\langle\mu_0\frac{\partial U^{(i)}}{\partial x}-\frac{W^{(i)}}{R^{(i)}}+\right.$$
$$+\left[\mu_0\frac{\partial U^{(i)}}{\partial x}-\frac{W^{(i)}}{R^{(i)}}\right]\left\{\frac{m}{E}\left(\frac{\sqrt{3}}{1+\mu_0}\right)^{1/2}\left[\mu_1\left(\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R}\right)^2\right)+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R}\right]^{1/4}+\right.$$
$$\left.\left.+\frac{m_2\sqrt{3}}{E(1+\mu_0)}\left[\mu_1\left[\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right]+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{1/2}\right\}\right\rangle=\rho_0h_0^{(i)}\frac{\partial^2W^{(i)}}{\partial t^2}-$$
$$-\left((-1)^{i-1}\left(q_n+U^{(i)}\frac{\partial q_n}{\partial x}-W^{(i)}\frac{\partial q_n}{\partial r}\right)+(i-1)\left(q_n^{cir}+U^{(i)}\frac{\partial q_n^{cir}}{\partial x}-W^{(i)}\frac{\partial q_n^{cir}}{\partial r}\right)\right)\Bigg|_{R^{(i)}},\quad i=1,2. \tag{5}$$

Note that in (1), (5) the right-hand sides, i. e., the load on the shells, represent the tangential and normal stresses of a viscous incompressible fluid written in the Lagrangian–Eulerian approach [31]. The load is carried on the undisturbed middle surfaces of the shells, as is common in hydroelasticity problems [32]. The expressions for $q_x^{(i)}$, q_n and q_x^{cir}, q_n^{cir} on the unperturbed middle surfaces of the shells are written in the Euler approach as [25]

$$q_x^{(i)}=-\rho\nu\left(\frac{\partial V_x}{\partial r}+\frac{\partial V_r}{\partial x}\right)\ \text{ at }\ r=R_\nu^{(i)},\quad q_n^{(i)}=-p+2\rho\nu\frac{\partial V_r}{\partial r}\ \text{ at }\ r=R_\nu^{(i)}. \tag{6}$$

Here we have in mind that that expressions for q_x^{cir}, q_n^{cir} coincide with (6) at $i=2$. In the case the physical properties of the fluid in the annular gap and the inner shell are different, then assuming in (6) $i=2$, we also denote the density and kinematic viscosity of the fluid in the inner shell as ρ_c and ν_c.

To determine (6) together with (5) it is necessary to consider the equations of dynamics of a viscous incompressible fluid between the shells and in the inner shell, which for the axisymmetric case have the form [33]

$$\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+V_x\frac{\partial V_r}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial r}=\nu\left(\frac{\partial^2 V_r}{\partial r^2}+\frac{1}{r}\frac{\partial V_r}{\partial r}+\frac{\partial^2 V_r}{\partial x^2}-\frac{V_r}{r^2}\right)$$
$$\frac{\partial V_x}{\partial t}+V_r\frac{\partial V_x}{\partial r}+V_x\frac{\partial V_x}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\left(\frac{\partial^2 V_x}{\partial r^2}+\frac{1}{r}\frac{\partial V_x}{\partial r}+\frac{\partial^2 V_x}{\partial x^2}\right),\quad \frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{\partial V_x}{\partial x}=0. \tag{7}$$

where V_x, V_r are the projections of the fluid velocity on the axes of the cylindrical coordinate system, p is the pressure in the fluid, ρ is the fluid density, ν is the kinematic viscosity coefficient.

Let us supplement (7) with boundary no-slip conditions at the shell's surfaces for the fluid between the shells (annular cross-section channel) and in the inner shell (circular cross-section channel). For the channel of annular cross-section, these conditions have the following form

$$V_x+U^{(i)}\frac{\partial V_x}{\partial x}-W^{(i)}\frac{\partial V_x}{\partial r}=\frac{\partial U^{(i)}}{\partial t},$$
$$V_r+U^{(i)}\frac{\partial V_r}{\partial x}-W^{(i)}\frac{\partial V_r}{\partial r}=-\frac{\partial W^{(i)}}{\partial t}\ \text{at}\ r=R_i-W^{(i)},\quad i=1,2. \tag{8}$$

For the channel of circular cross-section, we use (8) at $r=\left(R_i-h_0^{(i)}\right)-W^{(i)}$ and $i=2$. In addition, we use the conditions for the velocity components at the symmetry axis of the viscous fluid in the inner shell, which are justified and formulated in [25] by Mogilevich L. I. in the following form

$$rV_r=0,\quad r\frac{\partial V_x}{\partial r}=0\ \text{ at }\ r=0. \tag{9}$$

2. Asymptotic analysis of the hydroelasticity problem, system of evolution equations

Considering the wave process in shells, we assume that the following relations take place

$$\frac{h_0^{(i)}}{R^{(i)}}=\varepsilon\ll 1,\quad \frac{R^{(i)2}}{l^2}=O\left(\varepsilon^{1/2}\right),\quad \frac{w_m}{h_0^{(i)}}=O(1),$$
$$\frac{u_m}{l}\frac{R^{(i)}}{h_0^{(i)}}=O(1),\quad \frac{m}{E}=O(1),\quad \frac{m_2}{E}=O\left(\varepsilon^{-1/2}\right), \tag{10}$$

and use dimensionless variables

$$W^{(i)}=w_m u_3^{(i)},\quad U^{(i)}=u_m u_1^{(i)},\quad x^*=x/l,\quad t^*=tc_0/l,\quad r^*=r/R^{(i)}, \tag{11}$$

where $c_0=\sqrt{E/(\rho_0(1-\mu_0^2))}$ is the sound speed in the shell material, l is the wavelength taken as a characteristic linear scale, u_m, w_m are characteristic scales of elastic displacements of shells, ε is a small parameter of the problem.

Let us analyze equations (5) by perturbation method [34] considering asymptotic expansions of elastic displacement functions of shells

$$u_1^{(i)}=u_{10}^{(i)}+\varepsilon^{1/2}u_{11}^{(i)}+\ldots,u_3^{(i)}=u_{30}^{(i)}+\varepsilon^{1/2}u_{31}^{(i)}+\ldots \tag{12}$$

and introducing new independent variables ξ and τ

$$\xi=x^*-\sqrt{1-\mu_0^2}t^*,\quad \tau=\varepsilon^{1/2}t^*. \tag{13}$$

Writing (5) in dimensionless form taking into account (10)–(13) and restricting to the first term in (12) we have the system (zero approximation by ε)

$$\frac{\partial}{\partial\xi}\left\langle\frac{\partial u_{10}^{(i)}}{\partial\xi}-\mu_0u_{30}\right\rangle=\left(1-\mu_0^2\right)\frac{\partial^2u_{10}^{(i)}}{\partial\xi^2},\quad \mu_0\frac{\partial u_{10}^{(i)}}{\partial\xi}=u_{30}^{(i)},\quad i=1,2 \tag{14}$$

and restricting by the first two terms in (12) and bearing in mind the second equation (14), we obtain the system (first approximation by ε)

$$\frac{\partial}{\partial\xi}\left\langle\mu_0\left(\mu_0\frac{\partial u_{11}^{(i)}}{\partial\xi}-u_{31}^{(i)}\right)+\frac{m}{E}\left(\frac{\sqrt{3}}{1+\mu_0}\right)^{1/2}\left(1-\mu_0^2\right)\left(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2\right)^{1/4}\left(\frac{\partial u_{10}^{(i)}}{\partial\xi}\right)^{3/2}+\right.$$

$$\left.+\frac{m_2}{E}\varepsilon^{1/2}\frac{\sqrt{3}}{1+\mu_0}\left(1-\mu_0^2\right)\left(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2\right)^{1/2}\left(\frac{\partial u_{10}^{(i)}}{\partial\xi}\right)^2\right\rangle+2\sqrt{1-\mu_0^2}\frac{\partial^2u_{10}^{(i)}}{\partial\xi\partial\tau}=$$

$$=-\frac{l}{\varepsilon^{3/2}\rho_0h_0^{(i)}c_0^2}\left(q_x^{(i)}+(i-1)q_x^{cir}\right)\Bigg|_{R^{(i)}}, \tag{15}$$

$$\mu_0\frac{\partial u_{11}^{(i)}}{\partial\xi}-u_{31}^{(i)}-\mu_0\left(1-\mu_0^2\right)\frac{1}{\varepsilon^{1/2}}\left(\frac{R^{(i)}}{l}\right)^2\frac{\partial^3u_{10}^{(i)}}{\partial\xi^3}=$$

$$=-\frac{R^{(i)}}{\varepsilon^{3/2}\rho_0h_0^{(i)}c_0^2}\left((-1)^{i-1}q_n^{(i)}+(i-1)q_n^{cir}\right)\Bigg|_{R^{(i)}},\quad i=1,2.$$

Let us consider the system (14) and substitute the deflection from the second equation into the first one, resulting in the identity. Hence, the longitudinal displacement is an arbitrary function. In addition, we note that the first term of the expansion (12) corresponds to a linear wave process evolving at the sound speed in the shell material. The consideration of the second term allows to obtain the additive due to a nonlinear wave process.

Considering the system (15) we exclude from it u_{11}, u_{31} as a result we obtain

$$\frac{\partial^2u_{10}^{(i)}}{\partial\xi\partial\tau}+\frac{m}{E}\left(\frac{\sqrt{3}}{1+\mu_0}\right)^{1/2}\frac{3}{4}\sqrt{1-\mu_0^2}\left(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2\right)^{1/4}\left|\frac{\partial u_{10}^{(i)}}{\partial\xi}\right|^{1/2}\frac{\partial^2u_{10}^{(i)}}{\partial\xi^2}+$$

$$+\frac{m_2}{E}\varepsilon^{1/2}\frac{\sqrt{3}}{1+\mu_0}\sqrt{1-\mu_0^2}\left(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2\right)^{1/2}\frac{\partial u_{10}^{(i)}}{\partial\xi}\frac{\partial^2u_{10}^{(i)}}{\partial\xi^2}+\frac{\mu_0^2\sqrt{1-\mu_0^2}}{2}\frac{\partial^4u_{10}^{(i)}}{\partial\xi^4}=$$

$$=-\frac{1}{2\sqrt{1-\mu_0^2}}\frac{l}{\varepsilon^{3/2}\rho_0h_0^{(i)}c_0^2}\left[q_x^{(i)}+(i-1)q_x^{cir}-\mu_0\varepsilon^{1/4}\frac{\partial\left((-1)^{i-1}q_n+(i-1)q_n^{cir}\right)}{\partial\xi}\right]_{R^{(i)}},\quad i=1,2. \tag{16}$$

The equations of system (16) are the generalizations of the Schamel–Korteweg–de Vries equation for longitudinal deformation $\partial u_{10}^{(i)}/\partial\xi$. Note that the exclusion of the fluid between the shells and in the inner shell is equivalent to assuming $q_x^{(i)} = q_n = q_x^{cir} = q_n^{cir} = 0$. In this case, the system (16) decomposes into two independent Schamel–Korteweg–de Vries equations for the outer and inner shells.

To determine q_x^i, q_n, q_x^{cir}, q_n^{cir}, we analyze asymptotically the equations of fluid dynamics (7)–(9) between the shells and in the inner shell similarly [25]. For this purpose, for the fluid between the shells (annular channel) we introduce dimensionless variables of the following form

$$V_r = h_0^{(i)}\frac{c_0}{l}v_r, \quad V_x = h_0^{(i)}\frac{c_0}{\delta}v_x, \quad r^* = \frac{r - R^{(2)}}{\delta}, \quad t^* = \frac{c_0}{l}t, \quad x^* = \frac{x}{l}, \quad p = \frac{\rho\nu c_0 l h_0^{(i)}}{\delta^3}P, \quad (17)$$

and for the fluid in the inner shell (circular channel) we use the following dimensionless variables

$$V_r = h_0^{(i)}\frac{c_0}{l}v_r, \quad V_x = h_0^{(i)}\frac{c_0}{R^{(2)}}v_x, \quad r^* = \frac{r}{R^{(2)}}, \quad t^* = \frac{c_0}{l}t, \quad x^* = \frac{1}{l}x, \quad p = \frac{\rho_c\nu_c c_0 l h_0^{(i)}}{R^{(2)3}}P. \quad (18)$$

We assume that in the considered formulation for the annular channel the following relations take place

$$\psi = \frac{\delta}{R^{(2)}} = \varepsilon^{1/2}, \quad \lambda = \frac{h_0^{(i)}}{\delta} = \varepsilon^{1/2}, \quad \frac{h_0^{(i)}}{R^{(i)}} = \varepsilon, \quad \frac{h_0^{(i)}}{l} = \varepsilon^{5/4}, \quad \frac{\delta}{l} = \varepsilon^{3/4}, \quad (19)$$

and for the channel of circular cross-section we suppose

$$\frac{R^{(2)}}{l} = \psi_c = O\left(\varepsilon^{1/4}\right), \quad \lambda_c = \frac{h_0^{(i)}}{R^{(2)}} = \varepsilon. \quad (20)$$

Then passing in (6)–(9) to dimensionless variables (17) or (18) taking into account (19) or (20), for channels of the corresponding cross-section, we consider the following asymptotic expansions

$$P = P^0 + \varepsilon^{1/2}P^1 + ..., \quad v_r = v_r^0 + \varepsilon^{1/2}v_r^1 + ..., \quad v_x = v_x^0 + \varepsilon^{1/2}v_x^1 + ... \quad (21)$$

Restricting in (21) by the first term, we obtain linearized problems of viscous fluid dynamics in the corresponding channel.

For the channel of annular cross-section, the hydrodynamic equations will take the form of

$$\frac{\partial P^0}{\partial r^*} = 0, \quad \frac{\delta}{l}\frac{\delta c_0}{\nu}\frac{\partial v_x^0}{\partial t^*} + \frac{\partial P^0}{\partial x^*} = \frac{\partial^2 v_x^0}{\partial r^{*2}}, \quad \frac{\partial v_r^0}{\partial r^*} + \frac{\partial v_x^0}{\partial x^*} = 0, \quad (22)$$

with boundary conditions

$$v_r^0 = -\frac{\partial u_3^{(1)}}{\partial t^*}, \quad v_x^0 = 0 \ \text{ at } \ r^* = 1, \quad v_r^0 = -\frac{\partial u_3^{(2)}}{\partial t^*}, \quad v_x^0 = 0 \ \text{ at } \ r^* = 0. \quad (23)$$

For the channel of circular cross-section the hydrodynamic equations will be written as

$$\frac{\partial P^0}{\partial r^*} = 0, \quad \psi_c\frac{R_3 c_0}{\nu_c}\frac{\partial v_x^0}{\partial t^*} + \frac{\partial P^0}{\partial x^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial v_x^0}{\partial r^*}\right), \quad \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* v_r^0\right) + \frac{\partial v_x^0}{\partial x^*} = 0, \quad (24)$$

with boundary conditions

$$r^* v_r^0 = r^*\frac{\partial v_x^0}{\partial r^*} = 0 \ \text{ at } \ r^* = 0, \quad v_r^0 = -\frac{\partial u_3^{(2)}}{\partial t^*}, \quad v_x^0 = \frac{\partial u_1^{(2)}}{\partial t^*} \ \text{ at } \ r^* = 1. \quad (25)$$

The expressions for $q_x^{(i)}$, q_n, q_x^{cir}, q_n^{cir} will be written as

$$q_x^{(i)} = -\rho\nu \frac{h_0^{(i)} c_0}{\delta^2} \frac{\partial v_x^0}{\partial r^*},$$

$$q_n = -\frac{\rho\nu c_0 l h_0^{(i)}}{\delta^3} P^0 \quad \text{at} \quad r^* = 1 \quad (\text{for} \quad i = 1\ ) \quad \text{or} \quad (\text{for } i = 2\ ) \quad \text{at} \quad r^* = 0, \tag{26}$$

$$q_x^{cir} = -\lambda_c \frac{\nu_c}{R^{(2)} c_0} \rho_c c_0^2 \frac{\partial v_x^0}{\partial r^*}, \quad q_n^{cir} = -\frac{\lambda_c}{\psi_c} \frac{\nu_c}{R^{(2)} c_0} \rho_c c_0^2 P^0 \text{ at } r^* = 1. \tag{27}$$

The solution of problems (22)–(25) by the iteration method was carried out in [25] and the pressure P^0 and velocity gradients $\partial v_x^0/\partial r^*$ were determined. The expressions for these quantities in the annular channel are as follows

$$P^0 = \int \left[12\sqrt{1-\mu_0^2}\left(u_{30}^{(1)} - u_{30}^{(2)}\right) - \frac{6}{5}\,\text{Re}\,(1-\mu_0^2)\left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi}\right)\right] d\xi,$$

$$\frac{\partial v_x^0}{\partial r^*} = (2r^*-1)\left[6\sqrt{1-\mu_0^2}\left(u_{30}^{(1)} - u_{30}^{(2)}\right) - \frac{\text{Re}}{10}\,(1-\mu_0^2)\left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi}\right)\right], \quad \text{Re} = \frac{\delta}{l}\frac{\delta c_0}{\nu} \tag{28}$$

consequently

$$\frac{\partial P^0}{\partial \xi} = \left[12\sqrt{1-\mu_0^2}\left(u_{30}^{(1)} - u_{30}^{(2)}\right) - \frac{6}{5}\,\text{Re}\,(1-\mu_0^2)\left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi}\right)\right],$$

$$\left.\frac{\partial v_x^0}{\partial r^*}\right|_{r^*=1} = 6\sqrt{1-\mu_0^2}\left(u_{30}^{(1)} - u_{30}^{(2)}\right) - \frac{\text{Re}}{10}\,(1-\mu_0^2)\left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi}\right), \tag{29}$$

$$\left.\frac{\partial v_x^0}{\partial r^*}\right|_{r^*=0} = -\left.\frac{\partial v_x^0}{\partial r^*}\right|_{r^*=1}.$$

For a channel of circular cross-section in [25] it is defined that

$$\frac{\partial P^0}{\partial \xi} = \sqrt{1-\mu_0^2}\left[8\left(2u_{30}^{(2)} - \frac{\partial u_{10}^{(2)}}{\partial \xi}\right) - \frac{1}{3}\psi_c \frac{R^{(2)} c_0}{\nu_c}\left(8\frac{\partial u_{30}^{(2)}}{\partial \xi} - \frac{\partial^2 u_{10}^{(2)}}{\partial \xi^2}\right)\sqrt{1-\mu_0^2}\right],$$

$$\left.\frac{\partial v_x}{\partial r^*}\right|_{r^*=1} = \sqrt{1-\mu_0^2}\left[4\left(2u_{30}^{(2)} - \frac{\partial u_{10}^{(2)}}{\partial \xi}\right) - \frac{1}{6}\psi_c \frac{R^{(2)} c_0}{\nu_c}\left(2\frac{\partial u_{30}^{(2)}}{\partial \xi} - \frac{\partial^2 u_{10}^{(2)}}{\partial \xi^2}\right)\sqrt{1-\mu_0^2}\right]. \tag{30}$$

Then, using $u_{30}^{(i)} = \mu_0 \partial u_{10}^{(i)}/\partial \xi$ (the second equation of (14)), (26)–(30), considering the smallness of parameters ψ, λ, ψ_c, λ_c and assuming $R^{(1)} = R^{(2)} \approx R$, $h_0^{(1)} = h_0^{(2)} \approx h_0$,we determine the right parts of the equations system (16) and obtain the system of evolutionary equations of the following form

$$\frac{\partial^2 u_{10}^{(1)}}{\partial\xi\partial\tau}+\frac{m}{E}\frac{3}{4}\sqrt{1-\mu_0^2}\left(\frac{\sqrt{3}}{1+\mu_0}\right)^{1/2}(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2)^{1/4}\left|\frac{\partial u_{10}^{(1)}}{\partial\xi}\right|^{1/2}\frac{\partial^2 u_{10}^{(1)}}{\partial\xi^2}+$$
$$+\frac{m_2\sqrt{3}}{E(1+\mu_0)}\varepsilon^{1/2}\sqrt{1-\mu_0^2}\,(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2)^{1/2}\frac{\partial u_{10}^{(1)}}{\partial\xi}\frac{\partial^2 u_{10}^{(1)}}{\partial\xi^2}+\frac{\mu_0^2\sqrt{1-\mu_0^2}}{2}\frac{\partial^4 u_{10}^{(1)}}{\partial\xi^4}=$$
$$=-6\mu_0^2\frac{\rho l}{\rho_0 h_0}\frac{\nu}{Rc_0\varepsilon^{1/2}}\left(\frac{R}{\delta}\right)^3\left[\left(\frac{\partial u_{10}^{(1)}}{\partial\xi}-\frac{\partial u_{10}^{(2)}}{\partial\xi}\right)\left(1-\frac{1}{2}\frac{\delta}{\mu_0 R}\right)-\right.$$
$$\left.-\frac{1}{10}\,\mathrm{Re}\sqrt{1-\mu_0^2}\left(\frac{\partial^2 u_{10}^{(1)}}{\partial\xi^2}-\frac{\partial^2 u_{10}^{(2)}}{\partial\xi^2}\right)\left(1-\frac{1}{12}\frac{\delta}{\mu_0 R}\right)\right]. \tag{31}$$

$$\frac{\partial^2 u_{10}^{(2)}}{\partial\xi\partial\tau}+\frac{m}{E}\frac{3}{4}\sqrt{1-\mu_0^2}\left(\frac{\sqrt{3}}{1+\mu_0}\right)^{1/2}(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2)^{1/4}\left|\frac{\partial u_{10}^{(2)}}{\partial\xi}\right|^{1/2}\frac{\partial^2 u_{10}^{(2)}}{\partial\xi^2}+$$
$$+\frac{m_2\sqrt{3}}{E(1+\mu_0)}\varepsilon^{1/2}\sqrt{1-\mu_0^2}\,(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2)^{1/2}\frac{\partial u_{10}^{(2)}}{\partial\xi}\frac{\partial^2 u_{10}^{(2)}}{\partial\xi^2}+\frac{\mu_0^2\sqrt{1-\mu_0^2}}{2}\frac{\partial^4 u_{10}^{(2)}}{\partial\xi^4}=$$
$$=-6\mu_0^2\frac{\rho l}{\rho_0 h_0}\frac{\nu}{Rc_0\varepsilon^{1/2}}\left(\frac{R}{\delta}\right)^3\left[\left(\frac{\partial u_{10}^{(2)}}{\partial\xi}-\frac{\partial u_{10}^{(1)}}{\partial\xi}\right)\left(1-\frac{1}{2}\frac{\delta}{\mu_0 R}\right)-\right.$$
$$\left.-\frac{1}{10}\,\mathrm{Re}\sqrt{1-\mu_0^2}\left(\frac{\partial^2 u_{10}^{(2)}}{\partial\xi^2}-\frac{\partial^2 u_{10}^{(1)}}{\partial\xi^2}\right)\left(1-\frac{1}{12}\frac{\delta}{\mu_0 R}\right)\right]-$$
$$-\frac{1}{2\sqrt{1-\mu_0^2}}\frac{l}{\varepsilon^{1/2}\rho_0 h_0 c_0^2}\left\{\frac{\nu_c}{Rc_0}\rho_c c_0^2 4\sqrt{1-\mu_0^2}\,[1-2\mu_0]^2\frac{\partial u_{10}^{(2)}}{\partial\xi}-\right.$$
$$\left.-\frac{R}{l}\rho_c c_0^2\frac{1}{6}\left(1-\mu_0^2\right)\left[(1-2\mu_0)^2+12\mu_0^2\right]\frac{\partial^2 u_{10}^{(2)}}{\partial\xi^2}\right\}.$$

Note that the obtained system, in the case of exclusion of the fluid from consideration, i. e., when the right-hand sides are equal to zero, and when $m_2 = 0$, decomposes into two independent Schamel equations. These equations for the case of incompressible material, when $\mu_0 = 1/2$, a $\mu_1 = -\mu_2 = 1$, and $m < 0$ coincide with the equation obtained in [14], for the shell with internal stringers and skin made of incompressible material with softening fractional physical nonlinearity, with the height of stringers equal to zero.

3. Numerical modeling of the solitary strain waves evolution in coaxial shells

Let us represent the system of evolution equations (31) in the form of

$$\varphi_t^{(1)}+6\alpha_0\left|\varphi^{(1)}\right|^{1/2}\varphi_\eta^{(1)}+6\alpha_1\varphi^{(1)}\varphi_\eta^{(1)}+\varphi_{\eta\eta\eta}^{(1)}+\sigma_0\left(\varphi^{(1)}-\varphi^{(2)}\right)-\sigma_1\left(\varphi_\eta^{(1)}-\varphi_\eta^{(2)}\right)=0,$$
$$\varphi_t^{(2)}+6\alpha_0\left|\varphi^{(2)}\right|^{1/2}\varphi_\eta^{(2)}+6\alpha_1\varphi^{(2)}\varphi_\eta^{2}+\varphi_{\eta\eta\eta}^{(2)}+ \tag{32}$$
$$+\sigma_0\left(\varphi^{(2)}-\varphi^{(1)}\right)-\sigma_1\left(\varphi_\eta^{(2)}-\varphi_\eta^{(1)}\right)+\sigma_2\varphi^{(2)}-\sigma_3\varphi_\eta^{(2)}=0$$

by introducing the following notations

$$\frac{\partial u_{10}^{(1)}}{\partial \xi} = c_3 \varphi^{(1)}, \ \frac{\partial u_{10}^{(2)}}{\partial \xi} = c_3 \varphi^{(2)}, \ \eta = c_1 \xi, \ t = c_2 \tau, \ \sigma_0 = 6\mu_0^2 \frac{\rho l}{\rho_0 h_0} \left(\frac{R}{\delta}\right)^2 \frac{\nu}{\delta c_0} \frac{1}{\varepsilon^{1/2}} \left(1 - \frac{\delta}{2\mu_0 R}\right) \frac{1}{c_2},$$

$$\sigma_1 = 6\mu_0^2 \frac{\rho \delta}{\rho_0 h_0} \left(\frac{R}{\delta}\right)^2 \frac{1}{\varepsilon^{1/2}} \frac{\sqrt{1-\mu_0^2}}{10} \left(1 - \frac{\delta}{12\mu_0 R}\right) \frac{c_1}{c_2}, \quad \sigma_2 = \frac{\rho_c l}{\rho_0 h_0} \frac{\nu_c}{\varepsilon^{1/2} R c_0} 2 \left(1 - 2\mu_0\right)^2 \frac{1}{c_2},$$

$$\sigma_3 = \frac{\rho_c R}{\rho_0 h_0} \frac{1}{\varepsilon^{1/2}} \frac{\sqrt{1-\mu_0^2}}{12} \left[(1-2\mu_0)^2 + 12\mu_0^2\right] \frac{c_1}{c_2} \tag{33}$$

where

$$c_3 = \left[\frac{3}{4} \frac{m}{m_2 \varepsilon^{1/2}} \frac{1}{\left(\sqrt{3}/(1+\mu_0)\right)^{1/2} \left(\mu_1 + \mu_2\mu_0 + \mu_1\mu_0^2\right)^{1/4}}\right]^2,$$

$$c_1 = \left[\frac{c_3}{3\mu_0^2} \frac{m_2}{E} \varepsilon^{1/2} \frac{\sqrt{3}}{1+\mu_0} \left(\mu_1 + \mu_2\mu_0 + \mu_1\mu_0^2\right)^{1/2}\right]^{1/2},$$

$$c_2 = \left[\frac{c_3 c_1}{6} \frac{m_2}{E} \varepsilon^{1/2} \sqrt{1-\mu_0^2} \frac{\sqrt{3}}{1+\mu_0} \left(\mu_1 + \mu_2\mu_0 + \mu_1\mu_0^2\right)^{1/2}\right].$$

The subscript letters in the system (32) denotes the corresponding partial derivative, and the system describes the evolution of longitudinal nonlinear deformation waves in the considered shells. If we put $\alpha_0 = 0$, we pass to the system of generalized Korteweg–de Vries equations, at $\alpha_1 = 0$ we obtain the system of generalized Schamel equations, and at $\alpha_0 = \alpha_1 = 1$ we obtain the system of generalized Schamel–Korteweg–de Vries equations.

In the general case, system (32) has no exact solution and requires numerical solution. However, we note that in the special case when the fluid in the inner shell is excluded from consideration, i. e., when $\sigma_2 = \sigma_3 = 0$, the system of equations (32) has an exact solution in the form of a solitary wave

$$\varphi^{(1)}(t,\eta) = \varphi^{(2)}(t,\eta) = \frac{25}{4} k^4 \left(\alpha_0 + \sqrt{\alpha_0^2 + \frac{25}{8} k^2 \alpha_1 \operatorname{ch}(k(\eta - 4k^2 t))}\right)^{-2}. \tag{34}$$

In this solution, k is the wave number, which is arbitrary. The above exact solution can be used as an initial condition in the numerical solution of the system of evolution equations (32) by assuming $t = 0$ in (34) and taking $\alpha_0 = \alpha_1 = 1$. In this approach, the following options can be considered:

- at the initial moment of time, the solitary wave with the same wave number is excited in each of the shells

$$\varphi^{(1)}(0,\eta) = \varphi^{(2)}(0,\eta) = \frac{25}{4} k^4 \left(1 + \sqrt{1 + \frac{25}{8} k^2 \operatorname{ch}(k\eta)}\right)^{-2}; \tag{35}$$

- at the initial moment of time the solitary wave is excited only in the outer shell

$$\varphi^{(1)}(0,\eta) = \frac{25}{4} k^4 \left(1 + \sqrt{1 + \frac{25}{8} k^2 \operatorname{ch}(k\eta)}\right)^{-2}, \quad \varphi^{(2)}(0,\eta) = 0. \tag{36}$$

In addition, we can consider the excitation at the initial time moment of two waves with different wave numbers, i.e., with different velocities and amplitudes, in each of the shells. In this case, the initial conditions for the first solitary wave in the outer and inner shells are given as

$$\varphi^{(1)}(0,\eta) = \varphi^{(2)}(0,\eta) = \frac{25}{4}k_1^4\left(1+\sqrt{1+\frac{25}{8}k_1^2\mathrm{ch}(k_1\eta)}\right)^{-2}, \tag{37}$$

and for the second wave in the outer and inner shells we set in the form

$$\varphi^{(1)}(0,\eta) = \varphi^{(2)}(0,\eta) = \frac{25}{4}k_2^4\left(1+\sqrt{1+\frac{25}{8}k_2^2\mathrm{ch}(k_2\eta)}\right)^{-2}. \tag{38}$$

Here k_1, k_2 are the wave numbers corresponding to the first and second solitary waves excited in each of the shells.

To realize the numerical solution for the system of nonlinear evolution equations (32), we used the approach of generating new difference schemes for discretization of partial derivative equations using the Gröbner basis technique [35, 36]. The sequence of obtaining the difference scheme, checking its adequacy and stability is similar to [25], and the obtained new difference scheme for the system of generalized Schamel–Korteweg–de Vries equations (32), i. e., when $\alpha_0 = \alpha_1 = 1$, has the following form

$$\frac{u^{(1)}{}_j^{n+1} - u^{(1)}{}_j^{n}}{\tau} + 4\frac{(u^{(1)3/2}{}_{j+1}^{n+1} - u^{(1)3/2}{}_{j-1}^{n+1}) + (u^{(1)3/2}{}_{j+1}^{n} - u^{(1)3/2}{}_{j-1}^{n})}{4h} +$$
$$+3\frac{(u^{(1)2}{}_{j+1}^{n+1} - u^{(1)2}{}_{j-1}^{n+1}) + (u^{(1)2}{}_{j+1}^{n} - u^{(1)2}{}_{j-1}^{n})}{4h} + \frac{(u^{(1)}{}_{j+2}^{n+1} - 2u^{(1)}{}_{j+1}^{n+1} + 2u^{(1)}{}_{j-1}^{n+1} - u^{(1)}{}_{j-2}^{n+1})}{4h^3} +$$
$$+\frac{(u^{(1)}{}_{j+2}^{n} - 2u^{(1)}{}_{j+1}^{n} + 2u^{(1)}{}_{j-1}^{n} - u^{(1)}{}_{j-2}^{n})}{4h^3} + \sigma_0\left(\frac{u^{(1)}{}_j^{n+1} + u^{(1)}{}_j^{n}}{2} - \frac{u^{(2)}{}_j^{n+1} + u^{(2)}{}_j^{n}}{2}\right) -$$
$$-\sigma_1\left(\frac{(u^{(1)}{}_{j+1}^{n+1} - u^{(1)}{}_{j-1}^{n+1}) + (u^{(1)}{}_{j+1}^{n} - u^{(1)}{}_{j-1}^{n})}{4h} - \frac{(u^{(2)}{}_{j+1}^{n+1} - u^{(2)}{}_{j-1}^{n+1}) + (u^{(2)}{}_{j+1}^{n} - u^{(2)}{}_{j-1}^{n})}{4h}\right) = 0, \tag{39}$$

$$\frac{u^{(2)}{}_j^{n+1} - u^{(2)}{}_j^{n}}{\tau} + 4\frac{(u^{(2)3/2}{}_{j+1}^{n+1} - u^{(2)3/2}{}_{j-1}^{n+1}) + (u^{(2)3/2}{}_{j+1}^{n} - u^{(2)3/2}{}_{j-1}^{n})}{4h} +$$
$$+3\frac{(u^{(2)2}{}_{j+1}^{n+1} - u^{(2)2}{}_{j-1}^{n+1}) + (u^{(2)2}{}_{j+1}^{n} - u^{(2)2}{}_{j-1}^{n})}{4h} + \frac{(u^{(2)}{}_{j+2}^{n+1} - 2u^{(2)}{}_{j+1}^{n+1} + 2u^{(2)}{}_{j-1}^{n+1} - u^{(2)}{}_{j-2}^{n+1})}{4h^3} +$$
$$+\frac{(u^{(2)}{}_{j+2}^{n} - 2u^{(2)}{}_{j+1}^{n} + 2u^{(2)}{}_{j-1}^{n} - u^{(2)}{}_{j-2}^{n})}{4h^3} + \sigma_0\left(\frac{u^{(2)}{}_j^{n+1} + u^{(2)}{}_j^{n}}{2} - \frac{u^{(1)}{}_j^{n+1} + u^{(1)}{}_j^{n}}{2}\right) -$$
$$-\sigma_1\left(\frac{(u^{(2)}{}_{j+1}^{n+1} - u^{(2)}{}_{j-1}^{n+1}) + (u^{(2)}{}_{j+1}^{n} - u^{(2)}{}_{j-1}^{n})}{4h} - \frac{(u^{(1)}{}_{j+1}^{n+1} - u^{(1)}{}_{j-1}^{n+1}) + (u^{(1)}{}_{j+1}^{n} - u^{(1)}{}_{j-1}^{n})}{4h}\right) +$$
$$+\sigma_2\frac{u^{(2)}{}_j^{n+1} + u^{(2)}{}_j^{n}}{2} - \sigma_3\frac{(u^{(2)}{}_{j+1}^{n+1} - u^{(2)}{}_{j-1}^{n+1}) + (u^{(2)}{}_{j+1}^{n} - u^{(2)}{}_{j-1}^{n})}{4h} = 0.$$

Here we denote the grid mesh steps by $\tau = t_{n+1} - t_n, h = \eta_{j+1} - \eta_j$ and introduce the grid functions $u_j^{(1)^n} = \varphi^{(1)}(t_n, \eta_j), u_j^{(2)^n} = \varphi^{(2)}(t_n, \eta_j)$, where $\varphi^{(1)}(t_n, \eta_j), \varphi^{(2)}(t_n, \eta_j)$ are the grid values of the functions $\varphi^{(1)}(t, \eta), \varphi^{(2)}(t, \eta)$.

The software implementation of the difference scheme (39) requires linearization of the nonlinear grid power functions with exponent 3/2 and 2 for the next time layer. To implement this procedure, the following computational relations are proposed

$$\nu_{k+1}^{3/2} = \nu_{k+1}^{3/2} - \nu_k^{3/2} + \nu_k^{3/2} = \left(\nu_{k+1}^{1/2} - \nu_k^{1/2}\right)\left(\nu_{k+1} + \nu_{k+1}^{1/2}\nu_k^{1/2} + \nu_k\right) + \nu_k^{3/2} =$$

$$= \left(\nu_{k+1}^{1/2} - \nu_k^{1/2}\right)\left(\nu_{k+1}^{1/2} + \nu_k^{1/2}\right)\frac{\left(\nu_{k+1} + \nu_{k+1}^{1/2}\nu_k^{1/2} + \nu_k\right)}{\nu_{k+1}^{1/2} + \nu_k^{1/2}} + \nu_k^{3/2} \approx$$

$$\approx \left(\nu_{k+1} - \nu_k\right)\frac{3}{2}\nu_k^{1/2} + \nu_k^{3/2} = \frac{3}{2}\nu_k^{1/2}\nu_{k+1} - \frac{1}{2}\nu_k^{3/2},$$

$$\nu_{k+1}^2 = \nu_{k+1}^2 - \nu_k^2 + \nu_k^2 = \left(\nu_{k+1} - \nu_k\right)\left(\nu_{k+1} + \nu_k\right) + \nu_k^2 \approx \left(\nu_{k+1} - \nu_k\right)2\nu_k + \nu_k^2 = 2\nu_k\nu_{k+1} - \nu_k^2. \quad (40)$$

Using the difference scheme (39) with linearization by (40) the algorithm of numerical solution in Python programming language with the help of SciPy package (htt://scipy.org) has been implemented. The computational experiments on modeling the processes of propagation of solitary waves in the considered shells under the initial conditions of the form (35)–(38) was carried out. In the course of modeling, the following cases were considered: filling with fluid only the annular channel between the shells (equivalent to assuming $\sigma_2 = \sigma_3 = 0$); filling with viscous fluid the annular channel between the shells and the inner shell. In addition, we considered the case of incompressible shells material when the inner shell and the annular channel between the shells are filled with viscous fluid. To do this, in (33) we took $\mu_0 = 1/2$ and obtained, $\sigma_2 = 0$, and also assumed that $\sigma_3 = 0.4$.

The results of the calculations are shown in Figs. 2–7, namely:

- the evolution of solitary strain waves in the shells for the case $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = \sigma_3 = 0$ (absence of fluid in the inner shell) when a wave of the form (35) with $k = 0.2$ is excited at the initial moment of time in each of the shells (see Fig. 2);
- the evolution of solitary strain waves in the shells for the case $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = \sigma_3 = 0$ (absence of fluid in the inner shell) when a wave of the form (36) with $k = 0.2$ is excited at the initial moment of time in the outer shell (see Fig. 3);

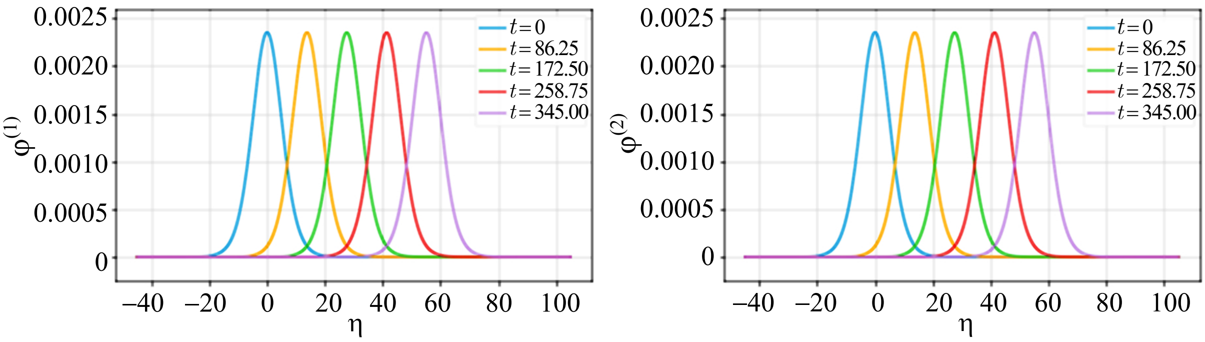

Fig. 2. The results of numerical solution of the system (32) at $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = \sigma_3 = 0$ with initial conditions (35) with wave number $k = 0.2$ (color online)

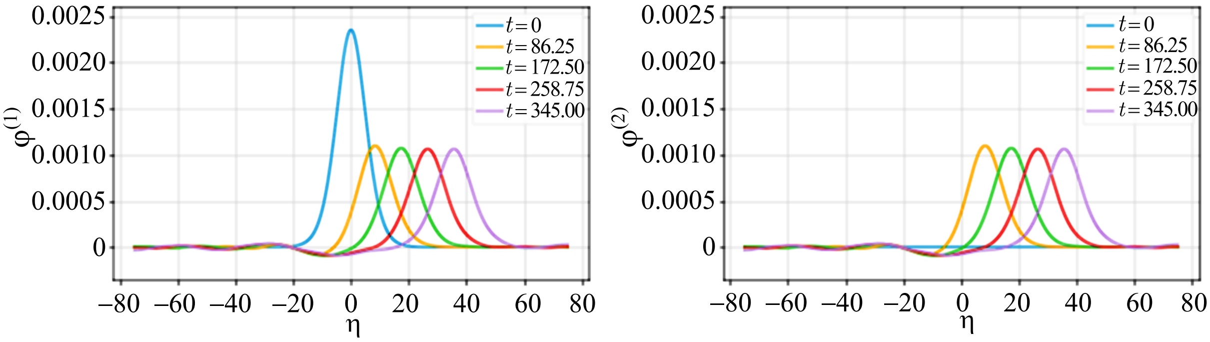

Fig. 3. The results of numerical solution of the system (32) at $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = \sigma_3 = 0$ with initial conditions (36) with wave number $k = 0.2$ (color online)

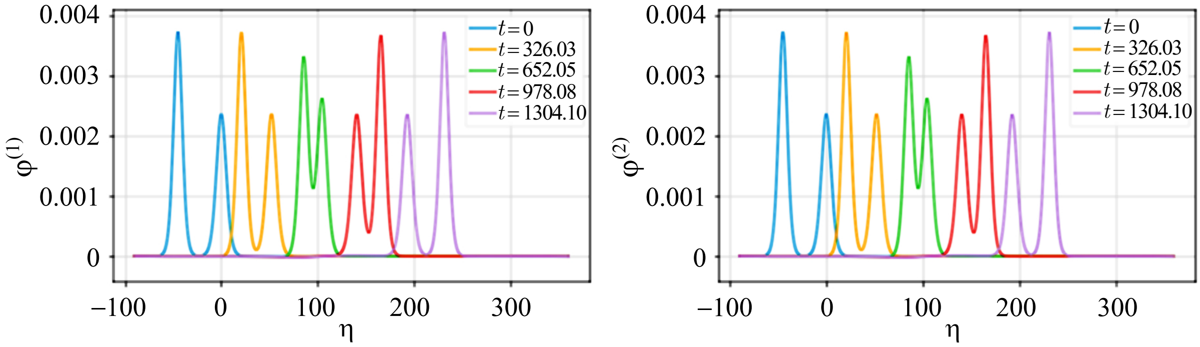

Fig. 4. The results of numerical solution of the system (32) at $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = \sigma_3 = 0$ with initial conditions of the form (37), (38): condition (37) with wave number $k = 0.225$ and initial value $\eta = -50$, condition (38) with wave number $k = 0.2$ and initial value $\eta = 0$ (color online)

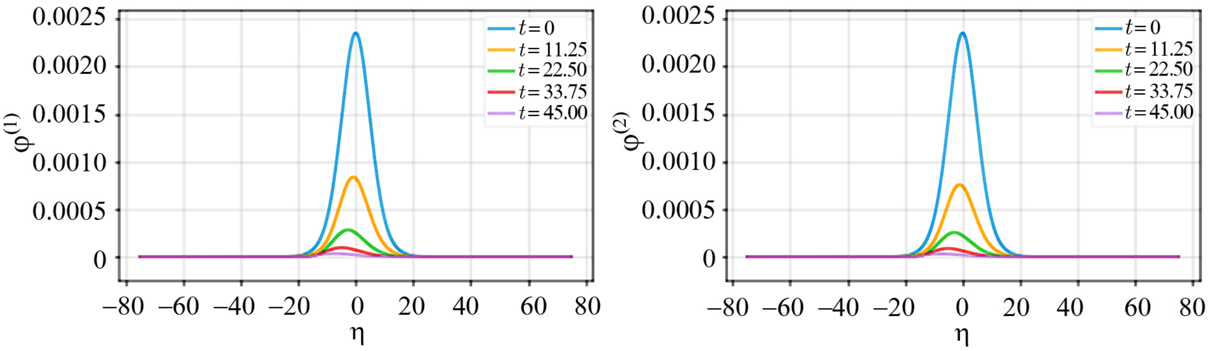

Fig. 5. The results of numerical solution of the system (32) at $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.4$ with initial conditions (35) with wave number $k = 0.2$ (color online)

- the evolution of solitary deformation waves with different amplitudes and velocities in each shell for the case $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = \sigma_3 = 0$ (no fluid in the inner shell) when two waves of the form (37), (38) are excited at the initial moment of time in each shell, (the first wave (37) with $k_1 = 0.225$ and initial value of the spatial variable $\eta = -50$, and the second wave (38) with $k_2 = 0.2$ and initial $\eta = 0$) (see Fig. 4);
- the evolution of solitary strain waves in the shells for the case $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.4$ (presence of viscous fluid in the annular gap and in the inner shell) when a wave

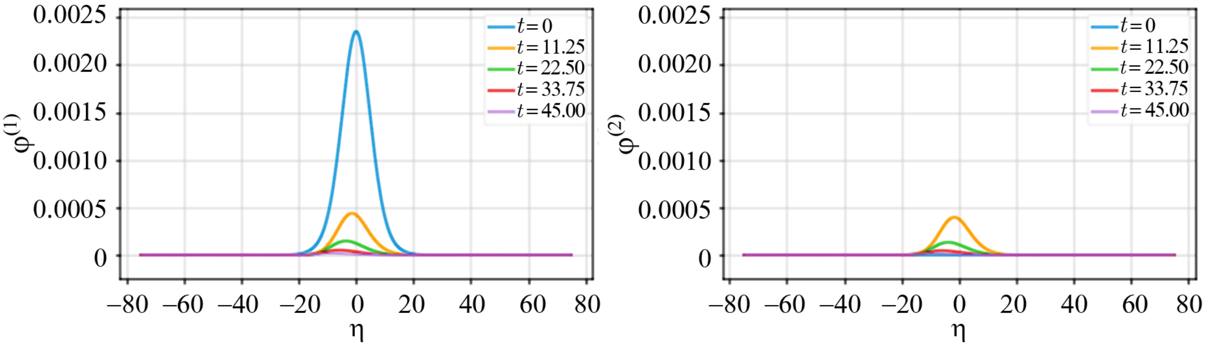

Fig. 6. The results of numerical solution of the system (32) at $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.4$ with initial conditions (36) with wave number $k = 0.2$ (color online)

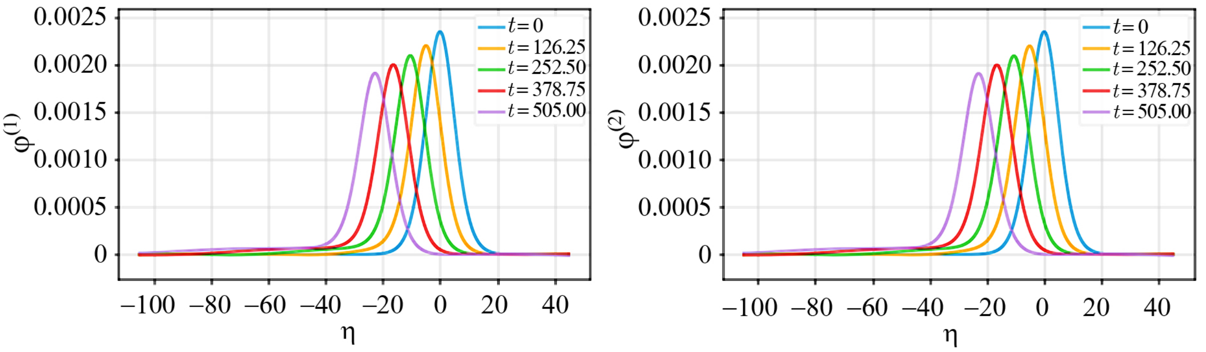

Fig. 7. The results of numerical solution of the system (32) at $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0$, $\sigma_3 = 0.4$ with initial conditions (35) with wave number $k = 0.2$ (color online)

of the form (35) with $k = 0.2$ is excited at the initial moment of time in each of the shells (see Fig. 5);

- the evolution of solitary deformation waves in the shells for the case $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.4$ (presence of viscous fluid in the annular gap and in the inner shell), when a wave of the form (36) with $k = 0.2$ is excited at the initial moment of time in the outer shell (see Fig. 6);
- the evolution of solitary strain waves in the shells for the case $\sigma_0 = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0$, $\sigma_3 = 0.4$ (shells of incompressible material, presence of viscous fluid in the annular gap and in the inner shell) when a wave of the form (35) with $k = 0.2$ is excited at the initial moment of time in each of the shells (see Fig. 7).

Summary and Conclusion

The calculations presented in Fig. 2–Fig. 4 show that for the cases when there is no fluid in the inner shell, the waves move to the right, i. e., the next term in (12), corresponding to the nonlinear wave process, is positive. Consequently, the propagation of solitary waves occurs at supersonic speed. The analysis of the curves in Fig. 2 indicates that the evolution of solitary strain waves in the shells occurs with constant velocity and amplitude. The calculations presented in Fig. 3 demonstrate that when a solitary strain wave is excited at the initial moment of time only in the outer shell, the wave is excited in the inner shell with the passage of time. At the initial stage, this process is accompanied by the drop in the amplitude of the solitary wave in the outer shell and the increase in the amplitude of the excited solitary wave in the inner shell. In the course of time, two waves of practically the same amplitude and velocity are observed in the shells. This indicates the energy transfer from the outer shell to the inner one through the viscous fluid filling the annular channel. The results of calculations in Fig. 4 allow us to conclude that two solitary waves with different speed and amplitude excited at the initial moment of time in each of the shells interact with each other during evolution. After the interaction, the waves keep their shape and speed, i.e., they interact as particles. This behavior indicates that in the considered cases, the solitary strain waves in the shells are supersonic solitons.

The calculations presented in Fig. 5 and Fig. 6 show that the presence of fluid in the inner shell significantly changes the evolution of the wave process, namely, there is a change in the direction of motion of solitary strain waves — they move to the left. This direction of motion indicates that the propagation of nonlinear strain waves occurs with subsonic velocity. In addition, in the considered cases, the drop in amplitude and velocity of solitary strain waves in the shells within a short time interval is observed compared to the calculations presented in Fig. 2–4. The evolution of the wave process at initial excitation of the solitary strain wave with the same wave number in each of the shells (Fig. 5) is accompanied by the intense drop in the amplitude and velocity of the waves and, eventually, by a rapid collapse of the strain solitons. For the case when at the initial time instant a solitary wave is excited only in the outer shell (Fig. 6), at the initial time step the excitation of a solitary wave in the inner shell is observed. This process is accompanied by the drop in the amplitude of the wave in the outer shell and the increase in the amplitude of the wave in the inner shell.If the shell material is incompressible (Fig. 7), then the movement of deformation waves to the left is observed. Consequently, the propagation of solitary waves occurs at subsonic speed. However, the attenuation of the deformation solitons persists, since the amplitude of the solitons in the shells decreases over time. This indicates energy transfer from the outer shell to the inner one through the viscous fluid in the annular channel. However, then, there is an intense drop in the amplitude of the deformation wave in both the outer and inner shells with subsequent collapse of the strain solitons in them. The results obtained suggest that the presence of viscous fluid in the inner shell leads to attenuation of strain solitons in the shells.

Summarizing the presented study, we note that in this paper we formulated the problem of hydroelasticity of two coaxial cylindrical shells made of material with the hardening combined quadratic-fractional nonlinearity. The system of evolution equations including two generalized Schamel–Korteweg–de Vries equations describing the nonlinear wave process in the shells is obtained on the bases on the asymptotic analysis of this issue. The new difference scheme using the Gröbner basis technique is derived to discretize the obtained system. The computational experiments allowed to evaluate the influence of viscous incompressible fluid between the shells and in the inner one on the evolution of nonlinear solitary strain waves in the shells. The results obtained in this work can be used as a fundamental basis for further development of methods of wave diagnostics of the state of pipelines filled with viscous fluid or vessels of the blood system of animals and humans.

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