Applied Problems of Nonlinear Oscillation and Wave Theory

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# Nonstationary scattering of elastic waves by a spherical inclusion

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***Abstract.*** Problems of elastic wave scattering by various types of inhomogeneities rank among the most complex and relevant topics in the field of deformable solid dynamics. From an applied perspective, this is due to the fact that information about the dynamic stress–strain state in the vicinity of such inhomogeneities is of significant interest for various engineering and physical applications. The *purpose* of this study is to investigate the nonstationary scattering of elastic waves by a spherical inclusion embedded in an infinite elastic medium. *Methods.* The analytical approach to the solution involves the application of Fourier integral transforms with respect to time. *Results.* It is established that the eigenfunctions of the considered problem cannot be treated as vectors in a Hilbert space, since they are not square-integrable due to their exponential growth with distance. This necessitates the use of generalized functions and specialized methods from scattering theory.

***Keywords***: spherical shell, wave scattering, wave amplitude, eigenfunctions, eigenfrequencies.

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## Introduction

The scattering of a plane wave by a single spherical obstacle is often found in many practical problems of geophysics and seismology [1,2]. In exploration geophysics, spherical objects provide a good approximation for real objects [3,4]. The analytical formulation of one sphere can be used to construct more complex solutions. In the oil industry, if oil is trapped in cavities, it is reasonable to assume that seismic energy can be trapped by liquid resonance. Such resonances are difficult to observe due to the impedance contrasts between rock and liquid [5,6]. Exact solutions for scattering problems are very relevant. Although analytical solutions exist for some types of obstacles (sphere, cylinder, or ellipsoid), the understanding gained is important [7,8]. The problems of diffraction of elastic waves on inhomogeneities are closely related to scattered waves, they are part of the classical fundamental problems of the dynamics of deformable bodies, and their solution requires a complex mathematical apparatus [9–11]. The addition of fillers is often used to affect the mechanical and thermal characteristics of materials [12,13].

The problem of scattering waves on spherical inhomogeneity was posed quite a long time ago when solving numerous scientific and technical problems related, in particular, to the diffraction of electromagnetic [14], sound [15] and elastic [16] waves. This problem is usually considered in a stationary formulation, when the incident wave is an infinite harmonic wave in space and time of the form $\exp\left[i\omega\left(t - z/c\right)\right]$.

In this case, a number of difficulties arise due to the fact that the eigenfunctions of the problem under study cannot be considered as vectors in a Hilbert space: they are not normalized due to exponential growth with distance. This fact, known in the general theory of scattering [17], follows from the following circumstance.

A scattered traveling wave going to infinity behaves like $u \sim r^{-1} \times \exp\left[i\omega^{(k)}\ \ (t - r/c)\right]$, where $\omega^{(k)} = \omega_R^{(k)} + i\omega_I^{(k)}$ are complex eigenfrequencies of the system «inclusion–medium» [18].

Thus, $u \sim r^{-1} \exp\left[-\omega_I^{(k)}\left(t - r/c\right)\right] \exp\left[i\omega_R^{(k)}\left(t - r/c\right)\right]$, and the amplitude of the scattered wave at a fixed point decreases over time due to radial losses. The spatial distribution of amplitudes at any given time increases exponentially with increasing $r$, since infinitely distant parts of the wave were excited at earlier time intervals when the amplitude of the oscillations of the inhomogeneity was infinitely large.

Naturally, the fact that the eigenfunctions increase infinitely does not have any real physical meaning, since, according to the principle of causality, there can be no signal at the point $r$ when $r > ct$. Since we are trying to replace the real situation with another one involving the introduction of a stationary process in an infinite space, we inevitably encounter an «exponential catastrophe».

To eliminate it, it is necessary to take into account that oscillations cannot exist for an infinitely long period of time, and, consequently, we come to the need to formulate and solve the problem of diffraction of a pulse of one form or another with a pronounced leading edge.

## 1. Methods

**1.1. Problem statement and methods of solution.** Let a plane displacement wave fall on a scattering center of radius $R$ placed at the origin of the spherical coordinate system $(r, \theta, \varphi)$, which is aligned with the Cartesian coordinate system $(x, y, z)$ in the usual way. The equations of motion of a spherical body $(k = 2)$ and its environment $(k = 1)$ have the following form:

$$\mu_k \operatorname{rot}\operatorname{rot}\vec{U} - (\lambda_k + 2\mu_k) \operatorname{grad}\operatorname{div}\vec{U} + \rho_k \frac{\partial^2 \vec{U}}{\partial t^2} = 0. \quad (1)$$

Here $\lambda_k, \mu_k$ are the Lame coefficients for the environment $(k = 1)$ and spherical bodies $(k = 2)$, $\rho_k$ are the densities of materials, $\vec{U}_k(u_{rk}, u_{\theta k}, u_{zk})$ is the displacement vector.

At the contact of two bodies at $r = R$, the equality of displacements and stresses is fulfilled (condition of rigid contact):

$$u_{r1} = u_{r2}, \quad u_{\theta 1} = u_{\theta 2}, \quad u_{\varphi 1} = u_{\varphi 2}, \quad \sigma_{rr1} = \sigma_{rr2}, \quad \sigma_{r\theta 1} = \sigma_{r\theta 2}, \quad \sigma_{r\varphi 1} = \sigma_{r\varphi 2}. \quad (2)$$

At infinity, the perturbations should attenuate:

$$\vec{U}_1 \to 0 \quad \text{with} \quad \sqrt{x^2 + y^2 + z^2} \to \infty.$$

The initial conditions are also set:

$$\vec{U}_k\Big|_{t=0} = 0, \quad \frac{\partial \vec{U}_k}{\partial t}\Big|_{t=0} = 0. \tag{3}$$

We denote by $c_{pk}$ and $c_{sk}$ (for $k = 1, 2$) the velocities of the longitudinal and transverse waves, respectively. For the sake of certainty, let us assume that the wave is moving in the positive direction of the $0z$ axis. Then the displacement vector

$$\vec{U}_1^{(p)} = \vec{e}_1\, U_0\left(\tau_q - \frac{z}{R} - 1\right) H\left(\tau_q - \frac{z}{R} - 1\right), \tag{4}$$

where $\vec{e}_1$ is the unit vector defining the wave polarization, $\tau_1 = c_1 e/R$ is a dimensionless time, $H(x)$ is a Heaviside step function.

To solve this problem, we use the Fourier integral time transformation, which is defined by the following formulas:

$$f_k(x_q) = \int_{-\infty}^{\infty} F_k(\tau_1)\, \exp(-ix_q\tau_1)\, d\tau_1, \quad F_k(\tau_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_k(x_q)\, \exp(ix_q\tau_1)\, dx_q, \tag{5}$$

where $x_q$ is a conversion parameter that has the meaning of a dimensionless frequency:

$$x_q = \frac{\omega R}{c_{p1}} = k_{q1}R.$$

$k_{11} = \omega/c_{p1}$; $\omega$ is frequency; $c_{p1} = \sqrt{\frac{\lambda_1+2\mu_1}{\rho_1}}$ is the velocity of propagation of longitudinal waves in the medium.

Lame equation (1) after applying transformation (5) takes the following form:

$$\mu_k\, \mathrm{rot}\, \mathrm{rot}\, \vec{u}_k - (\lambda_k + 2\mu_k)\, \mathrm{grad}\, \mathrm{div}\, \vec{u}_k - \rho_k\omega^2\vec{u}_k = 0. \tag{6}$$

Applying the Fourier transform to the incident pulse, we obtain

$$u_1^{(p)} = \vec{e}_1\, \exp(-ik_{q1}z)\, \eta(x_q). \tag{7}$$

Here

$$\eta(x_q) = \exp(-ix_q)\, g(x_q), \quad g(x_q) = \int U_0(T_q)\, \exp(-ix_qT_q)\, dT_q, \quad T_q = \tau_q - \frac{z}{R} - 1. \tag{8}$$

The solution of equations (4) is sought, as is known, by [7], in the form

$$\vec{u}_k = \frac{1}{k_{pk}}\, \mathrm{grad}\, \psi_{0k} + \frac{1}{k_{sk}}\, \mathrm{rot}\, \mathrm{rot}\, (\vec{r}\psi_{1k}) + \mathrm{rot}\, (\vec{r}\psi_{2k}), \tag{9}$$

moreover, the potentials $\psi_j$ satisfy the scalar Helmholtz equation, and the solution is expressed as follows:

$$(\psi_{0k}, \psi_{1k}, \psi_{2k}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (A_{mnk}, B_{mnk}, C_{mnk})\, b_n(lr)\, \Phi_n^m(0, \varphi), \quad k = \begin{cases} k_p, & j = 0, \\ k_s, & j = 1, 2. \end{cases} \tag{10}$$

Here $\Phi_n^{(m)}(\theta, \varphi) = P_n^m(\cos\theta)\exp(im\varphi)$, and $b_n(\xi)$ is a spherical Bessel function. For an external task as $b_n(\xi)$ it is necessary to take the Hankel function of the second kind: $b_n(\xi) = h_n^{(2)}(\xi) \equiv h_n(\xi)$, distinguishes divergent waves at infinity. For an internal problem $b_n(\xi) = j_n(\xi)$, which satisfies the boundedness condition at zero. An incident plane wave can be decomposed into regular vector eigenfunctions $\vec{L}^1_{\sigma mn}$, $\vec{M}^1_{\sigma mn}$, $\vec{N}^1_{\sigma mn}$ of the vector Helmholtz equations [19]:

$$\begin{aligned} \vec{e}_x\, \exp(-ikz) &= \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(-i)^n\left[M^1_{c1n} + iN^1_{c1n}\right], \\ \vec{e}_y\, \exp(-ikz) &= \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(-i)^n\left[M^1_{c1n} - iN^1_{s1n}\right], \\ \vec{e}_z\, \exp(-ikz) &= \sum_{n=0}^{\infty}(2n+1)(-i)^n\, L^1_{c0n}, \end{aligned} \tag{11}$$

where $s$ and $c$ denote $\sin\varphi$ and $\cos\varphi$ in the expressions for the eigenvectors.

The vectors $\vec{L}, \vec{M}, \vec{C}$ themselves are determined from the continuity conditions of the displacement vector:

$$\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_\varphi\vec{e}_\varphi,$$

as well as stress vectors:

$$\vec{\sigma}_n = \sigma_{rr}\vec{e}_r + \tau_{r\theta}\vec{e}_\theta + \tau_{r\varphi}\vec{e}_\varphi$$

at the boundary of the inhomogeneity and the host medium.

When $r = R$, the following relations must be fulfilled:

$$\vec{u}_1^{(p)} + \vec{u}_1 = \vec{u}_2, \qquad \vec{\sigma}_1^{(p)} + \vec{\sigma}_1 = \vec{\sigma}_2. \tag{12}$$

By calculating the displacements and stresses from the potentials $\psi_j$ and substituting the resulting expressions into the boundary conditions (12), as well as using the orthogonality of the spherical wave functions on the surface of the sphere, we arrive at a system of algebraic equations for determining the unknown coefficients. Note that in the case of scattering of a longitudinal wave pulse, as follows from (11), there is a degeneracy with respect to the parameter $m$ (for $m = 0$), and, therefore, there is no dependence on the azimuthal coordinate $\varphi$ in the general solution. In the case of scattering of a transverse wave, we will assume for definiteness that it is polarized in the direction $\vec{e}_z$. The scattered displacement field in the host medium is expressed by the following formulas (since we are only interested in the external field, we will omit the index 2 here and in the following):

$$\begin{aligned}
u_{r1} &= \eta(x_q)\cos(m\varphi)\sum_{n=m}^{\infty}\Omega_q\left[d_1(k_pr)\frac{A_{mn}}{k_pr} + n(n+1)h_n(k_sr)\frac{B_{mn}}{k_sr}\right]P_n^m(\cos\theta),\\
u_{\theta1} &= \eta(x_q)\cos(m\varphi)\sum_{n=m}^{\infty}\Omega_q\left\{\left[h_n(k_pr)\frac{A_{mn}}{k_pr} + d_2(k_sr)\frac{B_{mn}}{k_sr}\right]\tau_n(\theta) + ih_n(k_sr)\pi_n(\theta)C_{mn}\right\},\\
u_{\varphi1} &= -\eta(x_q)\sin(m\varphi)\sum_{n=m}^{\infty}\Omega_q\left\{\left[h_n(k_pr)\frac{A_{mn}}{k_pr} + d_2(k_sr)\frac{B_{mn}}{k_sr}\right]\tau_n(\theta) + ih_n(k_sr)\pi_n(\theta)C_{mn}\right\}.
\end{aligned} \tag{13}$$

Moreover, in the case of a longitudinal wave incident ($q = p$, $m = 0$):

$$\Omega_p = (-i)^{n+1}(2n+1), \quad C_{0n} = 0,$$

and in the case of a transverse wave incident ($q = s$, $m = 1$):

$$\Omega_p = (-i)^{n+1}\frac{2n+1}{n(n+1)}.$$

In equation (13), the following abbreviations are used:

$$d_1(k_pr) = nh_n(k_pr) - (k_pr)h_{n+1}(k_pr),$$

$$d_2(k_sr) = (n+1)h_n(k_sr) - (k_sr)h_{n+1}(k_sr),$$

$$\tau_n(\theta) = \frac{dP_n^m(\cos\theta)}{d\theta}, \qquad \pi_n(\theta) = \frac{P_n^m(\cos\theta)}{\sin\theta}.$$

The formulas for the scattering coefficients that characterize the external diffracted field can be represented as

$$\begin{aligned}
a_n &= \frac{\Delta_n^a(\omega)}{\Delta_n(\omega)}\cdot\frac{\gamma\, j_n(x_s)}{h_n(x_s)}, \qquad & b_n &= \frac{\Delta_n^b(\omega)}{\Delta_n(\omega)}\cdot\frac{j_n(x_s)}{h_n(x_s)},\\
c_n &= \frac{\Delta_n^c(\omega)}{\delta_n(\omega)}\cdot\frac{j_n(x_s)}{h_n(x_s)}, \qquad & A_n &= \frac{\Delta_n^A(\omega)}{\Delta_n(\omega)}\cdot\frac{j_n(x_p)}{h_n(x_p)},\\
B_n &= \frac{\Delta_n^E(\omega)}{\Delta_n(\omega)}\cdot\frac{j_n(x_p)}{\gamma\, h_n(x_s)}, \qquad & C_n &= 0.
\end{aligned} \tag{14}$$

Here $\gamma = c_{s2}/c_{p2}$, $a_n = A_{1n}$, $b_n = B_{1n}$, $c_n = C_{1n}$ are the scattering coefficients of the transverse wave, $A_n = A_{0n}$, $B_n = B_{0n}$, $C_n = C_{0n}$ are the scattering coefficients of the longitudinal wave, $\Delta_n^{a,bc,A,B}(\omega)$, $\Delta_n(\omega)$, $\delta_n(\omega)$ are some determinants whose elements depend on the ratios $j_{n+1}/j_n$ (Bessel functions) or $h_{n+1}/h_n$ (Hankel functions). It can be noted that the equations

$$\Delta_n(\omega) = 0, \quad \delta_n(\omega) = 0 \tag{15}$$

define, respectively, the eigen complex frequencies of the spheroidal and torsional oscillations of a spherical inhomogeneity in an infinite elastic medium [20].

The time domain solution is found by means of the inverse Fourier transform:

$$U(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\,\exp(ix\tau)\,dx, \tag{16}$$

where $u(x)$ is defined by formulas (13).

Let us write (16) explicitly in the wave zone approximation ($r \gg 1$). Neglecting terms of the order $r^{-2}$ and using the asymptotic representation of the Hankel function:

$$h_n(\xi) \equiv h_n^{(2)}(\xi) \sim \frac{1}{\xi} i^{n+1} e^{-i\xi},$$

we have

$$\begin{aligned}
U_r &= \frac{\cos m\varphi}{2\pi i}\frac{R}{r}\int_{-\infty}^{\infty} g(x_q)\frac{S_1(0,x_q)}{x_p}\exp\left(i\tau_q' x_q\right)dx_q,\\
U_\theta &= \frac{\cos m\varphi}{2\pi i}\frac{R}{r}\int_{-\infty}^{\infty} g(x_q)\frac{S_2(0,x_q)}{x_s}\exp\left(i\tau_q'' x_q\right)dx_q,\\
U_\varphi &= \frac{\sin m\varphi}{2\pi i}\frac{R}{r}\int_{-\infty}^{\infty} g(x_q)\frac{S_3(0,x_q)}{x_s}\exp\left(i\tau_q'' x_q\right)dx_q,
\end{aligned} \tag{17}$$

where the amplitude functions $S_j(0,x_q)$ are written as

$$\begin{aligned}
S_1(0,x_q) &= \sum_{n=m}^{\infty} i^{n+1}\,\Omega_q\,P_n^m(\cos\theta)\,A_{nm},\\
S_2(0,x_q) &= \sum_{n=m}^{\infty} i^{n+1}\,\Omega_q\left[\tau_n(\theta)\,B_{mn} + \pi_n(\theta)\,C_{mn}\right],\\
S_3(0,x_q) &= \sum_{n=m}^{\infty} i^{n+1}\,\Omega_q\left[\pi_n(\theta)\,B_{mn} + \tau_n(\theta)\,C_{mn}\right].
\end{aligned} \tag{18}$$

In the case of longitudinal pulse scattering

$$q = p, \quad m = 0, \quad C_{0n} = 0, \quad \tau_p' = \tau_p - \frac{r}{R} - 1, \quad \tau_p'' = \tau_p - \frac{r}{\gamma R} - 1.$$

For a transverse pulse

$$q = s, \quad m = 1, \quad C_{0n} = 0, \quad \tau_p' = \tau_s - \frac{\gamma r}{R} - 1, \quad \tau_s'' = \tau_s - \frac{r}{\gamma R} - 1.$$

In order to be able to calculate the integral (17) in a finite form, it is necessary to specify the form of the incident pulse (4). As a probing signal, we choose the Berlage pulse, which approximates the recordings of real seismic excitations with sufficient accuracy:

$$U_0(T_s) = T_s e^{-aT_s}\sin(bT_s),$$

where $a$ and $b$ are the parameters that define the pulse. Using (8), we obtain the signal spectrum:

$$g(x_s) = \frac{2b(a + ix_s)}{\left[(a + ix_s)^2 + b^2\right]^2}.$$

In principle, the integrals (16) and (17) can be calculated approximately using a computer by direct numerical integration. The theory of residues is used to obtain the expression for the displacements and stresses in the originals. This method provides a physically sound description of the process under consideration. By replacing the integration over $x_q$ with integration over a closed contour consisting of the real axis and a semicircle in the complex half-plane, the integrals are reduced to a sum of residues at the poles of the integrand. Some of these poles are the roots of the equations (15). This means that the expansion is based on functions whose arguments include complex eigenfrequencies, i.e., the eigenfunctions of the spherical elastic inhomogeneity. Let us rewrite (17) in the following form:

$$U_n = f(n,r,\theta,\varphi)\int_{-\infty}^{\infty} g(x_p)\frac{\Delta_n^A(x_p)j_n(x_p)}{x_p\,\Delta_n(x_p)h_n(x_p)}\exp\left(ix_p\left(\tau_p-\frac{r}{R}-1\right)\right)dx_p. \tag{19}$$

We choose the incident pulse such that its spectrum $g(z)\to 0$ as $|z|\to\infty$. Therefore, the expression in parentheses tends to zero uniformly.

Under the condition $\tau_p - r/R - 1 > 0$ the conditions of the Jordan lemma [21] are met, and the integration over the infinite semicircle can be neglected.

We note that the last inequality reflects the causality principle: the signal cannot arrive at the point $r+R$ before the time $t=(r+R)/c_p$. Thus, using the residue theory to calculate the integral (19), we obtain:

$$U_n = 2\pi i\, f(n,r,\theta,\varphi)\sum_{k,m}\operatorname{res}\left\{\frac{g(z)\,\Delta_n^A(z)\,j_n(z)}{z\,\Delta_n(z)\,h_n(z)}\right\}_{\substack{(k)z=z_1\\ z=z_2}}\exp(iz\tau_p')H(\tau_p'), \tag{20}$$

where $z_1^{(k)}$ are the poles of the functions $\Delta_n(z)h_n(z)$, and the poles of the function $g(z)$.

If the spherical inhomogeneity differs little from the surrounding medium, then the imaginary part of $z_1^{(k)}$ is small, and the poles lie close to the real axis.

In geophysical applications, it is often the case that the inclusion under consideration is sufficiently large in size $x=\omega R/c = 2\pi R/\lambda \geqslant 1$ and does not differ too much in its elastic and density properties from the surrounding medium (by 20–30% in velocities and by 3–5% in density).

In this case, the amplitude functions $S_j$ can be obtained using simple approximate formulas.

It is further shown in [21] that a weakly contrasting inhomogeneity is characterized by the following equalities:

$$S_1(\theta,x_p)\approx S_2(\theta,x_s)\approx S_3(\theta,x_s)\approx S(\theta,x_q),$$

and an expression for $S(\theta,x_q)$ is obtained in the form of a sum of two terms:

$$S(\theta,x_q) = S_d(\theta,x_q) + S_\tau(\theta,x_q).$$

The first term gives the well-known Fraunhofer diffraction pattern, and the second term is due to the rays that pass through the inclusion.

Physically, this is quite justified, since the rays that undergo reflections inside the sphere can be neglected due to the weak contrast of the inhomogeneity.

However, it should be noted that while the diffraction term

$$S_d(\theta,x_q) = x_q^2\sqrt{\frac{\theta}{\sin\theta}}\frac{I_1(\theta x_q)}{\theta x_q} \tag{21}$$

satisfies the requirements of geophysical accuracy [22], the second term

$$S_\tau(\theta,x_q) = -2ix_q\sqrt{\frac{\theta}{\sin\theta}}\frac{\alpha_q-1}{4(\alpha_q-1)^2+\theta^2}\exp\left[-ix_q\sqrt{4(\alpha_q-1)^2+\theta^2}\right], \tag{22}$$

where $\alpha_q = c_{q2}/c_{q1}$, is generally valid only in the small-angle approximation ($\theta\approx 0$) and is highly approximate.

Using the weak-contrast approximation, the scattering coefficients can be represented as follows:

$$A_n \approx b_n \approx c_n \approx \frac{1}{2}\left(1-e^{-\frac{2}{\varepsilon_\alpha}}\right), \tag{23}$$

where

$$\varepsilon_q = x\left[(\alpha \sin d_1 - \sin d) + (d - d_1)\cos d\right],$$
$$\cos d = \frac{\gamma}{x}, \quad \cos d_1 = \frac{\gamma}{\alpha x}, \quad \left(\gamma = n + \frac{1}{2} \leqslant x\right).$$

Substituting (23) into (22) and replacing the Legendre functions with their asymptotic representations, we obtain [23]:

$$S(\theta, x) \approx \sqrt{\frac{2}{\pi \sin\theta}} \sum \sqrt{\gamma} \cos\left(\gamma\theta - \frac{\pi}{4}\right)\left[1 - \exp(-2i\varepsilon_a)\right]. \tag{24}$$

The first term in the square brackets, which is equal to 1, gives the Fraunhofer diffraction pattern. To evaluate the second term, we replace the sum with an integral, which can be expressed as follows:

$$S_\tau(\theta, x) \approx \sqrt{\frac{2}{\pi \sin\theta}} \frac{1}{2}\left[\int_0^x \sqrt{\gamma}\, e^{i\varphi_\bullet(\gamma)}\, d\gamma + \int_0^x \sqrt{\gamma}\, e^{i\varphi_-(\gamma)}\, d\gamma\right], \tag{25}$$

where

$$\varphi_\bullet = 2x\left[\sin d - \alpha \sin d_1 + (d_1 - d)\cos d\right] + \left(\gamma\theta - \frac{\pi}{4}\right),$$
$$\varphi_- = 2x\left[\sin d - \alpha \sin d_1 + (d_1 - d)\cos d\right] - \left(\gamma\theta - \frac{\pi}{4}\right).$$

We calculate the integrals asymptotically using the stationary phase method [24]:

$$\int \sqrt{\gamma}\, e^{i\varphi(\gamma)}\, d\gamma \approx \sqrt{\frac{2\pi\gamma_0}{|\varphi''(\gamma_0)|}} \exp\left\{i\left[\varphi(\gamma_0) + \frac{\pi}{4}\operatorname{sign}\varphi''(\gamma_0)\right]\right\}. \tag{26}$$

First, we find the point $\gamma_0$ where the phase is stationary, which satisfies the condition $\varphi'_{\pm}(\gamma_0) = 0$, i.e., $2(d_1 - d) \pm \theta = 0$. To obtain an approximate scattering field, we use the method developed by Dubrovsky VA and Morozhnik VS [25].

The expression for the scattered transverse momentum displacement field can be expressed in terms of the calculated integrals as follows:

$$\begin{aligned} U_r &\approx 0,\\ U_\theta &= \frac{R}{r}\cos\varphi\left[\sqrt{\frac{\theta}{\sin\theta}} I_1^s - I_2^s\right],\\ U_\varphi &= -\frac{R}{r}\sin\varphi\left[\sqrt{\frac{\theta}{\sin\theta}} I_1^s - I_2^s\right]. \end{aligned} \tag{27}$$

Here

$$I_1^s = \frac{1}{2\pi i}\int_{-\infty}^{\infty} x_s\, g(x_s)\, \frac{J_1(\theta x_s)}{\theta x_s}\, \exp\left(i\tau_s'' x_s\right) dx_s. \tag{28}$$

In the case of $\tau_s'' \geqslant 0$ it can be calculated using residue theory, since the conditions of Jordan lemma are fulfilled. As a result, we get

$$I_1^s = \operatorname{Re}\left\{e^{i\tau_s''\nu}\left[iJ_0(\theta\nu) - \frac{J_1(\theta\nu)}{\theta\nu}\left(\tau_s''\nu + i\right)\right]\right\} \quad (\tau_s'' \geqslant 0), \quad \nu = b + ia.$$

If $\tau_s'' < 0$, then we will use the integral representation [24]:

$$\frac{J_1(\theta x)}{\theta x} = \frac{1}{\pi}\int_{-1}^{1} \exp(i\theta x w)\sqrt{1 - w^2}\, dw.$$

Substituting the last formula in (26) and changing the order of integration, we calculate the internal integral according to the theory of residues:

$$I_1^s = \frac{1}{\pi}\int_{-1}^{1} \exp(-\alpha y)\sqrt{1 - w^2}\left[(ay - 1)\sin(by) - by\cos(by)\right] H(y)\, dw,$$

where $y = \tau_s'' + \theta w$. In the case of longitudinal pulse scattering, we have

$$U_\theta \approx 0,$$
$$U_r \approx \frac{R}{r}\left[-\frac{i}{2\pi}\sqrt{\frac{\theta}{\sin\theta}}\int_{-\infty}^{\infty} g(z_s)\, z_s \frac{j_1(\theta z_s)}{\theta z_s} e^{i\tau_s'' z_s}\, dz_s - D_s(\tau_s'' - 2\delta_s)e^{-2\delta_s}\right], \tag{29}$$

where the integrals are calculated numerically by the Romberg method.

The numerical results were obtained on the basis of the integrated MATLAB software. The roots (poles) of the transcendental equation are found using the Muller method.

## 2. Results and analysis

The following parameter values are used in calculations:

$$\alpha_p = \frac{c_{p1}}{c_{p2}} = 0.91, \quad \alpha_s = \frac{c_{s1}}{c_{s2}} = 0.75, \quad \gamma = \frac{c_{p1}}{c_{s2}} = 0.72, \quad \eta = \frac{\rho_1}{\rho_2} = 0.94.$$

Fig. 1 shows the results of calculations of the scattered radial component of the displacement $u_r$ according to the developed methods for the case of scattering of a longitudinal wave pulse. The comparison results for $\theta = 10°$ with the exact data of [25] coincide with differences of up to 9%.

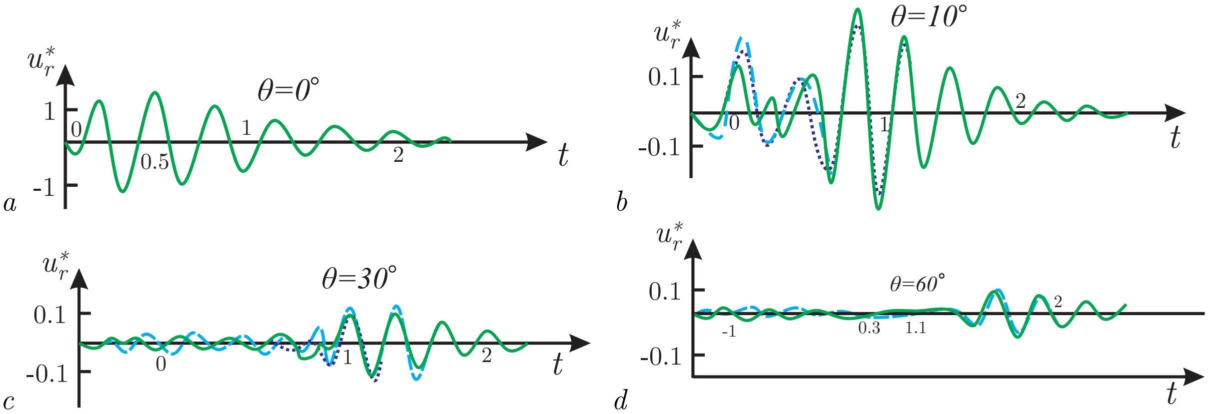


Fig. 1. Time dependence of the scattering of the shear component under the incidence of a longitudinal pulse

In accordance with the specified parameter $\alpha_p$, it is determined that approximate formulas can be used in the range of scattering angles

$$0 \leqslant \theta \leqslant 68°.$$

It should also be pointed out that at $\theta = 0°$, the moment of entry of the scattered wave turns out to be somewhat «blurred». This is due to the Gibbs phenomenon, and to eliminate it, it is necessary, generally speaking, to smooth the leading edge of the incident pulse.

Numerical calculations have shown that relatively simple approximate formulas can be used to solve the problem of unsteady wave scattering on a low-contrast inhomogeneity.

## Conclusion

A method has been developed for calculating wave scattering in a spherical body when longitudinal or transverse waves are incident. A method and algorithm for calculating special Bessel and Hankel functions with a complex argument have also been developed. The results obtained using the developed method have been compared with known methods.

It has been established that the proposed approximate formulas (Dubrovsky VA and Morozhnik VS) can be used in the range of scattering angles $0 \leqslant \theta \leqslant 68^\circ$.

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