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Local dynamics of aperiodic chains with unidirectional couplings

S. A. Kashchenko

P. G. Demidov Yaroslavl State University, Russia
E-mail: ✉kasch@uniyar.ac.ru

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Abstract. Chains of N unidirectionally coupled nonlinear first-order equations are considered, where the value of the last element is determined through the first element of the chain. The *aim* of this work is to investigate the local — in the neighborhood of the zero equilibrium state — dynamics of this system. Critical cases in the problem of equilibrium state stability are identified, and normal forms determining the local behavior of solutions are constructed. A detailed analysis is performed in the simplest cases, where $N = 2$ and $N = 3$. The most interesting part of the research concerns the case where the value of N is sufficiently large. It is shown that the critical cases then have infinite dimension. *Methods.* The standard research scheme, based on the use of the method of local invariant manifolds and the method of normal forms, turns out to be inapplicable. A special method of infinite-dimensional normalization developed by the author is used. The main *results* consist in the construction of so-called quasi-normal forms — analogs of normal forms for the infinite-dimensional case. It is important to emphasize that even for sufficiently large values of the number of chain elements N, the quasi-normal forms determining the dynamics of the original system significantly depend on variations in the value of N. Note that for certain values of the system coefficients, its dynamics can be quite complex.

Keywords: dynamics, differential equation, chain, normal form, stability.

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Problem statement

Let us consider one of the simplest nonlinear equations of the first order

$$\dot{u} + au = f(u), \tag{1}$$

where $a > 0$, and a sufficiently smooth function $f(u)$ has an order of smallness higher than the first one at zero:

$$f(u) = f_2u^2 + f_3u^3 + O(u^4).$$

A chain of N equations of the form (1) with unidirectional couplings is called a system of equations

$$\dot{u}_j + au_j = f(u_j) + bu_{j+1} \quad (b \neq 0), \tag{2}$$

in which $j = 1, \dots, N$ and at the right end of this chain for $u_{N+1}(t)$ the boundary condition is met

$$u_{N+1} = \gamma u_1 \quad (\gamma \neq 0). \tag{3}$$

Chains of the form (2) are important objects for research. They receive special attention. Such chains arise in the modeling of many applied problems in radiophysics [1–8], laser physics [9–13], mathematical ecology [14,15], neural network theory [16–21], optics [3,8,22,23], biophysics [24], etc. Relaxation oscillations in coupled chains with finite nonlinearity and delay for a small number of elements were studied in [25,26]. We also note the work [27], which considers the dynamics of a periodic chain with a large number of elements.

Let us set the task of investigating the behavior at $t \to \infty$ of all solutions of the chain (2), (3) with initial conditions from some sufficiently small neighborhood of the zero equilibrium state.

The linearized system of equations plays an important role in this problem:

$$\dot{u}_j + a u_j = b u_{j+1}, \quad u_{N+1} = \gamma u_1 \quad (j = 1, \dots, N). \tag{4}$$

The characteristic equation for system (4) has the form

$$[(\lambda + a) b^{-1}]^N = \gamma, \tag{5}$$

therefore, for the roots $\lambda_1, \dots, \lambda_N$ of this equation, the equalities are true

$$\ln \gamma = \ln |\gamma| + i \arg(\gamma), \tag{6}$$

where $\arg(\gamma) = 0$ for $\gamma > 0$ and $\arg(\gamma) = \pi$ for $\gamma < 0$.

Provided that all N roots of (6) have negative real parts, all solutions of system (4) and system (2), (3) with initial conditions from a sufficiently small neighborhood of the zero equilibrium state tend to zero at $t \to \infty$. If (6) has a root with a positive real part, then system (4) has an exponentially growing solution at $t \to \infty$ and the problem of dynamics of (2), (3) ceases to be local: its zero solution is unstable, and there cannot be an attractor in its sufficiently small neighborhood.

Below we will consider the critical case in the stability problem when (6) has no roots with a positive real part, but there is a root with a zero real part.

Since the parameter a is positive, for sufficiently small values of the parameter b, all the roots of (6) have negative real parts. Using b^+ we will denote the smallest positive value of the parameter b, at which (6) has a root with a zero real part. If such a value does not exist, then we assume $b^+ = \infty$. Accordingly, using b^-, we denote the largest negative value of b (if it exists, otherwise we put $b^- = -\infty$). Thus, for $b \in (b^-, b^+)$ all the roots of (6) have negative real parts.

Let us introduce two more values into consideration: γ^+ and γ^-, which are «similar» in meaning to b^+ and b^- , respectively. For small values of γ all the roots of (6) have negative real parts. Using γ^+ we will denote the smallest positive value of the parameter γ, at which (6) has a root with a zero real part. If such a value does not exist, then we assume $\gamma^+ = \infty$. Accordingly, using γ^- , we denote the largest negative value of γ (if it exists, otherwise we put $\gamma^- = -\infty$). Thus, for $\gamma \in (\gamma^-, \gamma^+)$ all the roots of (6) have negative real parts.

In sections 1 and 2, we will study two situations when $N = 2$ and $N = 3$. In section 3 , we present the results for an arbitrary N. In section 4, which is central to this work, it is assumed that the number of equations N is large enough, that is

$$N \gg 1. \tag{7}$$

In particular, the values of $b^\pm$ and $\gamma^\pm$ will be defined for these cases. Methodologically, the studies of local dynamics in sections 1–3 are based on the use of methods of local invariant integral manifolds and the method of normal forms (see, for example, [28, 29]). In the conditions of section 4, these methods are not directly applicable, since the critical cases then have an infinite dimension. A special infinite-dimensional normalization method [13, 14, 30] developed by the author is used. The main results consist in the construction of so-called quasi-normal forms, analogues of normal forms for the infinite-dimensional case.

In terms of one of the important generalizations of the chain model (2), (3), we point out that the results obtained extend to chains of equations (1) with other unidirectional couplings

$$\dot{u}_j + a u_j = f(u_j) + b(u_{j+1} - u_j),$$

in which, as for chain (2),

$$j = 1, \ldots, N; \quad u_{N+1} = \gamma u_1.$$

Note that in the most interesting case (7), chains for which the «periodicity» condition is met

$$u_{N+1} = u_1,$$

were studied in [27]. We should immediately emphasize that boundary condition (3) at $\gamma \neq 1$ fundamentally complicates the dynamic properties of system (2).

1. Case of $N = 2$

This case is the simplest. A system of two equations is considered

$$\begin{aligned} \dot{u}_1 + au_1 &= f(u_1) + bu_2, \\ \dot{u}_2 + au_2 &= f(u_2) + b\gamma u_1. \end{aligned} \tag{8}$$

For $\gamma < 0$, we have $b^{\pm} = \pm\infty$. Thus, for all b, the roots of (6) have negative real parts.

Let

$$\gamma > 0.$$

Then $b^{\pm} = \pm a(\sqrt{\gamma})^{-1}$ ($\sqrt{\gamma} > 0$ is the arithmetic root of γ). For $b = b^{\pm}$, the linear system (4) (for $N = 2$) has constant solutions

$$\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} = \begin{pmatrix} b^{\pm} \\ a \end{pmatrix} \cdot \text{const}.$$

We arbitrarily fix the value of b_1 and enter the small parameter $\varepsilon : 0 < \varepsilon \ll 1$. Let us put in (8)

$$b = b^{\pm} + \varepsilon b_1. \tag{9}$$

Then in (6) there is one negative (and separated from zero at $\varepsilon \to 0$) root and one root of $\lambda_0(\varepsilon)$, close to zero:

$$\lambda_0(\varepsilon) = \varepsilon b_1 \sqrt{\gamma} + O(\varepsilon^2).$$

For small ε in the phase space of system (8), there is a local invariant one-dimensional integral stable manifold (see, for example, [31]) on which system (8) (under some condition of non-degeneracy) up to terms of the order of $O(\varepsilon)$ takes the form of a scalar ordinary differential equation

$$\frac{d\xi}{d\tau} = b_1\sqrt{\gamma}\,\xi + a\big(1 + (\sqrt{\gamma})^{-1}\big)\xi^2, \tag{10}$$

where $\tau = \varepsilon t$ is «slow» time, and the function $\xi(\tau)$ is related to the solutions of (8) by the asymptotic equality

$$\begin{pmatrix} u_1(t,\varepsilon) \\ u_2(t,\varepsilon) \end{pmatrix} = \varepsilon\xi(\tau)\begin{pmatrix} b^{\pm} \\ a \end{pmatrix} + O(\varepsilon^2). \tag{11}$$

At $b_1 \neq 0$, equation (10) has a nonzero equilibrium state $\xi_0 = -b_1\sqrt{\gamma}\big[a + \big(1 + (\sqrt{\gamma})^{-1}\big)\big]^{-1}$. It is stable at $b_1 > 0$ and unstable at $b_1 < 0$. Therefore, system (8) has an equilibrium state for $\gamma > 0$, provided (9) and for sufficiently small ε

$$\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} = \varepsilon\ \xi_0\begin{pmatrix} b^{\pm} \\ a \end{pmatrix} + O(\varepsilon^2),$$

which is stable (unstable) at $b_1 > 0$ ($b_1 < 0$). In the considered close to critical case, equation (10) is called the normal form. The non-degeneracy condition mentioned above is that $f_2 \neq 0$. For $f_2 = 0$ and $f_3 \neq 0$, the changes are not significant. In normal form, the quadratic summand is replaced by a cubic one, and the asymptotic expansion, an analog of (11), proceeds in powers of $\varepsilon^{1/2}$.

Thus, the study of the local dynamics of system (8) is completed.

For system (8), we give the values γ^+ and γ^- :

$$\gamma^+ = \left(\frac{a}{b}\right)^2, \quad \gamma^- = -\infty.$$

2. Case of $N = 3$

System (2), (3) for $N = 3$ takes the form

$$\begin{aligned} \dot{u}_1 + au_1 =& f(u_1) + bu_2, \\ \dot{u}_2 + au_2 =& f(u_2) + bu_3, \\ \dot{u}_3 + au_3 =& f(u_3) + b\gamma u_1. \end{aligned} \tag{12}$$

For a linearized system

$$\dot{v} = A_\gamma v, \quad \text{where} \;\; v = (v_1, v_2, v_3), \quad A_\gamma = \begin{pmatrix} -a & b & 0 \\ 0 & -a & b \\ b\gamma & 0 & -a \end{pmatrix}, \tag{13}$$

roots λ_1, λ_2 and λ_3 of a characteristic equation are defined by the equalities

$$\lambda_1 + a = b\sqrt[3]{\gamma}, \quad \lambda_2 + a = b\sqrt[3]{\gamma}\Big(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\Big), \quad \lambda_3 + a = b\sqrt[3]{\gamma}\Big(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\Big), \tag{14}$$

where $\sqrt[3]{\gamma}$ is arithmetic root ($\sqrt[3]{\gamma} > 0$ for $\gamma > 0$ and $\sqrt[3]{\gamma} < 0$ for $\gamma < 0$).

The following equalities are true for values of $\gamma^\pm$

$$\gamma^+ = \begin{cases} \Big(\dfrac{a}{b}\Big)^3, & \text{if } b > 0, \\ \Big(\dfrac{2a}{|b|}\Big)^3, & \text{if } b < 0, \end{cases} \qquad \gamma^- = \begin{cases} -\Big(\dfrac{2a}{b}\Big)^3, & \text{if } b > 0, \\ \Big(\dfrac{a}{b}\Big)^3, & \text{if } b < 0. \end{cases}$$

Here are the values of $b^\pm$:

$$b^+ = \begin{cases} \dfrac{a}{\sqrt[3]{\gamma}}, & \text{if } \gamma > 0, \\ -\dfrac{2a}{\sqrt[3]{\gamma}}, & \text{if } \gamma < 0, \end{cases} \qquad b^- = \begin{cases} -\dfrac{2a}{\sqrt[3]{\gamma}}, & \text{if } \gamma > 0, \\ \dfrac{a}{\sqrt[3]{\gamma}}, & \text{if } \gamma < 0. \end{cases}$$

Under the conditions $\gamma \in (\gamma^-, \gamma^+)$ $\big(b \in (b^-, b^+)\big)$ roots (14) have negative real parts, and for $\gamma \in (-\infty, \gamma^-)$ and $\gamma \in (\gamma^+, \infty)$ $\big(b \in (-\infty, b^-)$ and $b \in (b^+, \infty)\big)$ among roots (14) there is a root with a positive real part. Under the conditions $\gamma = \gamma^\pm$ $\big(b = b^\pm\big)$ in the solution stability problem (12), critical cases of a zero root or critical cases of a pair of purely imaginary roots arise. Let us consider them.

2.1. The critical case of the zero root. This case occurs when $b > 0$ and $\gamma = \gamma^+$, or when $b < 0$ and $\gamma = \gamma^-$. Let us limit ourselves to considering only the first of these conditions, that is, we consider below that

$$b > 0 \;\; \text{and} \;\; \gamma = \gamma^+ = \Big(\frac{a}{b}\Big)^3.$$

Linear system (13) for $\gamma = \gamma^+$ has constant solutions $v = d_0 = \text{const}$, where $d_0 = \big(1, ab^{-1}, a^2b^{-2}\big)$.

We arbitrarily fix the value of γ_1 and put it in (12)

$$\gamma = \gamma^+ + \varepsilon\gamma_1, \quad 0 < \varepsilon \ll 1.$$

Then the roots λ_2 and λ_3 have negative real parts: $\operatorname{Re}\lambda_{2,3} = -\frac{1}{2}a + O(\varepsilon)$ for small ε, and for the root $\lambda_1(\varepsilon)$ holds the asymptotic equality

$$\lambda(\varepsilon) = \varepsilon\mu_1\gamma_1 + O(\varepsilon^2), \;\; \text{where} \;\; \mu_1 = b^3(3a^2)^{-1}.$$

It follows that in a sufficiently small and independent of ε neighborhood of the zero state of equilibrium of system (12), there exists a stable local invariant one-dimensional integral manifold on which this system can be represented as a normal form up to $O(\varepsilon)$ (under a certain non-degeneracy condition)

$$\frac{d\xi}{d\tau} = \alpha\xi + \beta\xi^2, \quad \tau = \varepsilon t. \tag{15}$$

To determine the coefficients α and β we substitute to (12) the solution $u = (u_1, u_2, u_3)$ in the form of an asymptotic series

$$u(t,\varepsilon) = \varepsilon\xi(\tau)d_0 + \varepsilon^2 U_2(\tau) + \ldots.$$

By collecting the coefficients at the first degree of ε in the resulting formal identity, we obtain the correct equality, and taking into account the coefficients at ε^2, we arrive at a system for determining the function $U_2(\tau)$:

$$A_{\gamma_+}U_2 = -d_0\frac{d\xi}{d\tau} + f_2\xi^2 d_0\cdot d_0 + b\gamma_1\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{16}$$

here and below, the multiplication of vectors is coordinate-wise.

System (16) is solvable if and only if its right side is orthogonal to the vector $h_0 = (1, ba^{-1}, b^2a^{-2})$, a nonzero solution of the homogeneous conjugate equation $A^*h_0 = 0$. Considering this, we obtain that in (15)

$$\alpha = \mu_1\gamma_1 = b^3(3a^2)^{-1}\gamma_1, \quad \beta = \frac{1}{3}f_2(d_0\cdot d_0, h_0). \tag{17}$$

The non-degeneracy condition mentioned above consists in fulfilling the inequality $f_2 \neq 0$. Using (17) in (15), we get a complete picture of the behavior of solutions (15), and hence solutions (12) in a small neighborhood of the zero equilibrium state.

2.2. The critical case of a pair of purely imaginary roots. This case occurs under the conditions

$$b < 0 \ \text{ and } \ \gamma^+ = \left(\frac{2a}{|b|}\right)^3, \quad \text{or} \quad b > 0 \ \text{ and } \ \gamma^- = -\left(\frac{2a}{b}\right)^3.$$

Let the first of these conditions be fulfilled

$$b < 0, \quad \gamma^+ = \left(\frac{2a}{|b|}\right)^3.$$

Then $\lambda_1 = -a + b\sqrt[3]{\gamma^+} < 0$ and $\lambda_{2,3} = \pm i\sqrt{3}a$. Linear system (13) has periodic solutions

$$v_0(t) = g_0\exp(ia\sqrt{3}t), \quad g_0 = \begin{pmatrix} -{\gamma^+}^{1/3}(1+i\sqrt{3}) \\ {\gamma^+}^{2/3} \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2}\end{pmatrix}.$$

We arbitrarily fix the value of γ_1 and put it in (12) and (13):

$$\gamma = \gamma^+ + \varepsilon\gamma_1, \quad 0 < \varepsilon \ll 1.$$

For all sufficiently small ε in a sufficiently small neighborhood of the zero equilibrium state (12) independent of ε, there exists (see, for example, [31]) a two-dimensional stable locally invariant integral manifold on which system (12) can be up to terms of the order ε are represented in the form of a normal form, a complex scalar ordinary differential equation of the first order of the form

$$\frac{d\xi}{d\tau} = \delta\xi + \sigma\xi|\xi|^2, \quad \tau = \varepsilon t. \tag{18}$$

To determine the coefficients δ and σ, we substitute the solution in (12) in the form of a formal series

$$U(t,\varepsilon) = \varepsilon^{1/2}\big(\xi(\tau)g_0\exp(ia\sqrt{3}t) + \bar{\xi}(\tau)\bar{g}_0\exp(-ia\sqrt{3}t)\big) + \varepsilon U_2(t,\tau) + \\ + \varepsilon^{3/2}U_3(t,\tau) + \ldots,$$

where the dependence on t is $2\pi(a\sqrt{3})^{-1}$-periodic. In the resulting formal identity, we will collect coefficients with the same degrees of ε. In the first step, collecting the coefficients for $\varepsilon^{1/2}$, we arrive at the correct equality. In the next step, we obtain a system of equations for determining the function $U_2(t,\tau) = U_{20}|\xi|^2 + U_{21}\xi^2\exp(2ia\sqrt{3}t) + \bar{U}_{21}\bar{\xi}\exp(-2ia\sqrt{3}t)$:

$$AU_{20} = f_2\begin{pmatrix} 4(\gamma^+)^{2/3} \\ (\gamma^+)^{4/3} \\ 1 \end{pmatrix}, \quad (A - 2ia\sqrt{3}I)U_{21} = f_2\begin{pmatrix} (\gamma^+)^{2/3}(4+2i\sqrt{3}) \\ (\gamma^+)^{4/3} \\ 1 - \frac{1}{2}i\sqrt{3} \end{pmatrix}.$$

Hence we find that

$$U_{20} = f_2A^{-1}\begin{pmatrix} 4(\gamma^+)^{2/3} \\ (\gamma^+)^{4/3} \\ 1 \end{pmatrix}, \quad U_{21} = f_2(A - 2ia\sqrt{3}I)^{-1}\begin{pmatrix} (\gamma^+)^{2/3}(4+2i\sqrt{3}) \\ (\gamma^+)^{4/3} \\ 1 - \frac{1}{2}i\sqrt{3} \end{pmatrix}.$$

In the third step, we collect the coefficients for $\varepsilon^{3/2}$. As a result, we arrive at a system of equations with respect to the vector function $U_3(t,\tau)$, which we will look for in the form

$$U_3(t,\tau) = U_{31}(\tau)\exp\left(ia\sqrt{3}t\right) + \overline{cc} + U_{33}(\tau)\exp\left(3ia\sqrt{3}t\right) + \overline{cc}.$$

Here and below, $\overline{cc}$ is used to denote the summand that is complex conjugate to the previous one.

The expression fo $U_{33}(\tau)$ is just found. It will not be needed further down, so we will omit it. For $U_{31}(\tau)$, we obtain a system of equations

$$\left(A_{\gamma_+} - ia\sqrt{3}I\right)U_{31}(\tau) = -b\gamma_1(\gamma^+)^{1/3}(1+i\sqrt{3})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\xi - g_0\frac{d\xi}{d\tau} + \xi|\xi|^2B, \tag{19}$$

where $B = 2f_2(g_0U_{20} + \bar{g}_0U_{21}) + 3f_3g_0\cdot g_0\cdot\bar{g}_0$.

A necessary and sufficient condition for the solvability of this system is the condition that the right-hand side (19) is orthogonal to the vector h, a nonzero solution of the homogeneous conjugate equation $A_{\gamma_+}h = -ia\sqrt{3}h$. We find that $h = \left((1+i\sqrt{3})^2a^2, (1+i\sqrt{3})ab, b^2\right)$.

As a result, to determine $\xi(\tau)$, we obtain the equation (18), in which

$$\delta = b^2(6a)^{-1}(1+i\sqrt{3})\gamma_1, \quad \sigma = (B,h)\left((g_0,h)\right)^{-1}.$$

For example, let us formulate one result.

Theorem 1. *Let the parameters* γ_1, f_2 *and* f_3 *be such that* $\operatorname{Re}\delta > 0$ *and* $\operatorname{Re}\sigma < 0$. *Then equation* (18) *has a stable cycle* $\rho_0\exp(i\varphi_0\tau)$, *where* $\rho_0 = \left(\operatorname{Re}\delta\cdot(\operatorname{Re}\sigma)^{-1}\right)^{1/2}$, $\psi = \sqrt{3}\gamma_1\operatorname{Re}\delta + \rho_0^2\operatorname{Im}\sigma$, *and system* (12) *with sufficiently small* ε *has a stable cycle*

$$u_0(t,\varepsilon) = \varepsilon^{1/2}\left(g_0\rho_0\exp\left((ia\sqrt{3} + \varepsilon i\psi + O(\varepsilon^2))t\right) + \overline{cc}\right) + O(\varepsilon).$$

3. The case of an arbitrary number of N

First of all, let us define the values of $\gamma^\pm$:

$$\gamma^+ = \begin{cases} \left(ab^{-1}\right)^N, & \text{if } b > 0, \\ \left(a|b|^{-1}\right)^N, & \text{if } b < 0 \quad \text{and} \quad N \text{ is odd}, \\ \left(a|b|^{-1}\right)^N\left(\cos\dfrac{\pi}{N}\right)^{-N}, & \text{if } b < 0 \quad \text{and} \quad N \text{ is even}; \end{cases}$$

$$\gamma^- = \begin{cases} -\left(ab^{-1}\right)^N\left(\cos\dfrac{\pi}{N}\right)^N, & \text{if } b > 0 \quad \text{and} \quad N \text{ is even}, \\ -\left(ab^{-1}\right)^N, & \text{if } b > 0 \quad \text{and} \quad N \text{ is odd}, \\ -\left(a|b|^{-1}\right)^N, & \text{if } b < 0 \quad \text{and} \quad N \text{ is odd}, \\ -\left(a|b|^{-1}\right)^N\left(\cos\dfrac{\pi}{N}\right)^N, & \text{if } b < 0 \quad \text{and} \quad N \text{ is even}. \end{cases}$$

Recall that for $\gamma \in (\gamma^-, \gamma^+)$ zero solution of system (20) is asymptotically stable, but at $\gamma < \gamma^-$ or $\gamma > \gamma^+$ is unstable. Critical cases in the zero equilibrium stability problem occur at $\gamma = \gamma^+$ or at $\gamma = \gamma^-$. In this section, we will consider the local dynamics of system (20) in cases close to critical.

Here are some formulas that will be needed later. Let γ_N be the arithmetic root of the Nth power of $|\gamma|$. Let us set

$$\gamma_0 = \begin{cases} \gamma_N, & \text{if } \gamma > 0, \\ \gamma_N \exp\left(i\dfrac{\pi}{N}\right), & \text{if } \gamma < 0, \end{cases}$$

and let

$$\alpha_k = \gamma_0 \exp\left(\frac{2\pi i k}{N}\right), \quad k = 1, \ldots, N.$$

Note that $\alpha_k^N = \gamma$. System (2), (3) will be written as

$$\dot{u} = Au + F(u), \tag{20}$$

where

$$A = \begin{pmatrix} -a & b & 0 & \ldots & 0 \\ 0 & -a & b & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & b \\ b\gamma & 0 & 0 & \ldots & -a \end{pmatrix}, \quad F(u) = f_2 u \cdot u + f_3 u \cdot u \cdot u + \ldots.$$

Here and below, the multiplication of vectors is coordinate-wise, $u = (u_1, \ldots, u_N)$.

The matrix A has eigenvalues

$$\lambda_k = -a + b\alpha_k \quad (k = 1, \ldots, N)$$

and corresponding eigenvectors

$$g_k = (1, \alpha_k, \alpha_k^2, \ldots, \alpha_k^{N-1}).$$

Note that the matrix A^* conjugate to A has corresponding eigenvectors $h_k = (1, \alpha_k^{-1}, \alpha_k^{-2}, \ldots, \alpha_k^{-(N-1)})$.

3.1. The case of an arbitrary number of N. Here we assume that $N > 2$ and the matrix A has a zero eigenvalue, that is

$$b > 0 \ \text{ and } \ \gamma = \gamma^+ = (ab^{-1})^N, \tag{21}$$

or

$$b < 0, \ \ \gamma = \gamma^- = (a|b|^{-1})^N \ \text{ and } \ N \text{ is odd}.$$

Let us briefly focus only on case of (21). The eigenvalues $\lambda_2, \ldots, \lambda_N$ have negative real parts. The eigenvalue $\lambda_1 = 0$ corresponds to the eigenvector $g^0 = (1, a/b, a^2/b^2, \ldots, a^{N-1}/b^{N-1})$. We arbitrarily fix γ_1 and put it in (1)

$$g = g^+ + \varepsilon\gamma_1, \ \text{ where } \ 0 < \varepsilon \ll 1. \tag{22}$$

To find the coefficients α and β of the normal form in this case, a scalar equation

$$\frac{d\xi}{d\tau} = \alpha\xi + \beta|\xi|^2, \quad \tau = \varepsilon t, \tag{23}$$

we will search for solutions $u(t, \varepsilon)$ of system (1) in the form of a formal series

$$u(t, \varepsilon) = \varepsilon\xi(\tau)g_0 + \varepsilon^2 U_2(\tau) + \ldots.$$

Then, to find $U_2(\tau)$, we obtain a system of equations

$$AU_2 = -g_0\frac{d\xi}{d\tau} + b\gamma_1 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \xi + f_2\xi^2 g^0 \cdot g^0.$$

For this system to be solvable, it is necessary and sufficient that its right side is orthogonal to the vector $h^0 = (1, b/a, b^2/a^2, \ldots, b^{N-1}/a^{N-1})$. From this we conclude that in equation (23)

$$\alpha = \frac{a\gamma_1}{N\gamma^+}, \quad \beta = \frac{1-\gamma^+}{N(1-ab^{-1})}. \tag{24}$$

So, it is shown that for sufficiently small ε, the dynamic properties of solutions (20) with initial conditions from some sufficiently small and independent of ε neighborhood of the zero equilibrium state are described by equation (23) with coefficients (24).

3.2. The critical case of a pair of purely imaginary roots. Here we assume that the matrix A has a pair of purely imaginary eigenvalues $\pm i\omega$ $(\omega > 0)$, and all other of its eigenvalues have negative real parts, that is, the conditions are met

$$b < 0, \quad \gamma^+ = (a|b|^{-1})^N \left(\cos\frac{\pi}{N}\right)^N \quad \text{and } N \textit{ is } \text{even}, \tag{25}$$

or

$$b > 0, \quad \gamma^- = -(ab^{-1})^N \left(\cos\frac{\pi}{N}\right)^N \quad \text{and } \ N \textit{ is } \text{even},$$

or

$$b < 0, \quad \gamma^- = -(a|b|^{-1})^N \left(\cos\frac{\pi}{N}\right)^N \quad \text{and } \ N \textit{ is } \text{even}.$$

Let's focus only on the case (25). The matrix A then has eigenvalues $\lambda^{\pm} = \pm i\omega$, where $\omega = a \operatorname{tg}\frac{\pi}{N}$. They are answered by the eigenvectors g_0 and $\bar{g}_0$, respectively, and $g_0 = (1, \alpha_{N/2}, \alpha^2_{N/2}, \ldots, \alpha^{N-1}_{N/2})$.

Let the equality (22) be fulfilled for γ. The normal form describing the dynamic properties of system (20), provided (22) and (25), is the scalar complex equation (18). To find the coefficients of this equation, consider the formal series

$$\begin{aligned} U(t,\varepsilon) = {} & \varepsilon^{1/2}\big(\xi(\tau) g_0 \exp(i\omega t) + \overline{cc}\big) + \varepsilon\big(|\xi|^2 U_{20} + \xi^2 U_{21}\exp(2i\omega t) + \overline{cc}\big) + \\ & + \varepsilon^{3/2}\big(\big(U_{31}\exp(i\omega t) + \overline{cc}\big) + \overline{cc} + \xi^3 U_{32}\exp(3i\omega t) + \overline{cc}\big) + \ldots . \end{aligned} \tag{26}$$

Let us substitute (26) into (20) and collect coefficients with the same degrees of ε. With $\varepsilon^{1/2}$, we get the correct equality. In the next step, we find that

$$U_{20} = 2f_2 A^{-1} g_0 \cdot \bar{g}_0, \quad U_{21} = f_2 (A - 2i\omega I)^{-1} g_0 \cdot g_0.$$

By collecting the coefficients for $\varepsilon^{1/2}$, we obtain the equations for U_{31} and U_{32}. The expression for U_{32} is simply defined. It will not be needed below, so we will not give it. To determine U_{31}, we come to the system

$$(A - i\omega I) U_{31} = B, \tag{27}$$

where

$$B = -g_0 \frac{d\xi}{d\tau} + \gamma_1 b \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \xi + 2f_2 g_0 U_{20} + 2f_2 \bar{g}_0 U_{21} + 3f_3 g_0 \cdot g_0 \cdot \bar{g}_0.$$

For the solvability of system (27), it is necessary and sufficient that the vector B becomes orthogonal to the vector h_0, the solution of the homogeneous conjugate equation $A^* h = -i\omega h$. As a result, for the coefficients of equation (18), we obtain the equalities

$$\delta = -b\gamma_1 (\gamma^+)^{(1/N-1)} N^{-1}, \tag{28}$$

$$\sigma = \frac{1}{N}\Big[2f_2\big((g_0 U_{20}, h_0) + (\bar{g}_0 U_{21}, h_0)\big) + 3f_3(g_0 \cdot g_0 \cdot \bar{g}_0, h_0)\Big]. \tag{29}$$

Under the conditions of non-degeneracy $\operatorname{Re}\delta \neq 0$ and $\operatorname{Re}\sigma \neq 0$ equation (18) with coefficients (28), (29) completely determines the local dynamics of equation (20). Using (26), we obtain an asymptotic representation of the solutions of (20) through the solutions of (18).

4. The case of sufficiently large values of N

The constructions for this case are much more complicated than the previous ones. Here we assume that the value of N is large enough, that is, the value is small enough

$$\varepsilon = N^{-1} \ll 1.$$

Let us examine the local dynamics of system (20) in this case.

First, we formulate one simple statement following from formula (6) for the roots of characteristic equation (5).

Lemma 1. *Let the inequality be fulfilled*

$$a|b|^{-1} < 1. \tag{30}$$

Then for all sufficiently small ε, all the roots of (6) have negative real parts that are separated from zero for $\varepsilon \to 0$. If

$$a|b|^{-1} > 1, \tag{31}$$

then, for sufficiently small values of ε, the root of equation (5) is found, the real part of which is positive and separated from zero for $\varepsilon \to 0$.

In case of (30) at small ε solutions of (20) with initial conditions from small and independent of ε at $\varepsilon \to 0$, the neighborhood of the zero equilibrium state tends to zero at $t \to \infty$. In case of (31), the null solution of (20) is unstable and the problem of dynamics of (20) becomes non-local. Therefore, we assume below that

$$|b| = a. \tag{32}$$

In particular, $b^{+} = a + O(\varepsilon)$, $b^{-} = -a + O(\varepsilon)$. Under condition (32), (5) has no roots with a positive real part separated from zero at $\varepsilon \to 0$, but there are infinitely many roots whose real parts tend to zero at $\varepsilon \to 0$. Thus, in the problem of stability of the zero state of equilibrium (20), a critical case of infinite dimension is realized. Let us consider separately the case when $b = a$ and when $b = -a$.

Let us note the works of the author of [13, 14, 17, 32], in which, in other situations, the dynamic properties of systems in infinite-dimensional critical cases were studied.

It is convenient to redefine the elements of $u_j(t)$ using a function of two variables $u_j(t) = u(t, x_j)$, where $x_j \in [0, 1)$ are points $x_j = 2\pi j/N = 2\pi i \varepsilon j$ $(j = 0, 1, \ldots, N)$ evenly distributed on the segment $[0, 1]$.

System (2), (3) for $x = x_j$ can then be written as an equation

$$\frac{\partial u}{\partial t} + au = bu(t, x + \varepsilon) - f(u) \tag{33}$$

with boundary conditions

$$u(t, 1) = \gamma u(t, 0), \tag{34}$$

and for the equation (33) linearized at zero, we obtain the expression

$$\frac{\partial v}{\partial t} + av = bv(t, x + \varepsilon), \tag{35}$$

$$v(t, 1) = \gamma v(t, 0). \tag{36}$$

Equations (33) and (35) cannot be considered for the continuous argument $x \in [0, 1]$, since the expressions $u(t, x+\varepsilon)$ and $v(t, x+\varepsilon)$ for $x+\varepsilon > 1$ are undefined. The exception is when $\gamma = 1$. It was reviewed in [27]. Then we assume that $x \in (-\infty, \infty)$, and the functions u, v were considered periodic in x with a period of 1. For the roots $\lambda_k(\varepsilon)$ $(k = 0, \pm1, \pm2, \ldots)$ of characteristic equation for (35) at $\gamma = 1$ has the formula

$$\lambda_k(\varepsilon) = -a + b\exp(2\pi k i \varepsilon),$$

and for the corresponding eigenfunctions $\varphi_k(t, x, \varepsilon)$, we get the expression

$$\varphi_k(t, x, \varepsilon) = \exp(\lambda_k(\varepsilon) t)\exp(2\pi k i \varepsilon).$$

Note that under the condition (32) there are infinitely many roots of $\lambda_k(\varepsilon)$ tends to zero at $\varepsilon \to 0$. It is important to emphasize that for $b = a + o(\varepsilon)$ functions $\varphi_k(t, x, \varepsilon)$ smoothly depends on ε and, which is the same thing, the condition of regularity is fulfilled

$$\varphi_k(t, x, \varepsilon) = \varphi_k(t, x, 0) + \varepsilon \frac{\partial \varphi_k(t, x, 0)}{\partial x} + \frac{1}{2}\varepsilon^2 \frac{\partial^2 \varphi_k(t, x, 0)}{\partial x^2} + o(\varepsilon^2).$$

If $b = -a + o(\varepsilon)$, then for those integers k for which $\lambda_k(\varepsilon)$ tends to zero at $\varepsilon \to 0$, we get that

$$\varphi_k(t, x, \varepsilon) = \exp(i\pi\varepsilon^{-1}x)\psi_k(t, x, \varepsilon),$$

where $\psi_k(t, x, \varepsilon)$ regularly depends on ε.

Let us return to the case of an arbitrary γ. For the roots $\lambda_k(\varepsilon)$ of equations (35) there is a formula

$$\lambda_k(\varepsilon) = -a + b \exp\big(\varepsilon(\ln\gamma + 2\pi k i)\big), \tag{37}$$

in which the number k takes the values $k = 0, \pm1, \pm2, \ldots$.

We arbitrarily fix the value of b_1, and let either

$$b = a + \varepsilon b_1, \tag{38}$$

or

$$b = -a(1 + \varepsilon b_1). \tag{39}$$

In case of (38), we apply the regularity condition in (35)

$$v(t, x + \varepsilon) = v(t, x) + \varepsilon \frac{\partial v(t, x)}{\partial x} + O(\varepsilon^2).$$

Then, with accuracy up to $O(\varepsilon^2)$ we arrive at the equation

$$\frac{\partial v}{\partial t} = \varepsilon b_1 v + \varepsilon a \frac{\partial v}{\partial x}, \quad v(t, 1) = \gamma v(t, 0).$$

In the irregular case, when condition (39) is fulfilled, we obtain that

$$v(t, x) = \exp(i\pi\varepsilon^{-1}x)\bar{v} + \overline{cc}$$

and

$$\frac{\partial \bar{v}}{\partial t} = \varepsilon b_1 \bar{v} + \varepsilon a \frac{\partial \bar{v}}{\partial x}, \quad (\bar{v}(t, 1) + \overline{cc}) \exp(i\pi N) = \gamma \bar{v}(t, 0) + \overline{cc}.$$

Let us put in (37) $\lambda_k(\varepsilon) = \varepsilon\lambda_{k1}(\varepsilon)$. The real parts of all $\lambda_{k1}(\varepsilon)$ has an asymptotic

$$b_1 + \ln|\gamma| + O(\varepsilon).$$

From this we obtain a criterion for the stability of the zero equilibrium state: at $b_1 + \ln|\gamma| > 0$, the equilibrium state is unstable, and at $b_1 + \ln|\gamma| < 0$, it is stable.

Note that in case of (6), all solutions of (33) (on condition (9)) from some sufficiently small and independent of ε neighborhood of the zero equilibrium state tend to zero at $t \to \infty$, and in case of (7), the problem of local dynamics of (33), (34) is not local. Below we will consider the critical case when the parameter $\gamma = \gamma_0$ is selected so that

$$|\gamma_0| \exp\left(a^{-1}b_1\right) = 1.$$

Let's consider separately the cases when the parameter b is close to the parameter a and when it is close to the parameter $-a$. In the first case, in sections 4.1 and 4.2 we will talk about regular solutions, and in the second case, in section 4.3, about irregular ones.

4.1. The case when the parameter b is close to the value of a and the parameter γ is positive.

In this section, we assume that equality (38) holds and

$$\gamma > 0 \quad \text{and} \quad f_2 \neq 0.$$

Then, for each integer k, the asymptotic equality holds

$$\lambda_k(\varepsilon) = \varepsilon[b_1 + a(\ln\gamma + 2\pi k i)] + O(\varepsilon^2), \tag{40}$$

and the eigenfunctions $v_k(t,x)$ corresponding to the root $\lambda_k(\varepsilon)$ can be represented as

$$v_k(t,x) = \exp(2\pi i k x + \lambda_k(\varepsilon)t). \tag{41}$$

Let us consider regular solutions of the boundary value problem (33), (34), that is, we assume

$$u(t, x+\varepsilon) = u(t,x) + \varepsilon\frac{\partial u(t,x)}{\partial x} + O(\varepsilon^2).$$

Using t_1, we denote the «slow» time $t_1 = \varepsilon t$ and replace $u(t,x) = \varepsilon u_1(t_1,x)$. Then, discarding terms of the order of ε^2, we arrive at the boundary value problem

$$\frac{\partial u_1}{\partial t_1} = a\frac{\partial u_1}{\partial x} + b_1 u_1 - f_2 u_1^2, \quad u_1(t_1,1) = \gamma u_1(t_1,0). \tag{42}$$

In (42), there is a zero equilibrium state of $u_1 \equiv 0$ and possibly non-zero

$$u_1 = u_0(x) = [-f_2 b_1^{-1} + c_0\exp(b_1 a^{-1}x)]^{-1}, \quad c_0 = f_2 b_1^{-1}(\gamma - 1)(\gamma\exp(b_1 a^{-1}) - 1),$$

if the conditions $\gamma > 0, \gamma \neq 1, -f_2 + c_0 b_1\exp(b_1 a^{-1}x) \neq 0$ are met for $x \in [0,1]$.

Boundary value problem (42) is a quasi-normal form for boundary value problem (33), (34). This means that according to the restricted solution $u_1(t_1,x)$ for $t_1 \to \infty, x \in [0,1]$, the function $u(t,x,\varepsilon) = \varepsilon u_1(\varepsilon t, x)$ is defined, which satisfies (33), (34) accurate to $O(\varepsilon^2)$.

We study the behavior of all solutions of (42) (and hence (33), (34)) from a certain sufficiently small neighborhood of the zero equilibrium state.

Let us formulate a simple statement.

Lemma 2. *Under the condition* $\exp(-b_1 a^{-1}) > \gamma$ $(< \gamma)$, *the zero equilibrium state in* (42) *and in* (33), (34) *is asymptotically stable (unstable).*

Consider the critical case when $\gamma = \gamma_0$, where

$$\gamma_0 = \exp(-b_1 a^{-1}), \tag{43}$$

and let us repeat the normalization scheme. We arbitrarily fix the value of b_2 and put

$$b = a + \varepsilon b_1 + \varepsilon^2 b_2. \tag{44}$$

For regular solutions of $u(t, x+\varepsilon)$, we take into account terms of the order ε^2 in (33). As a result, we get that

$$\frac{\partial u_1}{\partial t_1} = a\frac{\partial u_1}{\partial x} + (b_1 + \varepsilon b_2)u_1 + \frac{1}{2}a\varepsilon\frac{\partial^2 u_1}{\partial x^2} + f_2 u_1^2 + \varepsilon f_3 u_1^3, \quad u_1(t_1,1) = \gamma u_1(t_1,0). \tag{45}$$

The boundary value problem linearized at zero at $\varepsilon = 0$ has the form

$$\frac{\partial u_1}{\partial t_1} = a\frac{\partial u_1}{\partial x} + b_1 u_1, \quad u_1(t_1,1) = \gamma u_1(t_1,0). \tag{46}$$

Its characteristic equation

$$(\lambda - b_1)v = a\frac{dv}{dx}, \quad v(1) = \gamma v(0), \tag{47}$$

by virtue of (43), it has infinitely many roots $\lambda_k = 2\pi k i a$ $(k = 0, \pm 1, \pm 2, \ldots)$.

The root λ_k corresponds to its eigenfunction $v_k(x) = \exp(-b_1 a^{-1} x)\exp(\lambda_k a^{-1} x)$. Let us put $y = t_1 - a^{-1}x$ and $w_k(y) = \exp(\lambda_k y)$. Then $v_k(t_1, x) = \exp(-b_1 a^{-1} x)\exp(\lambda_k y) = \exp(-b_1 a^{-1} x) w_k(y)$.

Therefore, an arbitrary linear combination of functions

$$w(x, y) = \exp(-b_1 a^{-1} x) \cdot \sum_{k \to -\infty}^{+\infty} c_k w_k(y)$$

is also a solution of (46).

Based on the algorithm of the method of constructing quasi-normal forms [13, 14, 17, 32], we are looking for solutions to the nonlinear boundary value problem (45) in the form of a formal series

$$u_1(\tau, x, \varepsilon) = \varepsilon w(\tau, y)\exp(-b_1 a^{-1} x) + \varepsilon^2 U_2(\tau, x, y) + \dots, \tag{48}$$

where $\tau = \varepsilon t_1$, and for the variable y the condition of 1-periodicity is fulfilled.

Let us substitute (48) into (45) and collect coefficients with the same degrees of ε. In the first step, collecting the coefficients at zero degree ε, we obtain the correct equality. In the next step, we will collect the coefficients for ε^2. As a result, we obtain the ratio

$$a\frac{\partial U_2}{\partial x} + b_1 U_2 = \varphi(\tau, x, y), \quad U_2(1) = \gamma U_2(0).$$

Here

$$\varphi(\tau, x, y) = \Big[-\frac{\partial w}{\partial \tau} + \frac{1}{2}a\frac{\partial^2 w}{\partial y^2} - b_1\frac{\partial w}{\partial y} + (b_1^2(2a)^{-1} + b_2)w\Big]\exp(-b_1 a^{-1} x) + \\ + f_2 w \exp(-2b_1 a^{-1} x). \tag{49}$$

Let us use the following simple statement.

Lemma 3. *Let the function $\varphi(x)$ be continuous. Then, for the solvability of the boundary value problem*

$$a\frac{\partial \psi}{\partial x} + b_1\psi = \varphi(x), \quad \psi(1) = \gamma_0\psi(0) + \alpha$$

in the class of continuous functions, it is necessary and sufficient that the equality holds

$$\int_0^1 \varphi(s)\exp(b_1 a^{-1} s)ds = a \cdot \alpha\gamma_0. \tag{50}$$

Considering the equality of (49) in (50), we arrive at the boundary value problem for determining the function $w(\tau, y)$

$$\frac{\partial w}{\partial \tau} = \frac{1}{2}a\frac{\partial^2 w}{\partial y^2} - b_1\frac{\partial w}{\partial y} + (b_1^2(2a)^{-1} + b_2)w + f_2(1-\gamma)b_1^{-1}a_1 w^2, \tag{51}$$

$$w(\tau, y+1) \equiv w(\tau, y). \tag{52}$$

Let us formulate the main result of this section.

Theorem 2. *Let boundary value problem (51), (52) have a bounded solution $w_0(\tau, y)$ for $\tau \to \infty, y \in [0, 1]$. Then the function $u(t_1, x) = \varepsilon^2 w_0(\varepsilon^2 t, y)\exp(-b_1 a^{-1} x)$ satisfies boundary value problem (33), (34) accurate to $o(\varepsilon^3)$.*

Note that in the case when $f_2 = 0$ (or $\gamma = 1$), cubic nonlinearity appears in (51) instead of quadratic nonlinearity. It is important to emphasize that a stable solution to the boundary value problem (51), (52) can only be a homogeneous equilibrium state.

4.2. The case when the parameter b is close to a and the parameter γ is negative. Here we assume that

$$\gamma < 0. \tag{53}$$

Repeating the constructions of the previous section, that is, taking into account formulas (40)–(41), we also come to boundary value problem (42). Under condition (53), (42) has only a zero equilibrium state. Let us examine its stability.

Lemma 4. *Under the condition* $\exp(-b_1a^{-1}) > |\gamma|$ $(< |\gamma|)$, *the zero state of equilibrium in* (42) *and in* (33), (34) *is asymptotically stable (unstable).*

Consider the critical case when

$$|\gamma| = \gamma_0 = \exp(-b_1a^{-1}). \tag{54}$$

Let equality (44) hold. For regular solutions of $u(t, x + \varepsilon)$ we take into account terms of the order ε^2 in (33). Then we come back to boundary value problem (45). For linearized boundary value problem (46), we study the characteristic equation (47). Unlike the previous case, it has infinitely many roots $\lambda_k = \pi i(2k + 1)$ $(k = 0, \pm1, \pm2, \ldots)$.

As above, let us assume $y = t_1 - a^{-1}x$, $w_k(y) = \exp(\lambda_k y)$. Then $v_k(t, x) = \exp(-b_1a^{-1})w_k(y)$.

Solutions to nonlinear boundary value problem (45) in case of (53) are sought in the form

$$u_1(\tau, x, y) = \varepsilon^{1/2}w(\tau, y)\exp(-b_1a^{-1}) + \varepsilon U_2(\tau, x, y) + \varepsilon^{3/2}U_3(\tau, x, y) + \ldots, \tag{55}$$

where $\tau = \varepsilon t_1$, and the function $w(\tau, y)$ has the structure

$$w(\tau, y) = \sum_{k=-\infty}^{\infty} c_k(\tau)w_k(y).$$

Let us substitute formal expression (55) into (45) and collect coefficients with the same degrees ε. In the first step, collecting the coefficients for $\varepsilon^{1/2}$, we get the correct equality. In the next step, we will collect the coefficients at the first degree of ε. As a result, we come to the boundary value problem for finding $U_2(\tau, x, y)$:

$$a\frac{\partial U_2}{\partial x} + bU_2 = f_2\exp(-2b_1a^{-1}x)w^2(\tau, y), \quad U_2(\tau, 1, y) = \gamma U_2(\tau, 0, y).$$

Hence we find that

$$U_2 = \exp(-b_1a^{-1}x)\Big[c(\tau, y) + f_2w^2(\tau, y)\cdot ab_1^{-1}(1 - \exp(-b_1a^{-1}x))\Big],$$

where

$$c(\tau, y) = af_2(b_1(\gamma - 1))^{-1}w^2(\tau, y)|\gamma|(1 - |\gamma|).$$

In the third step, we obtain the equation for determining $U_3(\tau, x, y)$. From the condition of its solvability (according to lemma 3), we obtain a boundary value problem for finding the function $w(\tau, y)$:

$$\frac{\partial w}{\partial \tau} = \frac{1}{2}a\frac{\partial^2 w}{\partial y^2} - b_1\frac{\partial w}{\partial y} + (b_1^2(2a)^{-1} + b_2)w + \delta w^3, \tag{56}$$

$$w(\tau, y + 1) \equiv -w(\tau, y), \tag{57}$$

in which

$$\delta = 2f_2^2(b_1^{-2})\big[2a\gamma(1 + \gamma)^2 + 3a(|\gamma|^3 - 1)\big].$$

Here is the main result.

Theorem 3. *Let boundary value problem* (56), (57) *have a bounded solution* $w_0(\tau, y)$ *for* $\tau \to \infty, y \in (-\infty, \infty)$. *Then the function* $u(t_1, x, \varepsilon) = \varepsilon^{3/2}w_0(\varepsilon^2t_1, t_1 + a^{-1x})$ *satisfies boundary value problem* (33), (34) *accurate to* $o(\varepsilon^2)$.

Note that boundary value problem (56), (57) may have an inhomogeneous stable equilibrium state.

4.3. The case when the parameter b is close to the parameter $-a$. Let condition (39) be fulfilled for an arbitrarily fixed value of b_1. Consider linearized boundary value problem (35), (36). Its characteristic equation (37) under condition (39) has infinitely many roots $\lambda_k(\varepsilon)$ $(k = 0, \pm1, \pm2, \ldots)$, which tend to the imaginary axis at $\varepsilon \to 0$. Thus, the critical case of infinite dimension is realized. To the root of $\lambda_k(\varepsilon)$ is answered by its eigenfunction $\varphi_k(t, x, \varepsilon)$, which oscillates asymptotically rapidly over the spatial variable x. This means that the relevant solutions have an irregular structure.

In (35), (36) we assume

$$u(t, x) = v(t, x) \exp(i\pi\varepsilon^{-1}x) + \overline{cc}. \tag{58}$$

Considering that $u(t, 1) = v(t, 1) \exp(i\pi N) + \overline{cc}$ and that $\varepsilon = N^{-1}$, we get the equality

$$v(t, 1) + \overline{v}(t, 1) = \gamma(-1)^N (v(t, 0) + \overline{v}(t, 0)). \tag{59}$$

Then, for $v(t, x)$, we arrive at the equation

$$\frac{\partial v}{\partial t} + av = -bv(t, x + \varepsilon). \tag{60}$$

Using equality (39), we conclude that the function $v(t, x)$ is regular, that is

$$v(t, x + \varepsilon) = v(t, x) + \varepsilon \frac{\partial v(t, x)}{\partial t} + \frac{1}{2}\varepsilon^2 \frac{\partial^2 v}{\partial x^2} + \ldots .$$

Let us substitute this expression in (33), (34). Then, based on (58), we arrive at the equation

$$\frac{\partial v_1}{\partial t_1} + a\frac{\partial v_1}{\partial x} + b_1 v_1 = \exp(-b_1 a^{-1} x) f_2\big(v_1 \exp(i\pi\varepsilon^{-1}x) + \overline{cc}\big)^2 \tag{61}$$

with boundary condition (59). Here $t_1 = \varepsilon t$, $v = \varepsilon v_1$. For this boundary value problem, there is a statement similar to lemma 2.

Lemma 5. *On condition* $\exp(-b_1 a^{-1}) > |\gamma|$ $(< |\gamma|)$ *the zero equilibrium state in* (61), (59) *and in* (33), (34) *is asymptotically stable (unstable).*

Let us consider the critical case when equalities (54) are fulfilled. In this case, linearized boundary value problem (60), (59) it has infinitely many solutions that are periodic in y, $v_{1k} = \exp(-b_1 a^{-1} x) w_k(y)$, where $y = t_1 - a^{-1}x$, $w_k(y) = \exp(\lambda_k y)$, where $\lambda_k = 2\pi i k a$, if $\gamma(-1)^N > 0$ and $\lambda_k = \pi i a(2k + 1)$, if $\gamma(-1)^N < 0$.

Let us put

$$b = -(a + \varepsilon b_1 + \varepsilon^2 b_2) \quad \text{and} \quad \gamma = \gamma_0 + \varepsilon\gamma_1. \tag{62}$$

Let us consider the behavior under conditions (54) and (62) of all solutions of (33), (34) from some sufficiently small neighborhood of the zero equilibrium state.

Let us introduce the formal expression

$$\begin{aligned} v(t_1, x) ={}& \varepsilon\big(w(\tau, y) \exp(-b_1 a^{-1} x) \exp(i\pi\varepsilon^{-1}x) + \overline{cc}\big) + \\ &+ \varepsilon^2 \Big[u_{20}(t, x, y) + \overline{cc} + \exp(i\pi z) u_{21}(t, x, y) + \overline{cc} + \\ &+ \exp(2i\pi z) u_{22}(t, x, y) + \overline{cc} + \\ &+ \exp(3i\pi z) u_{23}(t, x, y) + \overline{cc}\Big] + \ldots , \end{aligned} \tag{63}$$

where $z = x\varepsilon^{-1}$, $u_1(t_1, x, y) = w(\tau, y) \exp(-b_1 a^{-1} x)$, $\tau = \varepsilon t_1$, $w(\tau, y) = \sum\limits_{k=-\infty}^{\infty} c_k(\tau) w_k(y)$, and for the variable y all functions from (63) are periodic.

Let us consider two cases separately. In the first case, we assume that

$$(-1)^N \gamma_0 > 0, \tag{64}$$

and in the second case, the inequality holds

$$(-1)^N \gamma_0 < 0. \tag{65}$$

4.3.1. Constructing the asymptotics of solutions under condition (64).

Let condition (64) be fulfilled. Substitute (63) into (33), (34) and collect the coefficients with the same powers of ε. As a result, we obtain the following equalities

$$a\frac{\partial u_1}{\partial x} + b_1 u_1 = 0, \tag{66}$$

$$2au_{20} = f_2|u_1|^2, \quad 2au_{22} = f_2 u_1^2, \tag{67}$$

$$\begin{aligned} a\frac{\partial u_{21}}{\partial x} + b_1 u_{21} = \Big[-(b_2 + b_1^2(2a)^{-1})w - \frac{\partial w}{\partial \tau} + \frac{a}{2}\frac{\partial^2 w}{\partial y^2} - b_1\frac{\partial w}{\partial y}\Big] \times \\ \times \exp(-b_1 a^{-1} x) + \big[f_2 u_{20} + 2f_2 u_{22} + 3f_3\big] \cdot w|w|^2 \exp(-3b_1 a^{-1} x), \end{aligned} \tag{68}$$

$$a\frac{\partial u_{23}}{\partial x} + b_1 u_{23} = [a^{-1} f_2^2 + f_3] u_1^3. \tag{69}$$

From the boundary conditions, we obtain the following relations

$$(-1)^N (u_1 + \overline{cc})\Big|_{x=1} = \gamma_0 (u_1 + \overline{cc})\Big|_{x=0}, \tag{70}$$

$$\begin{aligned} \Big(u_{20} + \overline{cc} + (-1)^N u_{21} + \overline{cc} + u_{22} + \overline{cc} + (-1)^N u_{23} + \overline{cc}\Big)\Big|_{x=1} = \\ = \gamma_0\Big(u_{20} + \overline{cc} + u_{21} + \overline{cc} + u_{22} + \overline{cc} + u_{23} + \overline{cc}\Big)\Big|_{x=0} + \gamma_1 (u_1 + \overline{cc})\Big|_{x=0}. \end{aligned} \tag{71}$$

Equations (66) and (70) are satisfied by the definition of u_1. From (67) and (68), we find that

$$u_{20} = (2a)^{-1} f_2 |u_1|^2, \quad u_{22} = (2a)^{-1} f_2 u_1^2, \tag{72}$$

and from (69),we obtain that

$$u_{23} = -(2b_1)^{-1}(f_2^2 + af_3)\Big[\exp\Big(-\frac{3b_1}{a}x\Big) - \exp\Big(-\frac{b_1}{a}x\Big)\Big] w^3. \tag{73}$$

Consider the question of solvability with respect to $u_{21}(\tau, x, y)$ of equation (68) with the boundary condition (71). According to lemma 3, the necessary and sufficient conditions for the solvability of this boundary value problem are the fulfillment of the equalities

$$\frac{\partial w}{\partial \tau} = \frac{a}{2}\frac{\partial^2 w}{\partial y^2} - b_1\frac{\partial w}{\partial y} + c_1 w + c_2|w|^2 + c_2 w^2 + c_3 w^3 + c_4 w|w|^2, \tag{74}$$

$$w(\tau, y+1) \equiv w(\tau, y), \tag{75}$$

where

$$c_1 = -(b_2 + (2a)^{-1} b_1^2) + a\gamma_0^2\gamma_1,$$

$$c_2 = \frac{1}{2}\gamma_0^2 f_2 (1 - \gamma_0), \quad c_3 = c_2,$$

$$c_4 = -a(2b_1)^{-1}(f_2^2 + af_3)\gamma_0^2(\gamma_0^2 - 1), \quad c_5 = 3(2b_1)^{-1} a(\gamma_0^2 - 1)\cdot[f_3 + a^{-1} f_2^2].$$

Let us formulate the main statement that follows from the above algorithm for constructing the asymptotics of solutions.

Theorem 4. *Let conditions (62) and (64) be fulfilled. Let function $w(\tau, y)$ be bounded for $\tau \to \infty$, $y \in [0, 1]$ to boundary value problem (74), (75). Then the function*

$$\begin{aligned} u(t, x, y) =& \varepsilon\big(w(\tau, y)\exp(-b_1 a^{-1} x)\exp(i\pi\varepsilon^{-1} x) + \overline{cc}\big) + \\ &+ \varepsilon^2\Big[u_{20}(t, x, y) + \overline{cc} + \exp(i\pi z) u_{21}(t, x, y) + \overline{cc} + \\ &+ \exp(2i\pi z) u_{22}(t, x, y) + \overline{cc} + \\ &+ \exp(3i\pi z) u_{23}(t, x, y) + \overline{cc}\Big] + \ldots, \end{aligned}$$

satisfies boundary value problem (33), (34) *accurate to* $O(\varepsilon^4)$.

4.3.2. Constructing the asymptotics of solutions under condition (65)**.** Let inequality (65) be fulfilled. In this case, the corresponding constructions become more complicated. Let us again consider asymptotic expression (63), but we will represent the function $u_{21}(\tau, x, y)$ appearing in it as the sum of two functions

$$u_{21}(\tau, x, y) = v_1(\tau, x, y) + v_2(\tau, x, y). \tag{76}$$

The first of them is 1-antiperiodic in y, like the function $w(\tau, y)$, that is, it contains only harmonics with odd numbers $\exp(i\pi(2k+1))$ $(k = 0, \pm1, \pm2, \ldots)$. The second function, $v_2(\tau, x, y)$, 1-periodic in y, that is, its Fourier series expansion contains only harmonics $\exp(2i\pi k)$ $(k = 0, \pm1, \pm2, \ldots)$. Let us substitute (63) with (76) in (33), (34) and perform the standard actions. As a result, we get equalities (66), (67), (69), (70). Equalities (66), (70) define function $u_1(\tau, x, y) = w(\tau, y)\exp(-b_1 a^{-1} x)$, and from (67) and (69) we find u_{20}, u_{22} and u_{23} according to formulas (72), (73). The equation for v_1 is obtained by replacing the function u_{21} and v_1 in the left part of equation (68), and the equation v_2 has the form

$$a\frac{\partial v_2}{\partial x} + b_1 v_2 = 0. \tag{77}$$

Based on the formula for boundary conditions (71), we define the boundary conditions for the functions v_1 and v_2 :

$$(-1)^N v_1\Big|_{x=1} = -(-1)^N u_{23}\Big|_{x=1} + \Big[\gamma_0 v_1 + \gamma_0 u_{23} + \gamma_0\gamma_1 u_1\Big]\Big|_{x=0}, \tag{78}$$

$$(-1)^N v_2\Big|_{x=1} = -u_{20}\Big|_{x=1} - u_{22}\Big|_{x=1} + \gamma_0\Big[v_2 + u_{20} + u_{22}\Big]\Big|_{x=0}. \tag{79}$$

From the boundary value problem (77), (79) we find that

$$v_2 = v_2(\tau, x, y) = (2a|\gamma_0|)^{-1} f_2\gamma_0(1-\gamma_0)w\Big[\bar{w} - \frac{1}{2}w\Big]\exp(-b_1 a^{-1} x).$$

For the solvability of boundary value problem (68) (with the replacement of u_{21} by v_1), (78), as follows from lemma 3, it is necessary and sufficient that the equality holds

$$\frac{\partial w}{\partial \tau} = \frac{a}{2}\frac{\partial^2 w}{\partial y^2} - b_1\frac{\partial w}{\partial y} + c_1 w + c_3 w^3 + c_4 w|w|^2$$

and 1-antiperiodic boundary conditions

$$w(\tau, y+1) \equiv -w(\tau, y). \tag{80}$$

Let us formulate the main result

Theorem 5. *Let conditions* (62) *and* (65) *be fulfilled. Let the function* $w(\tau, y)$ *be bounded for* $\tau \to \infty$, $y \in [0, 1]$ *by solving boundary value problem* (74), (75). *Then the function*

$$\begin{aligned} u(t, x, y) =& \varepsilon\big(w(\tau, y)\exp(-b_1 a^{-1} x)\exp(i\pi\varepsilon^{-1} x) + \overline{cc}\big) + \\ &+ \varepsilon^2\Big[u_{20}(t, x, y) + \overline{cc} + \exp(i\pi z)u_{21}(t, x, y) + \overline{cc} + \\ &+ \exp(2i\pi z)u_{22}(t, x, y) + \overline{cc} + \\ &+ \exp(3i\pi z)u_{23}(t, x, y) + \overline{cc}\Big] + \ldots, \end{aligned}$$

satisfies boundary value problem (33), (34) *accurate to* $O(\varepsilon^4)$.

Remark 1. *We can consider a problem in which the boundary conditions vary: instead of boundary conditions* (3)*, the equality is fulfilled*

$$u_N(t) = \gamma u_M(t),$$

where $M\,(M < N)$ *is an integer. Of the greatest interest is the study of the effect of these boundary conditions at sufficiently large values of* N*.*

First, we note that under condition $M \sim \mathrm{const}$ *(for* $\varepsilon \to 0$*) , the problem of the dynamics of the system under consideration is reduced to the case of a small perturbation of the parameter* γ *in problem* (33)*,* (34)*.*

Significant changes may occur in cases where the M *number is also large enough. For example, let* $M = \frac{m}{n}N$*, where* m *and* n *are natural numbers and* $m < n$*. Then the multiplier* $\exp(i\pi N)$ *and* $\exp\left(i\pi \frac{m}{n} N\right)$ *appear in boundary conditions* (70)*,* (71)*. It follows that for* $\varepsilon \to 0\,(N \to \infty)$ *there are about* n *different and alternating boundary conditions for* $N \to \infty$*. Thus, the dynamic properties of solutions are described with an increase of* N *by alternating* n *scenarios.*

Conclusion

The problem of the local dynamics of a system of N unilaterally coupled simplest nonlinear first-order equations in the vicinity of an equilibrium state is considered. Critical cases in the problem of stability of the equilibrium state are highlighted. It is shown that already at $N = 2$ a critical case of a zero root can occur, and at $N = 3$ critical cases of a single zero root or a pair of purely imaginary roots can occur. In these cases, the corresponding normal forms are constructed and bifurcation problems are considered. Constructions for an arbitrary value of N are given. In section 4, which is the main one, cases are considered when the value of N is large enough, that is, the parameter $\varepsilon = N^{-1}$ is small enough. In this case, a transition is made from a discrete system of N equations to a spatially continuous problem.

The values of the parameters at which critical cases can occur are determined. The main feature is that the critical cases have infinite dimension, that is, infinitely many roots of the characteristic equation of the linearized problem tend to the imaginary axis at $\varepsilon \to 0$.

Using the infinite-dimensional normalization method (the method of quasi-normal forms) developed in the works of the author [13,14,17,32], it was possible to construct special nonlinear partial differential equations of parabolic type with boundary conditions. These boundary value problems do not contain a small parameter, and their nonlocal dynamics determines the behavior of all solutions of the initial system from a sufficiently small neighborhood of the equilibrium state.

Under certain conditions, the corresponding equations may have a non-standard form and contain both quadratic and cubic nonlinearities. The dynamics of such boundary value problems can be quite complex (see, for example, [33]).

The asymptotics of the main terms of the asymptotic representation of solutions is constructed.

It is important to note that the solutions of the initial system may have a special «sensitivity» of dynamic properties to changes in the small parameter ε. This follows from the fact that the change in quantity (large) of N by just 1 can significantly change even the appearance of the corresponding partial differential equations and change periodic boundary conditions to antiperiodic ones.

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