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Transformation of caustic structures of the catastrophe type during the propagation of electromagnetic waves in cold plasma

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Abstract. The *purpose* of the work is to investigate the possibilities of formation of centers of topological caustic singularities of the catastrophe type during probing of a unimodal plasma layer by electromagnetic waves. The centers of caustic singularities correspond to the focusing regions of electromagnetic fields of different orders. Therefore, their study is an urgent task. *Methods.* The article develops a method for calculating the position of the centers of singularities in an extended parameter space, which in addition to coordinates include the height of the plasma layer, the angle of the ray exit, the ratio of the plasma frequency to the operating frequency, and cubicity. The Hamilton–Lukin bicharacteristic method is used to calculate the ray trajectories. *Results.* Mathematical modeling is performed using the example of a flat-layered plasma layer with a cubic dependence of the electron concentration on the height. Explicit expressions for the eikonal derivatives up to the eighth order inclusive are obtained, which makes it possible to determine the centers of the main cuspoid catastrophes. Graphs are constructed for the dependences of coordinates, the height of the trajectory reflection from the plasma layer, the distance from the radiation source to the plasma layer, the ray exit angle, and the ratio of the plasma frequency to the operating frequency on cubicity for a butterfly-type catastrophe. It is shown that the height of the trajectory reflection from the plasma layer, the distance from the radiation source to the plasma layer, and the height of the singularity reach maximum values for a parabolic layer. It is established that a butterfly-type singularity occurs even when not only the function itself describing the electron concentration is continuous, but also its derivative. *Conclusion.* The developed approach allows us to find the centers of not only the "butterfly" type catastrophe, but also the centers of other topological cuspoid singularities: "cusp", "swallow tail", "wigwam", "star", which is of great practical importance in studying the propagation of radio waves in ionospheric plasma.

Keywords: caustic structures, singularities, radio wave propagation, bicharacteristic system, wave catastrophes.

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Introduction

The tasks of studying the propagation and diffraction of electromagnetic and acoustic waves are relevant problems of radiophysics. Numerous studies have been devoted to these issues, among which it is necessary first of all to mention the works of Y.V. Gulyaev [1–4], as well as the works of V.L. Ginzburg [5], K.G. Budden [6], V.P. Maslov [7], K. Davis et al. [8].

In the asymptotic description of wave propagation and diffraction processes, when the wavelength is much smaller than the characteristic dimensions of the propagation medium, the most popular are ray methods: the method of geometric optics or the geometric theory of diffraction. In particular, ray methods are linear and make it possible to find solutions in multipath regions based on superposition. However, the superposition principle is violated in the vicinity of the envelopes of ray families of caustics and their singularities, which leads to nonlinear phenomena and the need for additional research [9–11]. Caustic surfaces are areas of increased field amplitude (focus regions) and divide space into zones with different numbers of rays.

The classification of the singularities of caustic structures is devoted to the theory of the singularities of differentiable maps, better known as the theory of catastrophes [12–15]. In the two-dimensional wave propagation problem, according to [13, 14, 16], only two singularities are stable: the caustic itself (catastrophe A_2) and the caustic tip (catastrophe A_3 is «cusp»). In three-dimensional space, three more singularities are added to them: «swallow tail» (catastrophe A_4), «elliptic umbilic» (catastrophe D_4^-) and «hyperbolic umbilic» (catastrophe D_4^+). However, in fact, in the presence of additional parameters, even in a two-dimensional problem, sections of caustic singularities of higher orders arise [17–20]. It turns out that at certain parameter values, not only singularities cross-sections arise, but also the focus centers themselves [21]. In this work, using the example of an inhomogeneous plasma layer (ionospheric or laboratory plasma layer), a method for determining the centers of caustic cuspoid singularities was developed and numerical modeling was performed.

1. Problem statement

Let us consider the propagation of electromagnetic waves in a plane-layered medium based on the ray approach. Let the dielectric constant of the media depend on only one of the three Cartesian coordinates $\mathbf{r} = (x, y, z)$ are coordinates of z: $\varepsilon(z, \mathbf{b})$, and the vector $\mathbf{b}$ is a set of additional parameters. Denote it as $\mu(S, \rho)$ a generalized eikonal in which S is the initial parameter of the ray output: $S = k_x/k_0$, k_x is the horizontal component of the wave vector k_x, and $k_0 = \omega/c$ is wavenumber in the void (or outside the plasma), ω is the circular frequency, c is the speed of light in a vacuum. The value of $\rho = (\mathbf{r}, \mathbf{b})$. The ray equations (the Hamilton–Lukin bicharacteristic system) have the form [22, 23]:

$$\frac{d\mathbf{r}}{dt} = -\frac{\partial \Gamma}{\partial \mathbf{k}} \Big/ \frac{\partial \Gamma}{\partial \omega}, \quad \frac{d\,\mathbf{k}}{dt} = \frac{\partial \Gamma}{\partial \mathbf{r}} \Big/ \frac{\partial \Gamma}{\partial \omega}, \quad \Gamma = (\mathbf{k}, \mathbf{k}) - \frac{\omega^2}{c^2}\varepsilon \tag{1}$$

$(\mathbf{k}, \mathbf{k}) - \frac{\omega^2}{c^2}\varepsilon = 0$ is the dispersion relation. If the propagation medium is plane-layered medium and isotropic, then the equations are simplified because

$$\frac{d\,k_x}{dt} = \frac{d\,k_y}{dt} = 0. \tag{2}$$

Therefore, the components of the wave vector k_x and k_y are constant. In the future, this allows us to consider the propagation of the signal in the (x, z) plane and assume that $k_z = 0$.

Thus, three equations remain from the bicharacteristic system (1):

$$\frac{d\,k_z}{dt} = \frac{\omega^2 {\varepsilon'}_z}{(\omega^2\varepsilon)'_\omega}, \quad \frac{d\,z}{dt} = \frac{2k_z c^2}{(\omega^2\varepsilon)'_\omega}, \quad \frac{d\,x}{dt} = \frac{2k_x c^2}{(\omega^2\varepsilon)'_\omega}, \tag{3}$$

from which time can be excluded t:

$$\frac{d\,k_z}{dx} = \frac{\omega^2 \varepsilon'_z}{2k_x c^2}, \quad \frac{dz}{dx} = \frac{k_z}{k_x}. \tag{4}$$

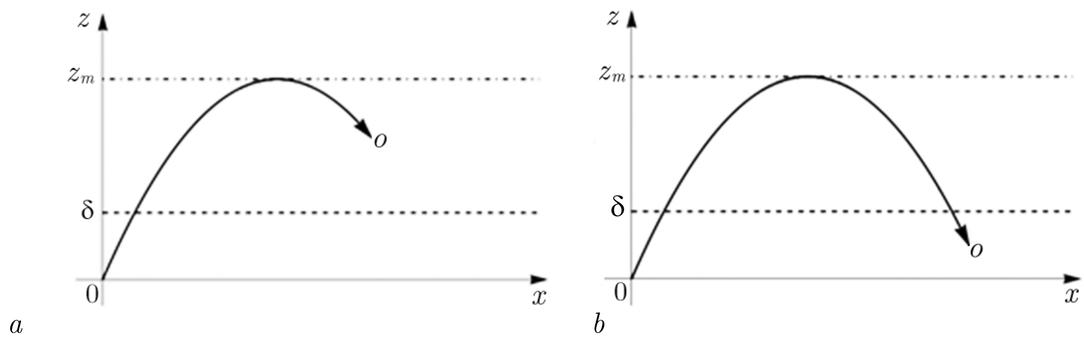


Fig. 1. Ray trajectory in the plasma layer, a — special point inside the layer, b — special point outside the layer

Given the form of the Hamiltonian and (2), system (4) can be represented as

$$\frac{dk_z}{dx} = \frac{k_0}{2}\frac{\varepsilon'_z}{S}, \quad \frac{dz}{dx} = \frac{\pm q}{S}, \quad q = \sqrt{\varepsilon - S^2}. \tag{5}$$

The sign «+» corresponds to the ascending part of the radial trajectory, and the sign «−» corresponds to the descending part (Fig. 1).

Although a bicharacteristic system supplemented by initial conditions, in accordance with Cauchy's theorem, has a unique solution in phase space, that is, in the space $(\mathbf{r}, \mathbf{k})$, in configuration space (that is, in coordinate space) the rays can intersect, forming regions interference, and have envelopes - caustics, which, in turn, have singularities of different orders, interpreted as focusing regions.

The center of focus in a plane-layered medium can be found as a solution to a system of equations [6, 21, 24, 25]:

$$\mu_1 = 0, \ \ \mu_2 = 0, \ \ \mu_3 = 0, \ \ \dots , \quad \mu_n = 0, \ \ \mu_{n+1} \neq 0, \tag{6}$$

where $\mu_n = \partial^n \mu / \partial S^n$. The first equation from list (6) defines the trajectory of the ray. If we add a second equation to it, then their solution is the caustic equation (according to catastrophe theory, the singularity A_2), the first three equations allow us to find the position of the caustic tip (the singularity A_3), four equations determine the center of the «swallow tail» (the singularity A_4), five equations determine the «butterfly» (A_5), six determine the «wigwam» (A_6), seven determine the «star» (A_7), etc. [21, 26–28]. The vanishing of the first n equations indicates the formation of the caspoid catastrophe A_n provided that $\mu_{n+1} \neq 0$. In order for the system of equations to have a solution, in addition to coordinates, additional parameters are needed, designated in the work as $\mathbf{b}$, which will provide the necessary focusing order (type of singularity). They are discussed in more detail below.

2. The method of determining the centers of cuspoid catastrophes

If the propagation medium is a cold isotropic plasma, then the effective dielectric constant can be represented as

$$\varepsilon = 1 - a^2 N(z). \tag{7}$$

We assume that the plasma layer begins at a height of $z = \delta$ (see Fig. 1) and has a maximum height of $z = z_M z$. Up to the height of $z = \delta$, the ray path is a straight line. Then we can assume that $a = \omega_p/\omega$, where ω_p is the value of the circular plasma frequency at the maximum of the layer, and the function $N(z)$ describes the normalized distribution of the electron concentration. In the future, let us assume that the coordinates x, z and δ are normalized to the half-thickness of the plasma layer $z_\Delta = z_M - \delta$. The ray is reflected from the ionosphere at an altitude of $z = z_m$ ($z_m < z_M$) and reaches the caustic (or its singularity) at the point o. In this case, the point o itself can be located as inside a layer (Fig. 1, a), and outside it (Fig. 1, b). It should be noted that caustics or their singularity does not leave any traces on the radial trajectory, since the appearance of caustics is not an individual, but a group property of radial trajectories.

The normalized eikonal in a plane-layered medium in the case when a special point is located in the layer (Fig. 1, a), it can be represented as [21,29]:

$$\mu = Sx_o + C\delta + \int\limits_{\delta}^{z_o} q dz + 2\int\limits_{z_o}^{z_m} q dz \tag{8}$$

or

$$\mu = Sx_o + C\delta + W_1 + 2V_1, \tag{9}$$

where

$$W_1 = \int\limits_{\delta}^{z_o} q dz, \quad V_1 = \int\limits_{z_o}^{z_m} q dz, \quad C = \sqrt{\varepsilon(0) - S^2}. \tag{10}$$

If the singular point is located under the layer, then

$$\mu = Sx_o + C(\delta - z_o) + 2\int\limits_{\delta}^{z_m} q dz. \tag{11}$$

If we multiply μ by a large parameter $\Lambda = k_0 z_\Delta = \frac{\omega}{c} z_\Delta$, then you can get the eikonal value in radians.

In Fig. 2 cross sections of the caustic and radial structures of the catastrophe A_5 are shown in the case when the caustic structure is developed and the focus center does not lie in the (x, z) plane.

Consider the expression (9). In order to obtain eikonal derivatives, it is necessary to differentiate (9) the required number of times (n) for the parameter S, that is, find

$$\mu_n = \mu_n^1 + \mu_n^2 + \mu_n^3 + \mu_n^4. \tag{12}$$

Differentiation of the first three terms is not difficult. The first term has only the first derivative other than zero. Therefore

$$\mu_1^1 = x_o, \quad \mu_n^1 = 0, \quad n > 1. \tag{13}$$

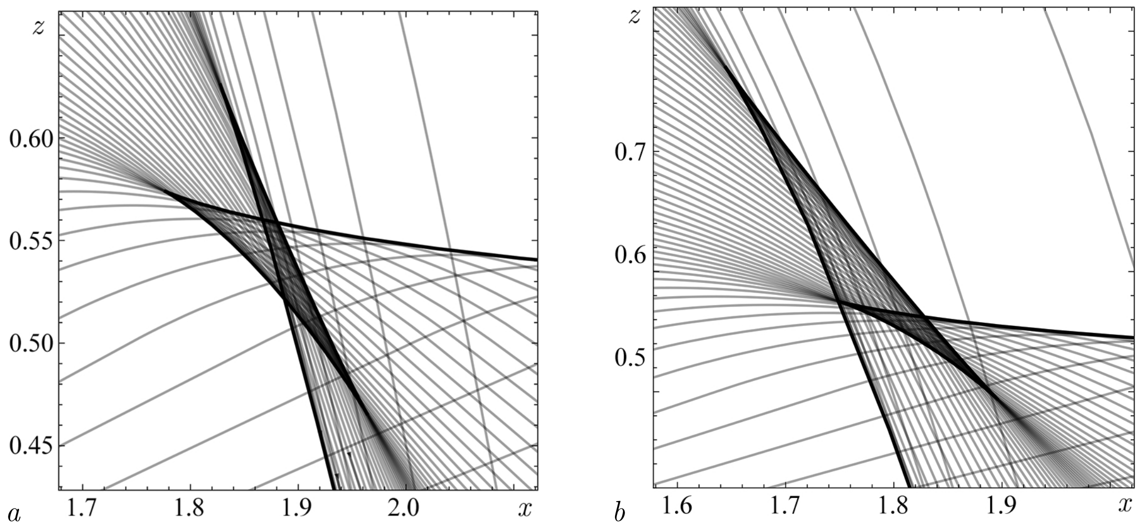


Fig. 2. Sections of the singularities of the caustic and ray structures of the catastrophe A_5

In the second term, C is differentiated. By entering the notation $h = S/C$, $C_n = \partial C/\partial S$, $\varepsilon_0 = \varepsilon(0)$, we find:

$$C_1 = -h, \quad C_2 = -\frac{\varepsilon_0}{C^3}, \quad C_4 = -(1+5h^2)\frac{3\varepsilon_0}{C^5}, \quad C_5 = -h(3+7h^2)\frac{15\varepsilon_0}{C^6},$$
$$C_6 = -(1+14h^2+21h^4)\frac{45\varepsilon_0}{C^7}, \quad C_7 = -h(5+30h^2+33h^4)\frac{315\varepsilon_0}{C^8}, \tag{14}$$
$$C_8 = -(5+135h^2+495h^4+429h^6)\frac{315\varepsilon_0}{C^9}.$$

Therefore

$$\mu_n^2 = \delta_n. \tag{15}$$

The recurrent formula is valid for differentiating the third term:

$$\frac{\partial W_n}{\partial S} = (2n-3)S \cdot W_{n+1}, \tag{16}$$

in which

$$W_n = \int\limits_{\delta}^{z_o} q^{3-2n} dz. \tag{17}$$

Then

$$\frac{\partial W_1}{\partial S} = -S \cdot W_2, \quad \frac{\partial^2 W_1}{\partial S^2} = -W_2 - S^2 \cdot W_3, \quad \frac{\partial^3 W_1}{\partial S^3} = -3SW_3 - 3S^3 \cdot W_4,$$
$$\frac{\partial^4 W_1}{\partial S^4} = -3(W_3 + 6S^2 \cdot W_4 + 5S^4 \cdot W_5), \quad \frac{\partial^5 W_1}{\partial S^5} = -45\,SW_4 - 150\,S^3 \cdot W_5 + 105\,S^5 \cdot W_6,$$
$$\frac{\partial^6 W_1}{\partial S^6} = -45(\,W_4 + 15\,S^2 \cdot W_5 + 35\,S^4 \cdot W_6 + 21\,S^6 \cdot W_7), \tag{18}$$
$$\frac{\partial^7 W_1}{\partial S^7} = -315\,S(\,5W_5 + 35\,S^2 \cdot W_6 + 63\,S^4 \cdot W_7 + 33\,S^6 \cdot W_8),$$
$$\frac{\partial^8 W_1}{\partial S^8} = -315\,(\,5 + 140\,S^2 \cdot W_6 + 630\,S^4 \cdot W_7 + 924\,S^6 \cdot W_8 + 429\,S^8 \cdot W_9).$$

Therefore,

$$\mu_n^3 = \frac{\partial^n W_1}{\partial S^n}. \tag{19}$$

Let us now proceed to the calculation of the fourth term. Let us differentiate V_1 by S. Considering that the vertical point of reflection of the wave from the plasma layer is determined from the condition

$$q = 0 \,\sim\, \varepsilon(z_m) = S^2, \tag{20}$$

we find

$$\frac{\partial V_1}{\partial S} = -S \int\limits_{z_o}^{z_m} \frac{dz}{q}, \tag{21}$$

and this is due to condition (20), it is an improper integral. Further differentiation (21) by S will increase the power of q in the denominator, and the integral will become divergent. To get rid of the singularity in the denominator, integrate (21) in parts. Then we get a convergent expression that can be further differentiated:

$$\frac{\partial V_1}{\partial S} = -\left.\frac{2S}{a^2 N_1}\right|_{z=z_o} + \frac{2S}{a^2} \int\limits_{z_o}^{z_m} \varphi_2\, q dz, \quad \varphi_2 = \frac{N_2}{N_1^2}, \quad N_n = \frac{\partial N(z)}{\partial z^n} \tag{22}$$

Repeating this procedure, we obtain the recurrent expression:

$$\frac{\partial V_n}{\partial S} = -\frac{2S}{a^2}\left(\left.\frac{\varphi_n\, q}{N_1}\right|_{z=z_o} + \int\limits_{z_o}^{z_m} \varphi_{n+1}\, q dz \right) = -\frac{2S}{a^2}\left(\left.\frac{\varphi_n\, q}{N_1}\right|_{z=z_o} + V_{n+1} \right), \tag{23}$$

in which the designation is entered

$$V_n = \int\limits_{z_o}^{z_m} \varphi_n\, q dz, \tag{24}$$

with

$$\varphi_{n+1} = \frac{\partial}{\partial z}\left(\frac{\varphi_n}{N_1}\right), \quad n \geqslant 2. \tag{25}$$

Let

$$E_n = \frac{1}{N_1^n}\frac{\partial^n N(z)}{\partial z^n}, \quad n \geqslant 2. \tag{26}$$

Then

$$\begin{aligned}
\varphi_1 &= 1,\\
\varphi_2 &= E_2,\\
\varphi_3 &= E_3 - 3E_2^2,\\
\varphi_4 &= E_4 - 10E_2E_3 + 15E_2^3,\\
\varphi_5 &= E_5 - 10E_3^2 - 15E_2E_4 + 105E_3E_2^2 - 105E_2^4,\\
\varphi_6 &= E_6 - 21E_2E_5 - 35E_3E_4 + 210E_4E_2^2 + 280E_2E_3^2 - 1260E_2^3E_3 + 945E_2^5,\\
\varphi_8 &= E_8 - 36E_2E_7 - 84E_3E_6 - 126E_4E_5 + 630E_6E_2^2 + 2520E_2E_3E_5 + 1575E_2E_4^2 +\\
&\quad +2100E_3^2E_4 - 34650E_2^2E_3E_4 - 6930E_3^2E_5 - 15400E_2E_3^3 + 138600E_2^3E_3^2 +\\
&\quad +51975E_4E_2^4 - 270270E_3E_2^5 + 135135E_2^7,\\
\varphi_9 &= E_9 - 45E_2E_8 - 120E_3E_7 - 210E_4E_6 - 126E_5^2 + 990E_2^2E_7 + 4620E_2E_3E_6 +\\
&\quad +6930E_2E_4E_5 + 4620E_3^2E_5 + 5775E_3E_4^2 - 13860E_2^3E_6 - 83160E_2^2E_3E_5 -\\
&\quad -138600E_2E_3^2E_4 - 51975E_2^2E_4^2 + 900900E_2^3E_3E_4 - 15400E_3^4 + 135135E_2^4E_5 +\\
&\quad +600600E_2^2E_3^3 - 3153150E_2^4E_3^2 - 945945E_2^5E_4 + 4729725E_2^6E_3 - 2027025E_2^8.
\end{aligned} \tag{27}$$

Differentiating the fourth term in formula (9), we finally find:

$$\mu_1^4 = -\frac{4S\,q}{a^2N_1} + S\,\tilde{V}_2, \quad N_1 = \frac{\partial N(z_o)}{\partial z}, \tag{28}$$

$$\mu_2^4 = \tilde{V}_2 - \frac{4\left(q^2 - S^2\right)}{a^2 q\, N_1} - \beta_2 S^2 q - S^2\,\tilde{V}_3, \tag{29}$$

$$\mu_3^4 = \frac{4\left(3d + d^3\right)}{a^2N_1} + \beta_2\,q^2\left(d^3 - 3d\right) + \beta_3\,q\,S^3 - 3S\,\tilde{V}_3 + S^3\,\tilde{V}_4, \quad d = \frac{S}{q}, \tag{30}$$

$$\mu_4^4 = \frac{\beta_1}{q^5} + \beta_2 q\left(6d^2 - 3 + d^4\right) + \beta_3 q^3\left(6d^2 - d^4\right) - \beta_4\,q\,S^4 - 3\,\tilde{V}_3 + 6S^2\tilde{V}_4 - S^4\tilde{V}_5, \tag{31}$$

$$\begin{aligned}\mu_5^4 = 5\frac{d\beta_1}{q^6} + \beta_2\left(15d + 10d^3 + 3d^5\right) + \beta_3\,q^2\left(15d - 10d^3 - d^5\right) + \beta_4\,q^4\left(d^5 - 10d^3\right) + \\ + \beta_5\,S^5 q + 15S\,\tilde{V}_4 - 10S^3\,\tilde{V}_5 + S^5\,\tilde{V}_6,\end{aligned} \tag{32}$$

$$\begin{aligned}\mu_6^4 = 5\frac{\beta_1}{q^7}\left(1 + 7d^2\right) + 15\beta_2\frac{\varepsilon^3}{q^7} + 3\beta_3\,q\left(5 - 15d^2 - 5d^4 - d^6\right) + \\ + \beta_4\,q^3\left(15d^4 + d^6 - 45d^2\right) + \beta_5\,q^5\left(15d^4 - d^6\right) - \beta_6 S^6 q + \\ + 15\,\tilde{V}_4 - 45S^2\,\tilde{V}_5 + 15S^4\,\tilde{V}_6 - S^6\,\tilde{V}_7,\end{aligned} \tag{33}$$

$$\begin{aligned}\mu_7^4 = 105\frac{\beta_1}{q^8}\left(d + 3d^3\right) + 105\beta_2\,S\frac{\varepsilon^3}{q^9} - 3\beta_3\left(35d + 35d^3 + 21d^5 + 5d^7\right) + \\ + \beta_4\,q^2\left(3d^7 + 21d^5 + 105d^3 - 105d\right) + \beta_5\,q^4\left(105d^3 - 21d^5 - d^7\right) + \\ + \beta_6\,q^6\left(d^7 - 21d^5\right) + \beta_7\,S^7 q - 105S\,\tilde{V}_5 + 105S^3\,\tilde{V}_6 - 21S^5\,\tilde{V}_7 + S^7\,\tilde{V}_8,\end{aligned} \tag{34}$$

$$\begin{aligned}\mu_8^4 = 105\frac{\beta_1}{q^9}\left(1 + 18d^2 + 33d^4\right) + 105\beta_2\left(1 + 9d^2\right)\frac{\varepsilon^3}{q^9} - 105\beta_3\frac{\varepsilon^4}{q^9} + \\ + 3\beta_4 q\left(5d^8 + 28d^6 + 70d^4 + 140d^2 - 35\right) + \beta_5\,q^3\left(420d^2 - 210d^4 - 28d^6 - 3d^8\right) + \\ + \beta_6 q^5\left(28d^6 + d^8 - 210d^4\right) + \beta_7\,q^7\left(28d^6 - d^8\right) - \beta_8 d^8 q^9 - 105\,\tilde{V}_5 + \\ + 420S^2\,\tilde{V}_6 - 210S^4\,\tilde{V}_7 + 28S^6\,\tilde{V}_8 - S^8\,\tilde{V}_9.\end{aligned} \tag{35}$$

In expressions (28)–(35) all values are calculated with $z = z_o$,

$$\tilde{V}_n = \frac{2^n}{a^{2(n-1)}}V_n, \quad \beta_1 = \frac{12\varepsilon^2}{a^2N_1}, \quad \beta_n = \frac{2^{n+1}}{a^{2n}N_1}\varphi_n, \quad n \geqslant 2. \tag{36}$$

Let us now turn to numerical results.

3. Numerical modeling

Consider a plasma layer with a cubic dependence of the normalized electron concentration on the height z:

$$N(z) = \begin{cases} Q\left(2 + p - (1 + 2p)\,Q + p\,Q^2\right), & Q = z - \delta, \quad z \geqslant \delta, \\ 0, & z \leqslant \delta. \end{cases} \tag{37}$$

If $p = 0$, then $N(z)$ is a parabolic plasma layer [22, 23]:

$$N(z) = \begin{cases} Q\left(2 - Q\right), & Q = z - \delta, \quad z \geqslant \delta, \\ 0, & z \leqslant \delta. \end{cases} \tag{38}$$

In order to find the center of the A_5 singularity («butterfly»), which is formally stable only in the four-dimensional space [19, 20], it is necessary that the first five derivatives of the eikonal (6) go to zero, and the sixth is not equal to zero. This is possible in the space of five parameters: x_o, z_o, δ, S, a. To do this, it is enough for us to numerically solve four equations

$$\mu_2 = 0, \;\; \mu_3 = 0, \;\; \mu_4 = 0, \;\; \mu_5 = 0 \tag{39}$$

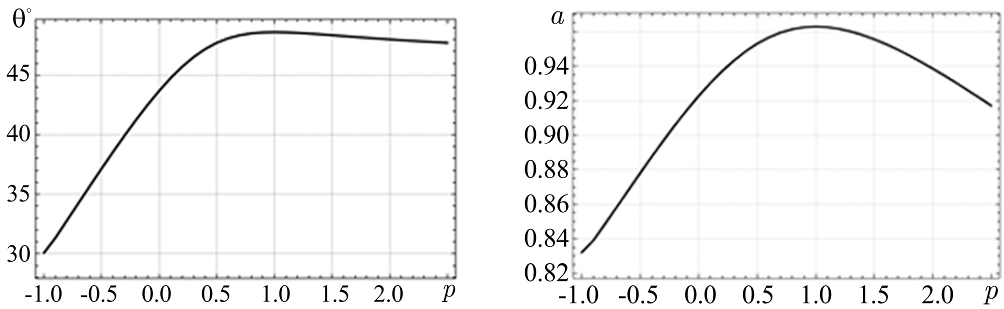
it is relative to the last four variables, since x_o is uniquely determined from the equation μ_1. In Fig. 3 and 4 show graphs of the dependencies of the studied parameters in the center of a topological singularity (catastrophe) of the «butterfly» type (that is, at the point with coordinates (x_o, z_o)) for different values of the parameter p, the cubicity parameter of the layer. In Fig. 3, a shows the dependence of the angle of exit of the ray θ from the radiation source on the cubicity of p ($\theta = \arccos S$). The behavior of θ is qualitatively different on the right and left sides of the drawing. If $p < 0.5$, the angle decreases sharply, and at $p > 0.7$, a slow decline is observed after the maximum ($\sim 49°$).

The dependencies $a(p)$ (Fig. 3, b), $\delta(p)$, $z_o(p)$ and $z_m(p)$ (Fig. 4) have a pronounced maximum, however, for different values of p. The function $a(p)$ has a maximum value at $p = 1$, when the ratio of plasma frequency to operating frequency slightly exceeds 0.96, and the dependences $\delta(p)$, $z_o(p)$ and $z_m(p)$ have a maximum at $p = 0$, when the plasma layer becomes parabolic. For the dependence of the horizontal coordinate of the singularities $x_o(p)$ the maximum is shifted to the region of negative values of p (Fig. 4, b).

In Fig. 5 shows the dependencies $\delta(x_o)$, $z_o(x_o)$ and $z_m(x_o)$, that is, the distance to the lower boundary of the layer, the height of the singularities, and the height of the reflection of the wave from the horizontal coordinate. All three curves have a maximum corresponding to (see Fig. 4, b) the parabolic layer.

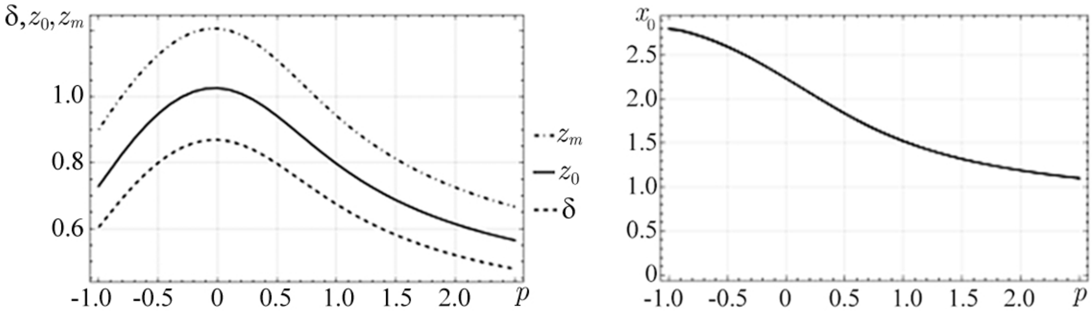
It may seem that the occurrence of high-order singularities in a simple unimodal layer is explained by the presence of a sharp layer boundary. However, this is not the case. In Fig. 6 here is an example of a cubic layer in which not only the function $N(z)$ itself is continuous, but also its derivative $\partial N(z)/\partial z$:

$$N(z) = \begin{cases} Q^2\,(3 - 2Q)\,, \;\; Q = z - \delta, & z \geqslant \delta, \\ 0, & z \leqslant \delta. \end{cases} \tag{40}$$



a *b*

Fig. 3. Dependence of the angle θ (a) and the frequency ratio a (b) on the cubicity parameter p



a *b*

Fig. 4. Dependence of the lower boundary of the layer δ, the position of the center of the singularity z_o and the height of the ray reflection z_m (a) and the coordinate of the singularity x_o on the cubicity p (b)

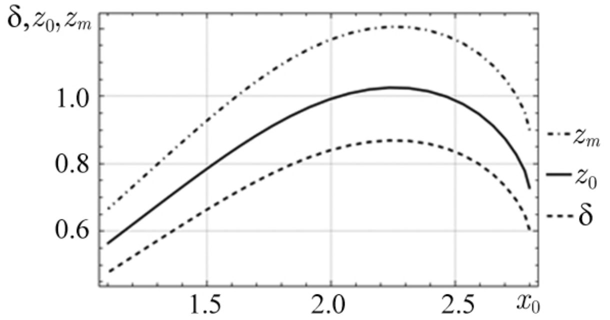


Fig. 5. Dependence of the lower boundary of the layer δ, the position of the center of the singularities z_o and the height of the ray reflection z_m on the coordinate x_o

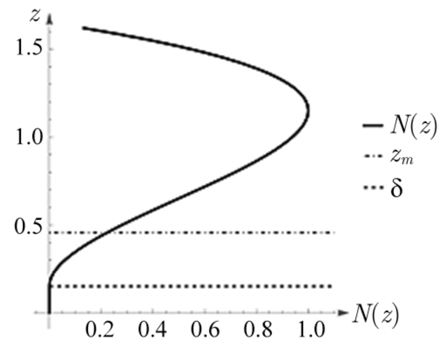


Fig. 6. An example of a cubic layer in which the function $N(z)$ and its derivative are continuous

However, with parameter values

$$a \approx 0.790607472644402, \quad \delta \approx 0.15076111644326395, \quad S \approx 0.9278253057832123 \tag{41}$$

at a point with coordinates: $z_o \approx 0.25804076672826454$, $x_o \approx 2.576153757958269$ «butterfly» type catastrophe is formed. In this case, the exit angle of the critical ray is $\theta \approx 21.901681711717607°$, and the reflection height is $z_m \approx 0.45597583256928126$.

Conclusion

Thus, the paper examines the possibilities of forming centers of topological caustic singularities such as catastrophes when probing the plasma layer with electromagnetic waves. In the vicinity of such singularities, the principle of superposition of rays is violated and focusing of wave fields occurs. A method for calculating the position of the centers of singularities in an expanded parameter space has been developed and mathematical modeling has been performed using the example of a plasma layer with a cubic dependence of the electron concentration on height. It is shown that when the ratio of the plasma frequency to the operating frequency, the height of the lower boundary of the plasma layer, and the angle of inclination of the ray output vary over a wide range, the center of the «butterfly» type singularities (catastrophe A_5) appears. These singularities occur even when not only the dependence of the electron concentration on height and its derivative is continuous.

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