

Reconstruction of self-oscillating systems with delay time modulation

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Abstract. *The aim* of our research is to study the possibility of reconstruction from time series the self-oscillating systems with variable time delay, demonstrating regimes of turbulent and laminar chaos. *Methods.* The object of study is self-oscillating systems described by delay-differential equations, in which the delay time is modulated by an external periodic signal. The possibility of estimating the parameters of systems with delay time modulation from their time series is considered using the known method for reconstructing systems with constant delay time, which is based on statistical analysis of time intervals between all possible pairs of extrema in time series. A new method for estimating the parameters of systems with variable delay time is proposed, based on statistical analysis of time intervals between two successive extrema in time series. *Results.* It is shown that in some cases the known methods for reconstructing systems with constant delay time are also effective for reconstructing systems with varying delay time. With their help, one can estimate the mean delay time and recover the nonlinear function of the system. The proposed method, aimed at application to time-delay systems with delay time modulation, allows one to estimate the frequency and amplitude of delay time modulation. *Conclusion.* The obtained results are of interest to various scientific disciplines that study systems with variable delay times based on their time series.

Keywords: systems with delay time modulation, laminar chaos, reconstruction of systems from time series, statistics of extrema.

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Introduction

Systems with delayed-feedback can generate chaotic oscillations with very high dimensionality even if they are described by a first-order differential equation with a delayed argument at a constant delay time [1–3]. The modulation of the delay time leads to the appearance of new, usually more complex regimes [4–6]. Relatively recently, a new phenomenon called laminar chaos has been discovered in systems with delay time modulation [7], in which laminar phases with an almost constant value of a dynamic variable are periodically interrupted by random bursts that transfer the system from one laminar phase to another with a different constant value of the dynamic variable.

Laminar chaos is observed in some areas of the frequency – amplitude parameter plane of the external harmonic signal modulating the delay time of the system, in which the period of the modulating signal is a multiple of the mean delay time [7, 8]. The appearance of these areas resembles synchronization languages in self-oscillating systems under external influence. Outside of these areas, the delayed system exhibits so-called turbulent chaos, which is described in detail in [9]. The phenomenon of laminar chaos has been investigated in various systems [10] not only theoretically and numerically, but has also been discovered in a physical experiment [8, 11–14].

The problem of reconstructing systems with a constant delay time from chaotic time series has been raised by many authors and is of interest not only in purely theoretical terms, but also for practical applications [15–17]. The discovery of laminar chaos regimes in systems with delay time modulation raises the question of the possibility of estimating their parameters using known methods for recovering systems with constant delay time and requires the development of new reconstruction methods focused on the class of systems with variable delay time.

In this paper, we develop methods for reconstructing systems described by a first-order differential equation with delay time modulation in the area of control parameters in which laminar chaos can be observed. Reconstruction methods for laminar and turbulent chaos regimes are proposed.

1. The system under study and methods

The object of the study is a system representing a first-order equation with a delay, described by equation (1):

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t - \tau(t))), \quad (1)$$

where x is a dynamic variable, ε is the inertia parameter, $\tau(t)$ is a variable delay time, $f(x)$ is a nonlinear function of the form $f(x) = \lambda - x^2$, where λ is a nonlinearity parameter. The value of $\tau(t)$ varies according to the periodic law $\tau(t) = \tau_0 + \tau_1 \sin(2\pi\nu t)$, where τ_0 is the mean value of the delay time, τ_1 is the amplitude (depth) of modulation, and ν is the modulation frequency of τ_0 .

For comparison, let us consider two groups of qualitatively different operating regimes of the generator. The first group includes regimes in which the system exhibits laminar chaos at different ratios of the delay time modulation period $T_m = 1/\nu$ and the mean delay time $\tau_0 = 100$. Three qualitatively different regimes with different modulation frequencies are considered $\nu = 0.011$, $\nu = 0.0185$ and $\nu = 0.0275$ at $\tau_1 = 15$ and $\tau_1 = 10$. In the second group of regimes, outside the regions of laminar chaos, two regimes are considered at $\nu = 0.013$ and $\nu = 0.024$ at $\tau_1 = 15$. The nonlinearity parameter λ is equal to 1.89, $\varepsilon = 1$ in all cases.

In the laminar chaos regime, we will analyze the levels x_i of horizontal plateaus in the temporal realization of a dynamic variable, which can provide information about the type of

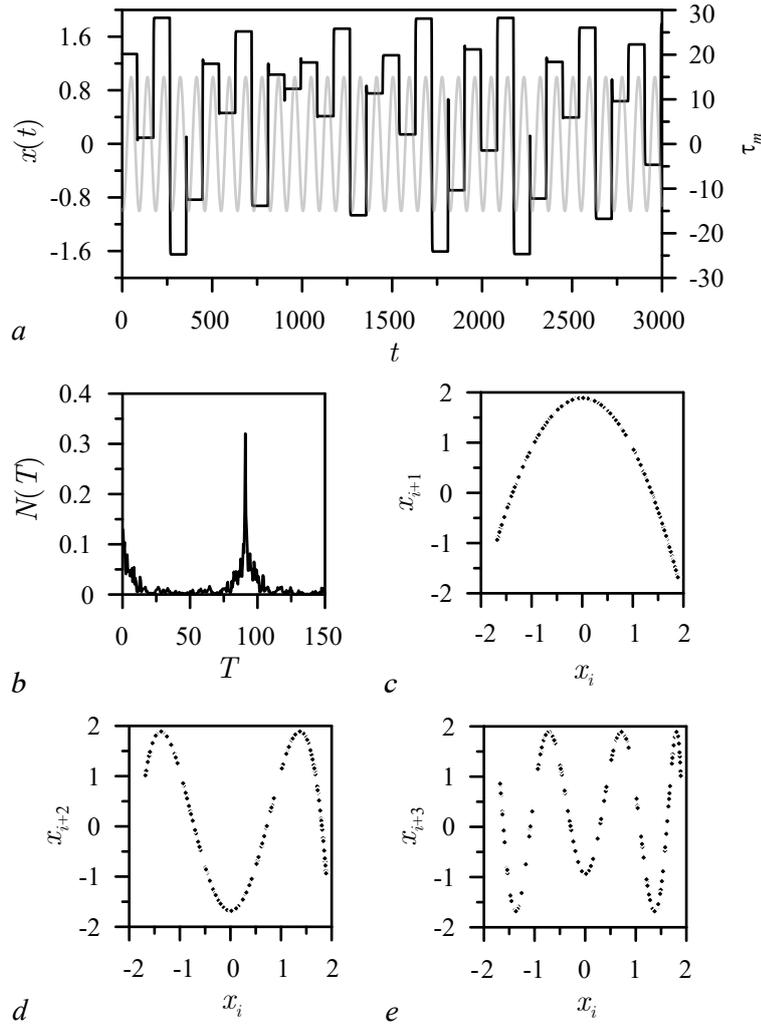


Fig 1. Time series of the system (black) and variation of the delay time τ_m (gray) (a), statistics of extrema of the time series (b), dependencies $x_{i+n}(x_i)$ for $n = 1$ (c), $n = 2$ (d), and $n = 3$ (e). Control parameters are as follows: $\nu = 0.011$, $\tau_1 = 15$, and $\tau_0 = 100$

nonlinear dependence of the dynamic variable on its delayed copy. At the same time, in some cases we expect to see a smooth dependence of the level of one plateau on the level of another in the time series. The delay time in this case is determined approximately. To estimate the mean delay time, we will use the previously developed method of reconstructing systems with a constant delay time, which is based on the analysis of statistics of time series extrema [18].

In the turbulent chaos regime, to estimate the mean delay time, we will also try the procedure for constructing statistics of extrema [18], and to estimate the frequency of time modulation of τ_0 a new procedure based on the analysis of changes in the distances between extrema over time is proposed.

2. Results

Fig. 1, a shows a time series of a time-delay system (solid line) in laminar chaos regime at $\nu = 0.011$, and a variable part of the delay $\tau_m = \tau_1 \sin(2\pi\nu t)$, where $\tau_1 = 15$. The remaining control parameters are fixed. Horizontal sections (plateaus) are visible on the time series, interrupted

by "bursts"(sudden changes in the dynamic variable $x(t)$ when moving from one plateau to another). Thus, we can enter a discrete time and denote as x_i the value of the dynamic variable at the plateau level, where i is the ordinal number of the plateau. The plateau lengths are the same, they are equal to the oscillation period of the delay time τ_m (see Fig. 1, a), but their length is not exactly equal to the delay time τ_0 . If the frequency ν of the delay time modulation is slightly varied, the plateau lengths will vary according to the period of the external signal. This effect resembles the synchronization of self-oscillating systems by an external signal. At the same time, the plateaus are limited to the left and right by sharp jumps, which determine the beginning and end of the plateau. If the length of the plateau is approximately equal to the period T_m of the external signal, this corresponds to synchronization on the fundamental harmonic. If two plateaus fit in the delay time, this corresponds to synchronization on the second harmonic, etc. With a strong change in the frequency of external influences (modulation frequency ν) a transition to the regime of turbulent chaos is observed. This effect was experimentally investigated in [8,19].

Fig. 1, b presents a measure of the statistics of extrema, which shows how many extrema occur in the time series of the variable x with a distance T between them [18]. Note that since

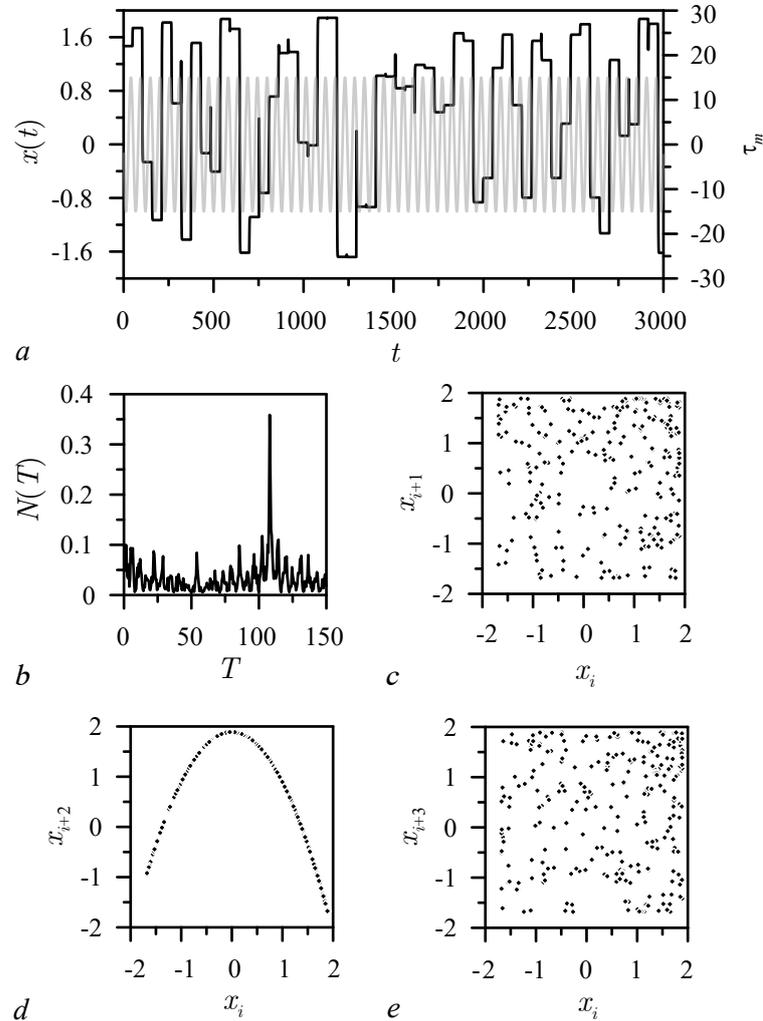


Fig 2. Time series of the system (black) and variation of delay time τ_m (gray) (a), statistics of extrema (b), dependencies $x_{i+n}(x_i)$ for $n = 1$ (c), $n = 2$ (d), and $n = 3$ (e). Control parameters are as follows: $\nu = 0.0185$, $\tau_1 = 15$, and $\tau_0 = 100$

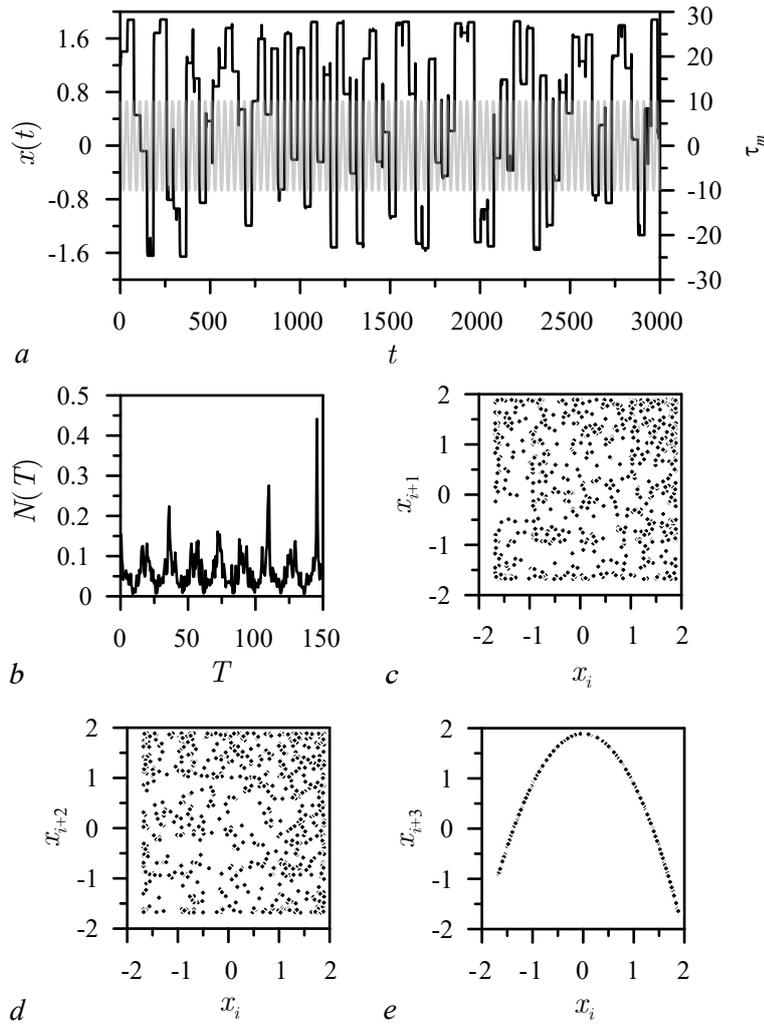


Fig 3. Time series of the system (black) and variation of delay time τ_m (gray) (a), statistics of extrema (b), dependencies $x_{i+n}(x_i)$ for $n = 1$ (c), $n = 2$ (d), and $n = 3$ (e). Control parameters are as follows: $\nu = 0.0275$, $\tau_1 = 10$, and $\tau_0 = 100$

the values of the dynamic variable are practically constant on the horizontal plateaus of the dependence $x(t)$, we did not take into account the extrema on the plateaus (even if they exist) when constructing Fig. 1, b. Only the extrema observed between the plateaus in the time series sections where there are sharp bursts of the dynamic variable are taken into account. The measure $N(T)$ is normalized to the total number of extrema in the time series. The maximum of $N(T)$ is observed at $T = 91$, which corresponds to the period of change in the delay time, and also gives a rough estimate of the mean delay time $\tau_0 = 100$. Fig. 1, c–e presents dependencies $x_{i+n}(x_i)$ for $n = 1$ (c), $n = 2$ (d), $n = 3$ (e). In Fig. 1, c points of the time series plotted on the plane (x_i, x_{i+1}) , fall on the nonlinear function $f(x)$, corresponding to the transformation in the delayed feedback. Fig. 1, d, e represent the second and third iterations of the function $f(x)$.

In Fig. 2 the research results are presented at $\nu = 0.0185$ and $\tau_1 = 15$. This regime corresponds to the case in which the horizontal plateaus in the temporal realization of the dynamic variable are about 2 times shorter than in the example discussed above. At the same time, one plateau fits in the period of oscillations of the delay time τ_m , but at the time approximately corresponding to the delay time τ_0 , there are already two plateaus (Fig. 2, a).

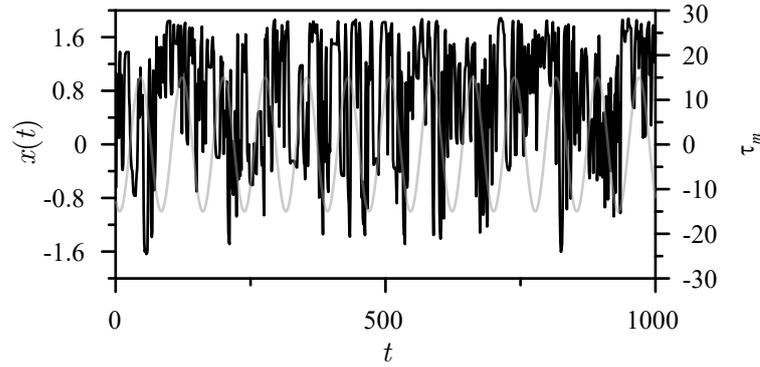


Fig 4. Time series of turbulent chaos in system (1) (black) and variation of delay time τ_m (gray) at $\nu = 0.013$ and $\tau_1 = 15$

The value of $N(T)$ in Fig. 2, *b* has a global maximum at $T = 108$, which corresponds to a doubled period of external action of $T_m = 1/\nu$ and approximately corresponds to the mean delay time τ_0 . Fig. 2, *c-e* demonstrates the dependencies of $x_{i+n}(x_i)$ for $n = 1$ (*c*), $n = 2$ (*d*), $n = 3$ (*e*). When analyzing them, it becomes obvious that the mean delay time τ_0 approximately corresponds to the length of two plateaus of the time series, since $x_{i+2}(x_i)$ demonstrates a clear dependence of the transformation of $f(x)$ in the delayed feedback. Fig. 3, *a* shows the time series at $\nu = 0.0275$ and $\tau_1 = 10$. This regime corresponds to the case in which three horizontal plateaus on the graph $x(t)$ fit in a time interval approximately corresponding to the delay time τ_0 . Thus, the length of the plateaus on the graph $x(t)$ in Fig. 3, *a* is about 3 times smaller than in Fig. 1, *a*. At the same time, only one plateau still fits in the period of oscillations of the delay time τ_m .

The value of $N(T)$ in Fig. 3, *b* demonstrates a maximum at the time $T = 109$. This approximately corresponds to the mean delay time τ_0 . Fig. 3, *c-e* shows the dependencies $x_{i+n}(x_i)$ for $n = 1$ (*c*), $n = 2$ (*d*), $n = 3$ (*e*). In Fig. 3, *c* and *d* the mapping shows a chaotic set of points, and for $n = 3$ in Fig. 3, *e* the points reconstruct the nonlinear function $f(x)$. This also confirms that the mean delay time τ_0 in the system approximately corresponds to the total length of three plateaus in the temporal realization of the dynamic variable x .

Let us now consider the possibilities of reconstruction in the regions of turbulent chaos (beyond the regions of laminar chaos). In this case, the time series of the variable x has the form of an irregular signal, generally similar to the signal of a time-delay system without delay time modulation (see Fig. 4). This figure shows a time series for $\nu = 0.013$, $\tau_1 = 15$.

The previously developed method of reconstructing the delay time in systems with a constant delay time, based on a statistical analysis of the time intervals between all possible pairs of extrema of the time series [18], is applicable to the time series of system (1) with a variable delay time in a turbulent chaos regime. In Fig. 5, *a, b* the dependences $N(T)$ are constructed for cases when the modulation frequency of the mean delay time τ_0 takes the values $\nu = 0.013$ and $\nu = 0.024$.

According to the classical statistics of extrema [18] the modulation frequency ν of the mean delay time τ_0 is easily determined. It corresponds to the highest peaks on the $N(T)$ graph. So, in Fig. 5, *a* the main maximum of $N(T)$ is observed at $T = 76$, which is close to $T_m = 1/\nu = 1/0.013 = 77$. In Fig. 5, *b* depending on $N(T)$ there is one global maximum at $T = 125$ and two smaller maxima at $T = 42$ and $T = 83$. The maximum at $T = 42$ corresponds to $T_m = 1/\nu = 1/0.024 = 42$. A maximum of $T = 83$ in Fig. 5, *b* correspond to a doubled modulation period, and a maximum of $T = 125$ corresponds to a tripled period.

The global minimum of $N(T)$ in Fig. 5, *a, b*, corresponding to the delay time τ_0 , is

significantly worse visible than for systems without delay time modulation, but some traces of it are visible on the plots. If we limit the search area of the delay time to a range of 70...130 (as a rule, such estimates are made for more general reasons in the study), in both cases considered, the global minimum of the dependence $N(T)$ is observed at $T = 100$, corresponding to the mean delay time $\tau_0 = 100$.

It can be noted that in Fig. 4 extrema in the time series of the variable x are located more frequently at positive values of the delay time τ_m , than at negative values of τ_m . Thus, the location of the extrema can provide information about the amplitude τ_1 and the modulation period $T_m = 1/\nu$ of the delay time τ_0 . Let's evaluate these parameters using our proposed new statistics of extrema, which is based on the following rule: along the axis of the abscissa, we postpone the moment of time t , at which an extremum is observed in the time series of $x(t)$. Along the ordinate axis, we plot the time distance Δt between the current extremum and the previous one. As a result, we get small values Δt in those places in the time series where the extrema are densely located and large values Δt in those places where the extrema are sparse. This is actually some kind of frequency modulation, where the delay time modulation acts as a control signal.

The new statistics of $\Delta t(t)$ are shown in Fig. 5, *c*, *d* for the same parameter values as in Fig. 5, *a*, *b*. The points of the resulting dependence $\Delta t(t)$ are indicated by crosses, which are interconnected by straight lines for clarity. The downward-pointing tongues correspond to a stronger consolidation of extrema in the time series of $x(t)$. Based on this characteristic, we can estimate the period of the external signal T_m , which is equal to the average distance between the minima of the dependence $\Delta t(t)$. In Fig. 5, *c* for the time $t = 500$, 6 deep minima (tongues) can be distinguished based on the dependence of $\Delta t(t)$ from $t = 30$ to $t = 490$. Therefore, the modulation period T_m of the delay time τ_0 is approximately $T_m = 460/6 = 77$. This estimate coincides with the true modulation period $T_m = 1/\nu = 1/0.013 = 77$. According to Fig. 5, *d* it is also possible to estimate the modulation period, which is approximately $T_m = 500/12 = 42$,

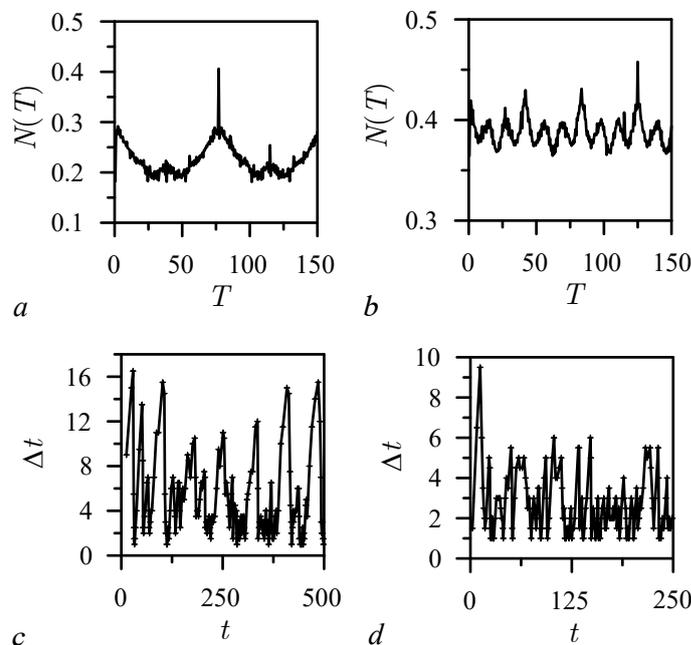


Fig 5. Classic statistics of extrema at $\nu = 0.013$, $\tau_1 = 15$ (*a*) and $\nu = 0.024$, $\tau_1 = 10$ (*b*) and new statistics at $\nu = 0.013$ (*c*) and $\nu = 0.024$ (*d*)

which practically coincides with the true modulation period. The amplitude of the modulation τ_1 roughly corresponds to the oscillation range of $\Delta t(t)$ in Fig. 5, *c*, *d*.

Conclusions

Thus, in this article, the possibility of reconstructing self-oscillatory systems with variable delay time is investigated using the well-known method of reconstructing systems with constant delay time, which is based on a statistical analysis of time intervals between all possible pairs of time series extrema. A new reconstruction method is proposed in application to systems with delay time modulation, in which qualitatively different oscillation regimes can be observed. The proposed method is based on a statistical analysis of the time intervals between two consecutive extrema of the time series.

In the case where laminar chaos exists in the system, a nonlinear function can be reconstructed. The nonlinear function is constructed as a dependence $x_{i+n}(x_i)$ of the values of the dynamic variable on the horizontal section (plateau) of the temporary realization of $x(t)$ with the number $i + n$ from the value of the variable on the plateau with the number i , where $n = 1, 2$ or 3 . The number n provides information about the region of laminar chaos in which the system is located. So, if the modulation period T_m is approximately equal to the mean delay time τ_0 , then the nonlinear function is reconstructed at $n = 1$, and the mean delay time can be estimated as the plateau length in the temporal realization of $x(t)$.

In the case where the delay time modulation period is about half the mean delay time, the nonlinear function is recovered at $n = 2$, and the mean delay time is approximately equal to the duration of two plateaus in the temporal realization of $x(t)$.

In the case where the delay time modulation period is about three times less than the mean delay time, the nonlinear function is recovered at $n = 3$, and the mean delay time is approximately equal to the duration of three plateaus in the temporal realization of $x(t)$. However, the depth of the modulation cannot be determined.

If the delay time is modulated in the system, but there is no laminar chaos, in some cases, in turbulent chaos regimes, the delay time can be estimated using the usual statistics of extrema described in [18]. In this case, the frequency and amplitude of the delay time modulation can be estimated using new statistics showing the time distance between the current and previous extremum of the time series.

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