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## On non-conservative perturbations of three-dimensional integrable systems

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**Abstract.** At present, non-conservative perturbations of two-dimensional nonlinear Hamiltonian systems have been studied quite fully. The *purpose* of the study is to generalize this theory to the three-dimensional case when the unperturbed system is nonlinear, integrable and has a region filled with closed phase trajectories. In this paper, autonomous perturbations are considered and the main attention is paid to the problem of limit cycles. *Methods.* The study is based on the construction of special coordinates in which the variables are divided into two slow and one fast, and in the first approximation with respect to a small parameter the equations for the slow variables are separated. *Results.* It is shown that hyperbolic equilibrium states of a truncated system determine closed phase trajectories, in the vicinity of which cycles appear under the perturbation. *Conclusion.* Thus, the problem is reduced to the study of solutions of the “generating” system of two algebraic or transcendental equations, similar to the generating Poincaré–Pontryagin equation for two-dimensional systems. As examples, we consider a three-dimensional van der Pol type system and the Lorentz system in the case of large Rayleigh numbers.

**Keywords:** averaging, limit cycles, three-dimensional systems, the generating function.

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### Introduction

The theory of nonlinear resonance in Hamiltonian and nearly Hamiltonian systems is an important section of oscillation theory. At present, non-conservative perturbations of two-dimensional Hamiltonian systems have been studied most thoroughly. For systems with  $3/2$  degrees of freedom, the foundations of this theory were laid by A. D. Morozov and L. P. Shilnikov in their work [1]. Further development was continued in the works of A. D. Morozov and his students: resonances of various types (including degenerate ones), questions related to synchronization of oscillations in the resonance zone and basic examples were studied (see the monograph [2] and references therein). In the works [3–7] the results on systems with  $3/2$  degrees

of freedom are generalized to the case of quasiperiodic perturbations. As the studies [1–7] show, when considering resonances in systems close to Hamiltonian, the problem of limit cycles in autonomous systems of this type plays an important role. This problem was solved by L. S. Pontryagin in the note [8], where sufficient conditions for the birth of a limit cycle from a closed phase trajectory of an unperturbed system are given.

Of interest is the generalization of the theory of non-conservative perturbations to the three-dimensional case when the unperturbed system is nonlinear, integrable and has a domain filled with closed phase trajectories. In this paper, we will turn to autonomous systems and discuss the application of the averaging principle to solving the problem of limit cycles. Generalizing the technique associated with the construction of the Poincaré–Pontryagin generating equation (see [8], also [9]), we obtain a «generating» system of two algebraic or transcendental equations, simple solutions of which determine the limit cycles of the original system<sup>1</sup>. As in the two-dimensional case, the problem of limit cycles plays an important role in the study of non-autonomous (periodic or quasiperiodic) perturbations of three-dimensional integrable systems.

Consider a system

$$\begin{aligned}\dot{x} &= \frac{\partial H(x, y, z)}{\partial y} + \varepsilon f_1(x, y, z), \\ \dot{y} &= -\frac{\partial H(x, y, z)}{\partial x} + \varepsilon f_2(x, y, z), \\ \dot{z} &= \varepsilon f_3(x, y, z),\end{aligned}\tag{1}$$

where  $0 < \varepsilon \ll 1$  is a parameter,  $H, f_i$  ( $i = \overline{1, 3}$ ) are sufficiently smooth (e.g., analytic) functions in some domain  $G \subset \mathbb{R}^3$ . Suppose that the perturbation is non-conservative:

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \neq 0, \quad (x, y, z) \in G.$$

At  $\varepsilon = 0$  system (1) is integrable. The first integrals  $z = c$ ,  $H(x, y, z) = h$  define two one-parameter families of surfaces in  $\mathbb{R}^3$ . Suppose that for each  $c \in (c^-, c^+)$  and each  $h \in (h^-, h^+)$  the surfaces of these families intersect along closed phase trajectories<sup>2</sup> and denote by  $D$  the domain (solid torus) consisting of these trajectories. Let  $D \subset G$  and, in addition,  $D$  be separated from separatrix surfaces and equilibria. The main goal of the paper is to analyze the behavior of solutions of system (1) in the domain  $D$ .

Three-dimensional integrable systems of general form

$$\begin{aligned}\dot{x} &= P(x, y, z), \\ \dot{y} &= Q(x, y, z), \\ \dot{z} &= R(x, y, z),\end{aligned}$$

under certain conditions can be reduced to the form of the system (1) at  $\varepsilon = 0$ . Thus, if  $H_1(x, y, z) = c_1$  is the first integral and  $\frac{\partial H_1}{\partial z} \neq 0$  in the neighborhood of a closed trajectory, then, making the substitution  $\tilde{z} = H_1(x, y, z)$ , we arrive in this neighborhood at the system

$$\begin{aligned}\dot{x} &= \tilde{P}(x, y, \tilde{z}), \\ \dot{y} &= \tilde{Q}(x, y, \tilde{z}), \\ \dot{\tilde{z}} &= 0.\end{aligned}$$

<sup>1</sup>For a direct application of the Poincaré small parameter method for the analysis of limit cycles in some three-dimensional systems close to integrable, see [10].

<sup>2</sup>It follows from the implicit function theorem that if system (1) at  $\varepsilon = 0$  has at least one closed phase trajectory, then there also exists a two-parameter family of such trajectories.

For the corresponding  $\tilde{z} = \text{const}$ , the system of the first two equations has a domain filled with closed phase curves. In this domain it is conservative in the sense of the existence of an integral invariant, but does not necessarily have a Hamiltonian form [9]. However, by means of a smooth one-to-one change of variables this system can be reduced to such a form (see [11], p. 35, Theorem 6)<sup>3</sup>.

The paper is organized as follows. In Section 1, the averaged first-approximation system is derived and the relationship between its equilibrium states and the limit cycles of system (1) is substantiated. In Section 2, the presented method is applied to the analysis of limit cycles in two examples – a three-dimensional van der Pol system and a Lorenz system. The discussion and conclusions are given in the Conclusion.

## 1. Averaging. Limit Cycles

Let us turn to system (1) at  $\varepsilon = 0$ . Substituting  $z = c \in (c^-, c^+)$  into the first two equations, we arrive at the Hamiltonian system

$$\begin{aligned} \dot{x} &= \frac{\partial H(x, y, c)}{\partial y}, \\ \dot{y} &= -\frac{\partial H(x, y, c)}{\partial x}. \end{aligned} \quad (2)$$

By assumption, system (2) has an annular domain  $G_c$  filled with closed curves  $H(x, y, c) = h \in (h^-, h^+)$  and separated from separatrices and equilibria. In this region, we pass from the variables  $(x, y)$  to the «action–angle» variables  $(I, \theta)$  using the canonical transformation:

$$x = X(\theta, I, c), \quad y = Y(\theta, I, c). \quad (3)$$

In the variables  $(I, \theta)$ , the Hamiltonian  $\tilde{H} = \tilde{H}(I, c)$  does not depend on  $\theta$  and the equations (2) take the form

$$\begin{aligned} \dot{I} &= 0, \\ \dot{\theta} &= \omega(I, c). \end{aligned}$$

Here  $\omega(I, c) = \partial \tilde{H}(I, c) / \partial I$  is the frequency of oscillations on closed curves. For example, if  $H(x, y, c) = y^2/2 + U(x, c)$ , then  $\omega(I, c) = 2\pi/T(I, c)$ , where

$$T(I, c) = \sqrt{2} \int_{x_1(I, c)}^{x_2(I, c)} \frac{dx}{\sqrt{h(I) - U(x, c)}},$$

and  $x_1 < x_2$  are real roots of the equation  $h - U(x, c) = 0$ .

Now let  $\varepsilon \neq 0$ . In (1) we make the replacement

$$x = X(\theta, I, z), \quad y = Y(\theta, I, z),$$

where  $X, Y$  are the same functions as in (3). The system takes the form

$$\begin{aligned} \dot{I} &= \varepsilon B_1(\theta, I, z), \\ \dot{z} &= \varepsilon B_2(\theta, I, z), \\ \dot{\theta} &= \omega(I, z) + \varepsilon B_3(\theta, I, z), \end{aligned} \quad (4)$$

<sup>3</sup>See also the example in Section 2.2.

where

$$\begin{aligned} B_1(\theta, I, z) &= X'_\theta \tilde{f}_2 - Y'_\theta \tilde{f}_1 + \tilde{f}_3(X'_z Y'_\theta - X'_\theta Y'_z), \quad B_2(\theta, I, z) = \tilde{f}_3, \\ B_3(\theta, I, z) &= -X'_I \tilde{f}_2 + Y'_I \tilde{f}_1 + \tilde{f}_3(X'_I Y'_z - X'_z Y'_I), \\ \tilde{f}_i(\theta, I, z) &= f_i(X(\theta, I, z), Y(\theta, I, z), z), \quad i = 1, 2, 3. \end{aligned}$$

$B_s$  ( $s = 1, 2, 3$ ) are sufficiently smooth functions of their arguments,  $2\pi$ -periodic in  $\theta$ . The phase space of system (4) is the solid torus  $(I(h^-), I(h^+)) \times (c^-, c^+) \times S^1$ . Suppose that in the region under consideration the functions  $B_s$  and  $\omega$  are bounded together with their derivatives up to the second order. For each  $(I, z) \in (I(h^-), I(h^+)) \times (c^-, c^+)$  we expand the functions  $B_s$  into uniformly converging in  $\theta \in [0, 2\pi]$  Fourier series:

$$B_s(\theta, I, z) = \sum_{k=-\infty}^{\infty} B_{sk}(I, z) \exp(ik\theta), \quad s = 1, 2, 3.$$

Let us introduce new variables  $(w_1, w_2, v)$  into system (4) using the formulas

$$\begin{aligned} I &= w_1 + \frac{\varepsilon}{i\omega} \sum_{k \neq 0} \frac{1}{k} B_{1k} \exp(ikv), \\ z &= w_2 + \frac{\varepsilon}{i\omega} \sum_{k \neq 0} \frac{1}{k} B_{2k} \exp(ikv), \\ \theta &= v + \frac{\varepsilon}{i\omega} \sum_{k \neq 0} \frac{1}{k} B_{3k} \exp(ikv) - \frac{\varepsilon}{\omega^2} \sum_{k \neq 0} \frac{1}{k^2} (\omega'_I B_{1k} + \omega'_z B_{2k}) \exp(ikv), \end{aligned} \quad (5)$$

where  $\omega, \omega'_I, \omega'_z, B_{sk}$  are calculated at  $(w_1, w_2)$ . Note that in the domain  $D$  the inequality  $\omega \geq \omega_* > 0$  holds, therefore there is no «small denominator» in substitution (5) and the series converge uniformly in  $v$ . Moreover, it is easy to show that for sufficiently small  $\varepsilon$  this transformation is reversible. As a result, we arrive at the system

$$\begin{aligned} \dot{w}_1 &= \varepsilon B_{10}(w_1, w_2) + \varepsilon^2 S_1(v, w_1, w_2), \\ \dot{w}_2 &= \varepsilon B_{20}(w_1, w_2) + \varepsilon^2 S_2(v, w_1, w_2), \\ \dot{v} &= \omega(w_1, w_2) + \varepsilon B_{30}(w_1, w_2) + \varepsilon^2 S_3(v, w_1, w_2), \end{aligned} \quad (6)$$

where the functions  $S_i$  are sufficiently smooth and  $2\pi$ -periodic in  $v$ , and  $B_{s0}$  are determined by the formulas

$$B_{s0}(w_1, w_2) = \frac{1}{2\pi} \int_0^{2\pi} B_s(\theta, w_1, w_2) d\theta, \quad s = 1, 2, 3.$$

Thus, substitution (5) allows us to exclude the cyclic variable from the equations of motion (4) in the first approximation with respect to  $\varepsilon$ . If we now neglect terms of order  $\sim \varepsilon^2$  in system (6), we obtain a system in which the first two equations are separated:

$$\begin{aligned} \dot{u}_1 &= \varepsilon B_{10}(u_1, u_2), \\ \dot{u}_2 &= \varepsilon B_{20}(u_1, u_2). \end{aligned} \quad (7)$$

System (7) is obtained from system (4) by averaging the first two equations over the fast variable  $\theta$ , so we will call it the *averaged system* (of the first approximation). It is known

(see, for example, [12]) that for  $0 < t < 1/\varepsilon$  the difference between the averaged variables  $(u_1, u_2)$  and the original variables  $(I, z)$ , which coincide at  $t = 0$ , has the order  $\sim \varepsilon$ :  $|I - u_1| < M\varepsilon, |z - u_2| < M\varepsilon, M > 0$ . Indeed, the right-hand sides of system (7) and the first two equations of system (6) differ by a value of the order  $\sim \varepsilon^2$ . Over time  $t \sim 1/\varepsilon$ , the difference between the corresponding solutions becomes of the order  $\sim \varepsilon$ . Moreover, as follows from substitution (5), the difference between  $(I, z)$  and  $(w_1, w_2)$  is also of the order  $\sim \varepsilon$ , so the difference between  $(I, z)$  and  $(u_1, u_2)$  will be of the same order (rigorous estimates can be obtained using Gronwall's lemma, see [12, 13]). Now let system (7) have a hyperbolic equilibrium  $(u_{10}, u_{20})$ . Then system (6), if we discard terms of order  $\sim \varepsilon^2$ , has a hyperbolic periodic solution with frequency  $\omega(u_{10}, u_{20}) + \varepsilon B_{30}(u_{10}, u_{20})$ , which is structurally stable (see, for example, [14]). However, the additions of order  $\sim \varepsilon^2$  are not arbitrarily small with respect to the system where they are discarded and at  $\varepsilon = 0$  the system has no structurally stable solutions, so one cannot formally use the roughness property to justify the preservation of the periodic solution<sup>4</sup>. Nevertheless, this solution is preserved. Indeed, choosing in (6) the variable  $v$  as the new time, we arrive at a system in the standard form of the averaging method. Then, applying the first theorem of Bogolubov [13], we obtain that there exists a hyperbolic periodic solution with period  $2\pi$  in  $v$  (in  $t$  the period is  $2\pi/\omega_0$ ), where  $\omega_0 = \omega(u_{10}, u_{20}) + O(\varepsilon)$ . In the phase space of system (1) the image of the specified solution is the limit cycle  $L_\varepsilon$ , where  $L_\varepsilon \rightarrow L_0$  as  $\varepsilon \rightarrow 0$ , where  $L_0$  is the phase curve of the unperturbed system (system (1) at  $\varepsilon = 0$ ), determined by the values of the first integrals  $I = u_{10}, z = u_{20}$ . The stability of the cycle is determined by the stability of the equilibrium state. By analogy with the Poincaré–Pontryagin generating equation in the two-dimensional case, we will call the system of equations

$$B_{s0}(u_1, u_2) = 0, \quad s = 1, 2,$$

which determines the limit cycles of system (1) in the domain  $D$ , *generating*.

Thus, we arrive at the theorem.

**Theorem 1.** *Let the conditions be satisfied*

$$B_{s0}(u_{10}, u_{20}) = 0, \quad s = 1, 2, \tag{8}$$

$$\Delta \neq 0, \quad \sigma \neq 0, \tag{9}$$

where

$$\Delta = \frac{\partial(B_{01}, B_{02})}{\partial(u_1, u_2)}, \quad \sigma = \frac{\partial B_{10}}{\partial u_1} + \frac{\partial B_{20}}{\partial u_2} \quad \text{at } u_1 = u_{10}, u_2 = u_{20}.$$

Then there exists  $\varepsilon_* > 0$  such that for all  $0 < \varepsilon < \varepsilon_*$ : 1) system (1) has in the  $O(\varepsilon)$ -neighborhood of the unperturbed phase curve  $L_0$ , determined by the values of the first integrals  $I = u_{10}, z = u_{20}$ , a unique limit cycle  $L_\varepsilon$ ; 2)  $L_\varepsilon \rightarrow L_0$  as  $\varepsilon \rightarrow 0$ ; 3) the cycle  $L_\varepsilon$  is asymptotically (orbitally) stable if  $\sigma < 0, \Delta > 0$ , and unstable otherwise.

Indeed, conditions (8) mean that  $(u_{10}, u_{20})$  is an equilibrium state of the system (7). The characteristic equation is of the form

$$\lambda^2 - \sigma\lambda + \Delta = 0.$$

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<sup>4</sup>For illustration, we can give a simple one-dimensional example:  $\dot{x} = x^2 + \varepsilon x + \varepsilon^2$ . If we discard the term of order  $\sim \varepsilon^2$ , there are two hyperbolic equilibria — stable and unstable (the system is rough), but when this term is taken into account, there are no equilibria.

Conditions (9) mean that the roots of this equation have non-zero real parts and the equilibrium  $(u_{10}, u_{20})$  is hyperbolic. From this we obtain statements 1 and 2. Applying the Rauss–Hurwitz criterion, we are convinced of 3.

Under the conditions of the theorem, the first approximation of the limit cycle is the unperturbed phase curve  $x = X(\omega(u_{10}, u_{20})t, u_{10}, u_{20}), y = Y(\omega(u_{10}, u_{20})t, u_{10}, u_{20}), z = u_{20}$ . If we substitute  $w_1 = u_{10}, w_2 = u_{20}, v = (\omega(u_{10}, u_{20}) + \varepsilon B_{30}(u_{10}, u_{20}))t$  into formulas (5), then we obtain the so-called «improved» first approximation satisfying system (4) up to terms of order  $\sim \varepsilon^2$ . By means of a suitable change of coordinates one can eliminate the cyclic variable up from the equations of motion to any fixed order in  $\varepsilon$  and obtain an “averaged” system of higher approximation, but these changes are cumbersome, so in practice one usually restricts oneself to an averaged system of the first or second<sup>5</sup> approximation.

Before proceeding to examples, we note that the averaged system (7) itself can have a limit cycle. It can be shown that if this cycle is hyperbolic, then it corresponds to a two-dimensional invariant torus in system (1) whose stability coincides with the stability of the cycle (see Bogolubov’s second theorem [13], see also [15]). A sufficient condition for the absence of an invariant torus in system (1) is the constancy of sign of its divergence.

## 2. Examples

**2.1. Van der Pol type system.** Consider the system

$$\begin{aligned} \ddot{x} + (1 + z^2)x &= \varepsilon(\alpha - x^2)\dot{x}, \\ \dot{z} &= \varepsilon f(x, \dot{x}, z), \end{aligned} \tag{10}$$

where  $0 < \varepsilon \ll 1, \alpha$  are parameters. This system is an example of a system with automatic regulation,  $z$  is a control parameter and the function  $f$  describes the feedback. For example, let  $f = x^2 - \beta\dot{x}^2$ , where  $\beta$  is a parameter. Rewrite (10) as a system of three first-order equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(1 + z^2)x + \varepsilon(\alpha - x^2)y, \\ \dot{z} &= \varepsilon(x^2 - \beta y^2). \end{aligned} \tag{11}$$

At  $\varepsilon = 0$ , system (11) has two first integrals:  $z = c, H(x, y, z) = y^2/2 + (1 + z^2)x^2/2 = h$ . Surfaces  $z = c$  and  $H(x, y, z) = h, h > 0$  intersect along closed trajectories in the form of an ellipse  $y^2/(2h(1 + c^2)) + x^2/(2h) = 1$ , the  $z$  axis is filled with non-isolated equilibria. Note that if  $\varepsilon = 0$  and  $z = c$  is fixed, then first two equations of system (11) form a linear oscillator with the natural frequency  $\omega(c) = \sqrt{c^2 + 1}$ , but the three-dimensional system (11) is nonlinear even for  $\varepsilon = 0$ .

Now let  $\varepsilon \neq 0$ . Let’s make a substitution:

$$\begin{aligned} x &= X(\theta, I, z) \equiv \sqrt{2I}/(1 + z^2)^{1/4} \sin \theta, \\ y &= Y(\theta, I, z) \equiv \sqrt{2I}(1 + z^2)^{1/4} \cos \theta. \end{aligned}$$

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<sup>5</sup>If the equilibrium state  $(u_{10}, u_{20})$  of the averaged first approximation system has complex conjugate eigenvalues, then the second approximation system can be useful for determining the stability of the limit cycle.

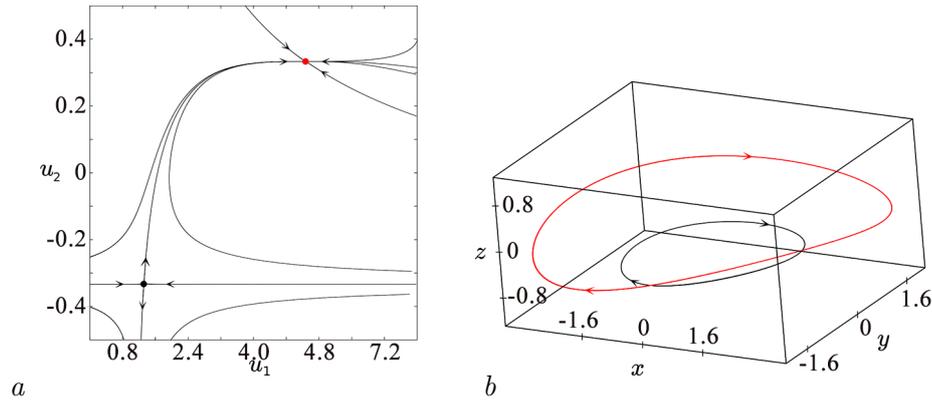


Fig 1. *a* – The phase portrait of system (12) at  $\alpha = 1, \beta = 0.9$ ; *b* – limit cycles of system (11) at  $\varepsilon = 0.05, \alpha = 1, \beta = 0.9$  (color online)

We obtain a system of form (4), in which

$$B_1 = 2I \cos^2 \theta \left( \alpha - \frac{2I}{\sqrt{1+z^2}} \sin^2 \theta \right) - \frac{2I^2 z \cos 2\theta (\sin^2 \theta - \beta(1+z^2) \cos^2 \theta)}{(1+z^2)^{3/2}},$$

$$B_2 = \frac{2I}{\sqrt{1+z^2}} (\sin^2 \theta - \beta(1+z^2) \cos^2 \theta).$$

The averaged system of the first approximation takes the form

$$\begin{aligned} \dot{u}_1 &= \varepsilon u_1 \left( \alpha - \frac{u_1}{2(1+u_2^2)^{3/2}} (1 - (1+\beta)u_2 + u_2^2 - \beta u_2^3) \right), \\ \dot{u}_2 &= \varepsilon \frac{u_1}{\sqrt{1+u_2^2}} (1 - \beta - \beta u_2^2). \end{aligned} \quad (12)$$

If one of the inequalities  $\alpha \leq 0$ ,  $\beta < 0$  or  $\beta > 1$  is satisfied, system (12) has no equilibrium states in the region  $u_1 > 0$ . If  $\alpha \leq 0$  we have  $\dot{u}_1 < 0$  and the trajectories of the original system (11) tend to non-isolated equilibria located on the  $z$  axis. In addition,  $\dot{u}_2 > 0$  for  $\beta < 0, \alpha > 0$  and  $\dot{u}_2 < 0$  for  $\beta > 1, \alpha > 0$ , therefore, for these values, system (11) experiences a «drift» of trajectories along the  $z$  axis. When  $\beta = 1, \alpha > 0$ , a complex equilibrium state (saddle–node) is formed in the averaged system (12), which splits into a node  $O_1 \left( \frac{2\alpha}{\sqrt{\beta}(\sqrt{\beta}-\sqrt{1-\beta})^2}, \sqrt{\frac{1-\beta}{\beta}} \right)$  and a saddle  $O_2 \left( \frac{2\alpha}{\sqrt{\beta}(\sqrt{\beta}+\sqrt{1-\beta})^2}, -\sqrt{\frac{1-\beta}{\beta}} \right)$  for  $\beta < 1$ . These equilibria exist for  $\beta \in (0, 1), \beta \neq 0.5$ . At  $\beta = 0.5$  there is a single saddle point  $O_2$ , and at  $\beta = 0$  both equilibria «go to infinity». In system (11) the point  $O_1$  corresponds to an asymptotically stable limit cycle, and  $O_2$  corresponds to a saddle cycle to a saddle point. For  $0 < \varepsilon \ll 1$ , the limit cycles are close to the ellipses  $\beta y^2/(2h) + x^2/(2h) = 1$ ,  $h = \frac{2\alpha}{\beta(\sqrt{\beta} \pm \sqrt{1-\beta})^2}$ , located in the planes  $z = \pm \sqrt{\frac{1-\beta}{\beta}}$ , and the number  $\omega_0 = 1/\sqrt{\beta}$  is an asymptotic approximation of the oscillation frequency on the cycles. Fig. 1, *a* shows the phase portrait of the averaged system (12) with two equilibrium states, and Fig. 1, *b* shows limit cycles in system (11) (the stable limit cycle is highlighted in red).

**2.2. Lorenz system.** Let's consider the Lorenz system:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = -zx + rx - y, \quad \dot{z} = xy - bz, \quad (13)$$

where  $\sigma > 0$  is the Prandtl number,  $r > 0$  is the relative Rayleigh number,  $0 \leq b < 4$  is a parameter. As V. I. Yudovich [16] pointed out (see also [17, 18]), by changing variables and time

$$x \rightarrow \sqrt{2\sigma(r-1)}x, \quad z \rightarrow (r-1)(z+x^2), \quad t \rightarrow \frac{t}{\sqrt{\sigma(r-1)}}$$

the system is reduced to the form

$$\begin{aligned} \ddot{x} + (z-1)x + x^3 &= -\varepsilon\gamma\dot{x}, \\ \dot{z} &= \varepsilon(-az + \beta x^2), \end{aligned} \tag{14}$$

where  $\varepsilon = 1/\sqrt{r-1}$ ,  $a = b/\sqrt{\sigma}$ ,  $\beta = (2\sigma - b)/\sqrt{\sigma}$ ,  $\gamma = (\sigma + 1)/\sqrt{\sigma}$  — parameters. It's obvious that

$$a > 0, \quad \gamma = \frac{\beta + a}{2} + \frac{2}{\beta + a} \geq 2, \quad a + \beta = 2\sqrt{\sigma} > 0.$$

Consider the case when  $r \gg 1$ . Then  $0 < \varepsilon \ll 1$  and system (14) is close to integrable:

$$\begin{aligned} \ddot{x} + (z-1)x + x^3 &= 0, \\ \dot{z} &= 0. \end{aligned} \tag{15}$$

From the last equation of system (15) we obtain  $z = q = \text{const}$ . Thus, we arrive at the Duffing equation

$$\ddot{x} + (q-1)x + x^3 = 0. \tag{16}$$

The phase portrait of this equation is qualitatively different for  $q < 1$  and for  $q > 1$ . For  $q > 1$  equation (16) has a unique equilibrium state of the center type at the origin, surrounded by closed phase curves

$$\dot{x}^2/2 + (q-1)x^2/2 + x^4/4 = h, \quad h > 0. \tag{17}$$

When  $q < 1$ , the equilibrium state  $(0, 0)$  becomes a saddle and two more centers  $(0, \pm\sqrt{1-q})$  appear. The separatrices of the saddle  $(0, 0)$  form two symmetric homoclinic trajectories (homoclinic “figure eight”). Closed phase curves lying inside the figure eight correspond to values  $h$  satisfying the inequality  $-(1-q)^2/4 < h < 0$ , and outside the figure eight — the inequality  $h > 0$ . Homoclinic trajectories correspond to the Hamiltonian level  $h = 0$ . The solution on closed phase curves is determined through the Jacobi elliptic functions by the known formulas:

$$X(\theta, k, q) = \pm \sqrt{\frac{2(1-q)}{2-k^2}} \text{dn}(\mathbf{K}\theta/\pi), \quad k^2 = \frac{2\sqrt{4h+(1-q)^2}}{1-q+\sqrt{4h+(1-q)^2}}, \tag{18}$$

for  $q < 1, -(1-q)^2/4 < h < 0$  (Case A);

$$X(\theta, k, q) = \sqrt{\frac{2(q-1)k^2}{1-2k^2}} \text{cn}(2\mathbf{K}\theta/\pi), \quad k^2 = \frac{1-q+\sqrt{4h+(1-q)^2}}{2\sqrt{4h+(1-q)^2}}, \tag{19}$$

for  $q < 1, h > 0$  and  $q > 1, h > 0$  (Case B).

Here  $\mathbf{K}(k)$ ,  $\mathbf{E}(k)$  are complete elliptic integrals of the first and second kind,  $k$  is their modulus,  $\theta = \omega t$  is the angular coordinate,  $\omega = \omega(k, q)$  is the natural frequency. In Case A, the values of  $k^2$  belong to the interval  $(0, 1)$ , and in Case B — to the interval  $(0.5, 1)$  for  $q < 1$  and to the interval  $(0, 0.5)$  for  $q > 1$ . The natural frequency  $\omega(k, q)$  is found as follows:

$$\omega = \frac{\pi}{\mathbf{K}(k)} \sqrt{\frac{1-q}{2-k^2}}$$

— in Case A,

$$\omega = \frac{\pi}{2\mathbf{K}(k)} \sqrt{\frac{1-q}{2k^2-1}}$$

– in Case B.

Now let  $\varepsilon \neq 0$ . For the considered parameter values, the system (14) has three equilibria:  $O(0, 0, 0)$  with eigenvalues  $\lambda_{1,2} = -\varepsilon\gamma/2 \pm \sqrt{1 + (\varepsilon\gamma/2)^2}$ ,  $\lambda_3 = -\varepsilon a$  and  $O^\pm(\pm\sqrt{a/(a+\beta)}, 0, \beta/(a+\beta))$  with eigenvalues  $\lambda_{1,2} = -\frac{\varepsilon(\beta-\gamma)}{2} \pm i\sqrt{\frac{2a}{a+\beta}} + o(\varepsilon)$ ,  $\lambda_3 = -\varepsilon(\beta+a) + o(\varepsilon)$ . The point  $O$  is a saddle-node with a two-dimensional stable and one-dimensional unstable manifold and  $O^\pm$  are foci that are asymptotically stable for  $\beta > \gamma$ .

Let us pass from variables  $x, \dot{x}, z$  to variables  $\theta, I, z$ . We obtain a system of the form (4), in which

$$\begin{aligned} B_1 &= (-az + \beta X^2)(X'_z Y'_\theta - X'_\theta Y'_z) - \gamma Y X'_\theta, \\ B_2 &= -az + \beta X^2, \end{aligned}$$

and the function  $X(\theta, k(I), z)$  is defined by formula (18) for Case A and by formula (19) for Case B,  $Y = \omega X'_\theta$ . From the first integral (17) and from equation (16) we find  $X'_\theta = \pm 1/\omega \sqrt{2h + (1-z)X^2 - X^4/2}$ ,  $X''_{\theta^2} = 1/\omega^2((1-z)X - X^3)$ . Using these expressions, we obtain  $X'_z Y'_\theta - X'_\theta Y'_z = X^2/(2\omega)$ . Since the right-hand sides of the resulting system depend on  $I$  implicitly, it is more convenient to pass from the variable  $I$  to the variable  $k$ . To do this, the derivative  $\frac{\partial I}{\partial k}$  is needed. Obviously,  $\frac{\partial I}{\partial k} = \frac{\partial I}{\partial h} \frac{\partial h}{\partial k} = \frac{1}{\omega} \frac{\partial h}{\partial k}$ , where the function  $h = h(k, z)$  is found by applying the formulas for  $k^2$  in (18) and (19), in which  $z$  should be substituted for  $q$ . Averaging the resulting system over  $\theta$  and retaining the previous notations for the averaged variables for clarity, we arrive at a system of the first approximation

$$\begin{aligned} \dot{k} &= \varepsilon \left( -az \int_0^{2\pi} X^2 d\theta + \beta \int_0^{2\pi} X^4 d\theta - 2\gamma \int_0^{2\pi} Y^2 d\theta \right) / (4\pi h'_k), \\ \dot{z} &= \varepsilon \left( -az + \frac{\beta}{2\pi} \int_0^{2\pi} X^2 d\theta \right). \end{aligned}$$

Using (17) and (16), we find

$$\begin{aligned} \int_0^{2\pi} X^4 d\theta &= \frac{8}{3} \pi h(k, z) + \frac{4}{3} (1-z) \int_0^{2\pi} X^2 d\theta, \\ \int_0^{2\pi} Y^2 d\theta &= \frac{8}{3} \pi h(k, z) + \frac{1}{3} (1-z) \int_0^{2\pi} X^2 d\theta. \end{aligned}$$

Thus, the calculation of the right-hand sides of the averaged system is reduced to finding the integral  $\int_0^{2\pi} X^2 d\theta$ . Calculating this integral, we arrive at the averaged system

$$\begin{aligned} \dot{k} &= \varepsilon \frac{(2-k^2)^2}{3k^3} \left( \left( \frac{-3az}{2(1-z)} + 2\beta - \gamma \right) \frac{\mathbf{E}}{\mathbf{K}} + (\beta - 2\gamma) \frac{k^2 - 1}{2 - k^2} \right), \\ \dot{z} &= \varepsilon \left( -az + \frac{2\beta(1-z)}{2 - k^2} \frac{\mathbf{E}}{\mathbf{K}} \right), \end{aligned} \tag{20}$$

– in Case A,

$$\begin{aligned} \dot{k} &= \varepsilon \frac{(2k^2 - 1)^2}{3k} \left( \left( \frac{3az}{2(1-z)} + \gamma - 2\beta \right) \frac{\mathbf{E}}{\mathbf{K}} + \frac{k^2 - 1}{2k^2 - 1} \left( \frac{3az(2k^2 - 1)}{2(1-z)} - \beta(3k^2 - 2) - \gamma \right) \right), \\ \dot{z} &= \varepsilon \left( -az + \frac{2\beta(1-z)}{2k^2 - 1} \left( k^2 - 1 + \frac{\mathbf{E}}{\mathbf{K}} \right) \right), \end{aligned} \tag{21}$$

— in Case B.

From the fact that the Lorenz system (13) is dissipative (the divergence of the system (13) is equal to  $-\sigma - b - 1 < 0$ ), it follows that it cannot have two-dimensional invariant tori. The averaged systems (20) and (21) do not have limit cycles and the problem is reduced to the study of their equilibrium states. Equating the right-hand sides of system (20) to zero, we obtain the generating equation for cycles in the domain  $z < 1$ ,  $-(1-z)^2/4 < h < 0$ :

$$\left( -3\beta \frac{\mathbf{E}}{\mathbf{K}} + (2\beta - \gamma)(2 - k^2) \right) \mathbf{E} + (\beta - 2\gamma)(k^2 - 1)\mathbf{K} = 0, \quad k^2 \in (0, 1).$$

For each root  $k$  the corresponding value  $z$  is found by the formula  $z = (2\beta\mathbf{E})/(a(2-k^2)\mathbf{K} + 2\beta\mathbf{E})$ . For cycles in the regions  $z < 1$ ,  $h > 0$  and  $z > 1$ ,  $h > 0$  we obtain the equation

$$\begin{aligned} \left( 3\beta \left( \frac{\mathbf{E}}{\mathbf{K}} + k^2 - 1 \right) + (\gamma - 2\beta)(2k^2 - 1) \right) \mathbf{E} + \\ + \left( 3\beta \left( \frac{\mathbf{E}}{\mathbf{K}} + k^2 - 1 \right) - \beta(3k^2 - 2) - \gamma \right) (k^2 - 1)\mathbf{K} = 0, \end{aligned}$$

where  $k^2 \in (0.5, 1)$  for the region  $z < 1$ ,  $h > 0$  and  $k^2 \in (0, 0.5)$  for the region  $z > 1$ . For  $z$  we have the formula  $z = 2\beta((k^2 - 1)\mathbf{K} + \mathbf{E})/((a(2k^2 - 1) + 2\beta(k^2 - 1))\mathbf{K} + 2\beta\mathbf{E})$ .

Expressing from the obtained equations the ratio  $\nu = \gamma/\beta$  as a single-valued function of  $k$  and changing  $k$  within the appropriate limits, we find the values of the parameter  $\nu$  at which limit cycles appear in system (14). We have

$$\nu(k) = \frac{2(2 - k^2)\mathbf{E} + (k^2 - 1)\mathbf{K} - 3\mathbf{E}^2/\mathbf{K}}{(2 - k^2)\mathbf{E} + 2(k^2 - 1)\mathbf{K}}$$

in Case A,

$$\nu(k) = \frac{(3\mathbf{E}/\mathbf{K} + k^2 + 1)\mathbf{E} + (-3\mathbf{E}/\mathbf{K} + 1)(k^2 - 1)\mathbf{K}}{(2k^2 - 1)\mathbf{E} - (k^2 - 1)\mathbf{K}}$$

in Case B.

Fig. 2 shows the graphs of the functions  $\nu = \nu(k)$  for each of the regions under consideration. From the analysis of these functions, the following conclusions follow about the limit cycles of system (14).

1. In the region  $z < 1$ ,  $-(1-z)^2/4 < h < 0$  for  $1 < \nu < 2$  (in the initial parameters  $(2b+1)/3 < \sigma < b+1$ ) there is a pair of symmetric limit cycles, each of which covers one of the stable foci  $O^\pm$ . The corresponding equilibrium state of the averaged system is a saddle, therefore these cycles are unstable (saddle), see Fig. 3, a. As  $\nu \rightarrow 1$  ( $\sigma \rightarrow b+1$ ) the cycles stick to the foci  $O^\pm$  (subcritical Andronov–Hopf bifurcation). In another limiting case, as  $\nu \rightarrow 2$  ( $\sigma \rightarrow (2b+1)/3$ ) the cycles stick to symmetric homoclinic loops  $h = 0$ ,  $z = 0$  and then disappear.

2. In the region  $z < 1$ ,  $h > 0$  for  $0.5 < \nu < 2$  ( $\sigma > (2b+1)/3$ ) there is a unique asymptotically stable limit cycle that encircles both foci. The phase portrait of averaged system (21) in the neighborhood of the stable equilibrium state is shown in Fig. 3, b. The inequality  $\nu > 0.5$  follows

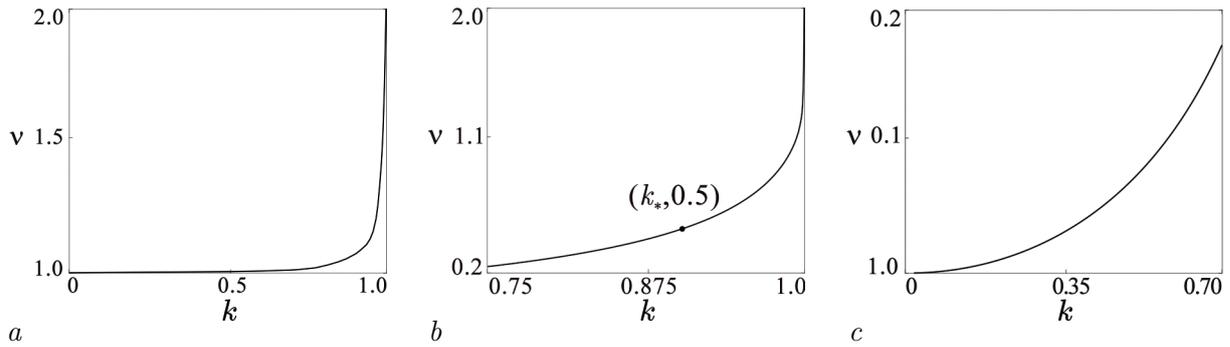


Fig 2. The graphs of  $v = v(k)$  for the regions:  $z < 1, -(1-z)^2/4 < h < 0$  (a);  $z < 1, h > 0$  (b);  $z > 1, h > 0$  (c)

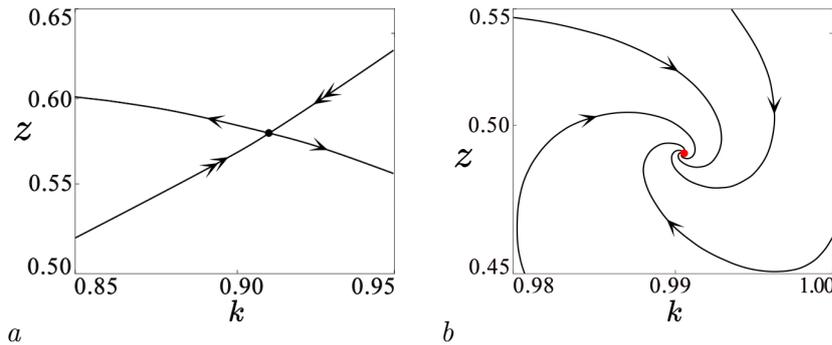


Fig 3. The equilibrium state of averaged system (20) at  $r = 1000, b = 2.6, \sigma = 3.4$  (a); the equilibrium state of averaged system (21) at  $r = 1000, b = 2.6, \sigma = 3.4$  (b)

from the conditions on the parameters  $v = \frac{\sigma+1}{2(\sigma-b/2)} > 0.5$ , where  $v \rightarrow 0.5$  as  $\sigma \rightarrow +\infty$ . Note that when  $\sigma \rightarrow +\infty$  the value of  $k$  corresponding to the cycle tends to  $k_* \approx 0.91$ . As  $v \rightarrow 2$  ( $\sigma \rightarrow (2b+1)/3$ ) the cycle sticks to the separatrix loops  $h=0, z=0$ .

3. In the region  $z > 1$  (with  $h > 0$ ) the inequality  $0 < v(k) < 0.5$  is satisfied, therefore for the considered values of the parameters there are no cycles in this region.

The limit cycles of the Lorenz system for  $b = 2.6, \sigma = 3.3, r = 500$  are shown in Fig. 4, where the stable cycle is highlighted in red and the saddle cycles are highlighted in black.

The value  $\sigma = (2b+1)/3$  is an asymptotic value as  $r \rightarrow +\infty$  corresponding to the formation of two symmetric homoclinic loops in system (13). If  $r$  is fixed, then when they are destroyed, in addition to the two saddle limit cycles inside the “figure-eight”, an invariant set with complex dynamics appears, containing a countable set of saddle cycles that are not detected by the averaging method<sup>6</sup>. This bifurcation plays an important role in the appearance of the Lorenz attractor [19]. The value  $\sigma = b+1$  is an

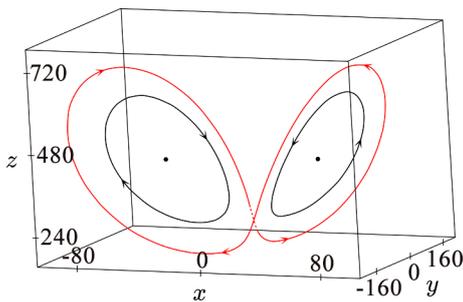


Fig 4. The limit cycles of system (13) at  $b = 2.6, \sigma = 3.3, r = 500$ . The stable cycle is highlighted in red, the saddle ones are highlighted in black (color online)

<sup>6</sup>This emphasizes the inapplicability of the averaging method in the neighborhood of the separatrix surface.

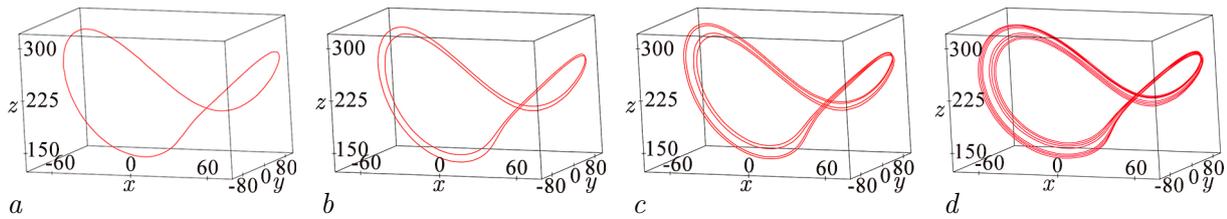


Fig 5. The stable limit cycle of system (13) at  $b = 2.6, \sigma = 10$  and  $r = 250$  (a),  $r = 230$  (b),  $r = 222$  (c),  $r = 220.4$  (d)

asymptotic value as  $r \rightarrow +\infty$  corresponding to the contraction of saddle limit cycles into stable foci (the inverse Andronov–Hopf bifurcation). When  $r$  is fixed, this bifurcation can lead to a hard regime of stochastic excitation [19]. Note also that for fixed values of  $\sigma, b$  and increasing  $r$ , the Lorenz attractor degenerates into a quasi-attractor, and then into a stable limit cycle, which is detected by the averaging method. As computer calculations show, with the inverse change of  $r$  a cascade of doubling bifurcations occurs with this cycle (see Fig. 5).

## Conclusion

The paper considers non-conservative autonomous perturbations of three-dimensional integrable systems. The main attention is paid to the application of the averaging method for the analysis of limit cycles. It is shown that hyperbolic equilibrium states of the averaged first-approximation system determine closed phase trajectories of the unperturbed system, in the neighborhood of which cycles appear under the action of perturbation. Thus, the problem under consideration leads to the study of the generating system of two algebraic or transcendental equations by analogy with the Poincaré–Pontryagin generating equation for two-dimensional systems close to Hamiltonian ones. The article considers two examples — a three-dimensional van der Pol system and a Lorenz system. For the examples, the analysis of limit cycles is carried out by the method described in the paper: the averaged systems are calculated and their equilibrium states are studied. The corresponding phase portraits are constructed.

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