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A method for constructing a complete bifurcation picture of a boundary value problem for nonlinear partial differential equations: application of the Kolmogorov–Arnold theorem

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Abstract. The *purpose* of this study is to develop a numerical method for bifurcation analysis of nonlinear partial differential equations, based on the reduction of partial differential equations to ordinary ones, using the Kolmogorov–Arnold theorem. *Methods.* This paper describes a method for reducing partial differential equations to ordinary ones using the Kolmogorov–Arnold theorem, as well as methods for the bifurcation analysis of nonlinear boundary value problems for ordinary differential equations. *Results.* The paper presents a new method for solving and bifurcation analysis of nonlinear boundary value problems for partial differential equations, which allow variational formulation. The method was applied to a nonlinear two-dimensional Bratu problem with Dirichlet-type boundary conditions. *Conclusion.* A new method of bifurcation analysis for nonlinear partial differential equations has been developed. Specifically, a method has been proposed for reducing partial different equations to ordinary equations, which allows the use of the developed apparatus of bifurcation analysis for boundary value problems of ordinary differential equations. This method allows conducting bifurcation analysis for arbitrary nonlinear partial differential equations.

Keywords: bifurcation analysis, nonlinear partial differential equations, boundary value problems, Kolmogorov–Arnold theorem.

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Introduction

Solving a significant number of natural science problems makes it necessary to construct, mainly within the framework of numerical methods, a complete bifurcation picture for boundary value problems for systems of nonlinear partial differential equations (nonlinear boundary value problems). At the same time, if we can specify a significant number of methods (finite element method, finite difference method, etc.) to find solutions to problems of this class, then the task of creating a method for constructing a complete bifurcation picture (including branches of primary, secondary, tertiary, etc. branching) remains far from being solved. Strictly speaking, there is a certain gap between the deep theoretical constructions of bifurcation theory (and catastrophe theory) [1–3] and methods that allow us to construct a complete bifurcation picture for nonlinear partial differential equations, as well as analyze it within the framework of these theories. In this paper, we propose a method that allows us to construct a complete bifurcation picture in a wide class of nonlinear boundary value problems for partial differential equations that allow variational formulation.; The method generalizes the approach of N. I. Obodan and V. A. Gromov [4–8] to the general case of representing unknown functions of a problem within the framework of Kolmogorov's theorem–Arnold's.

The further presentation is structured as follows. The next section provides an overview of methods for solving and bifurcation analysis of partial differential equations based on a particular representation of a function of many variables as a superposition of functions of one variable. Section 2 presents the formulation of the problem, as well as the equations of the model problem (the Bratu problem); section 3 describes the method proposed in this article for solving nonlinear boundary value problems for both ordinary differential equations and partial differential equations. The method of reducing partial differential equations using Kolmogorov's theorem is described in Sections 3.5-3.8. Section 4 describes the results of a numerical experiment for the Bratu problem. The last section presents the conclusions.

1. Literature review

1.1. Representation of functions of many variables as a superposition of functions of one variable. Within the framework of the considered method, reduction is assumed (as part of an iterative process) The boundary value problem for nonlinear partial differential equations refers to a sequence of boundary value problems for systems of nonlinear ordinary differential equations; moreover, the bifurcation criterion for the boundary value problem for nonlinear partial differential equations is a combination of bifurcation criteria for the corresponding boundary value problems for ordinary differential equations. It should be noted that for ordinary differential equations in [9], a methodology was proposed for the numerical construction of a complete bifurcation structure and its analysis within the framework of catastrophe theory, based on the Keller-Antman theorem [10].

The above circumstance necessitates the representation of a function of many variables as a superposition of functions of one variable. The classical Fourier method assumes the representation of an unknown function in the form

$$u(x_1, x_2) \approx \sum_i h_i^*(x_1) g_i^*(x_2). \tag{1}$$

Here and further, the specified functions are marked with an asterisk; without an asterisk, the functions to be determined within the framework of a particular numerical algorithm appear. In the works of L. V. Kantorovich [11], a method was proposed in which the unknown functions of

the problem were represented as

$$u(x_1, x_2) \approx \sum_i h_i^*(x_1) g_i(x_2). \tag{2}$$

Substituting the representation (2) into a variational formulation of the problem followed by variation over unknown functions $g(x_2)$ reduces the problem of bifurcation analysis of boundary value problems for a system of nonlinear partial differential equations to the problem of bifurcation analysis of boundary value problems of systems of ordinary differential equations. The article [12] provides an overview of existing methods for reducing partial differential equations to ordinary differential equations (Fourier method, Galerkin method, etc.). In the works of N. I. Obodan and V. A. Gromov [4–8, 13, 14] an iterative generalized Kantorovich method was proposed (IGKM), based on the representation of the form

$$u(x_1, x_2) \approx \sum_i h_i(x_1) g_i(x_2). \tag{3}$$

Here, the definition (within the framework of the iterative process) is already subject to both functions $h_i(x_1)$, and $g_i(x_2)$.

The representation (3) allows us to reduce the problem of bifurcation analysis of boundary value problems for nonlinear partial differential equations to the problem of bifurcation analysis of boundary value problems for nonlinear ordinary differential equations [5]. The results obtained here are in good agreement[1] with theoretical results obtained within the framework of the theory of singularities of differentiable maps (catastrophe theory) - the order of degeneracy of singular points, acceptable variations in the number of maxima and minima of a potential function at the intersection of a bifurcation set in the parameter space, etc. [3]. In particular, the complete bifurcation structure here was constructed for the Karman equations (shell theory equations) [5]. This representation proved to be effective in a very wide range of tasks. It seems to us that there are two reasons for this. On the one hand, such representations can be considered as the construction of a basis that is «optimal» for a given task: here we do not «impose» a certain predefined basis (for example, the Fourier basis), but we are looking for the basis that is best for this task. As a result, the number of terms in the representation (3) is usually very small compared to methods based on Fourier series expansion, where it is assumed that the function is decomposed according to a basis that is the same for all tasks. Two or three terms are usually sufficient here. On the other hand, in a significant number of problems, the solution has the form (3) or close to it.

However, the representation (3) is obviously not the most general representation of a function of many variables using functions of a single variable. The most general representation is given by the Kolmogorov-Arnold theorem [15][2]. As formulated by D. Spreher, the Kolmogorov-Arnold theorem takes the form [17, 18]:

$$u(x_1, x_2, \ldots, x_n) = \sum_{q=0}^{2n} \Phi_q \Big(\sum_{p=1}^{n} \alpha_p \psi(x_p + aq) \Big), \tag{4}$$

where the constants $a, \alpha_p, p = 1, ..., n$ and the internal function $\psi(x)$ are the same for all continuous $u(x_1, ..., x_n)$. It should be emphasized that the representation (4), unlike the representations

[1] This is far from always the case for methods based on finite differences, finite elements, etc.

[2] We also note the work of Majorov–Pinkus [16], which gives a formula for the best approximation (not an exact representation!) functions of many variables are a superposition of differentiable functions of one variable.

(1)–(3), is an exact equality; however, the number of terms here is small and is determined only by the number of arguments to the function u.

In the Hedberg-Kahane formulation, the Kolmogorov-Arnold theorem takes the form [19]:

$$u(x_1, x_2, \ldots, x_n) = \sum_{q=0}^{2n} \chi\Big(\sum_{p=1}^{n} \lambda_p \psi_q(x_p) \Big). \tag{5}$$

Unlike the previous formulations, Hedberg's formulation is true for «almost all» sets of functions $(\psi_0(x), ..., \psi_{2n}(x))$, where ψ_i belongs to the class of continuous non-decreasing functions satisfying the conditions $\psi_i(0) = 0, \psi_i(1) = 1$, for which there are integer independent constants $\lambda_1, ..., \lambda_n$, $\sum_{i=1}^{n} \lambda_i = 1$.

Finally, in the formulation of R. Doss [20] the Kolmogorov–Arnold theorem takes the form

$$u(x_1, \ldots, x_n) = \sum_{i=1}^{2n+1} \varphi\Big(\prod_{p=1}^{n} \psi_{i,p}(x_p) \Big). \tag{6}$$

This formulation is also valid for "almost all" $n(2n+1)$ sets $\varphi_{i,p} \in C(I)^{n(2n+1)}$. For the convenience of constructing the computational process, a generalization of Doss's formulation was used (7):

$$u(x_1, \ldots, x_n) = \sum_{i=1}^{2n+1} \varphi_i\Big(\prod_{p=1}^{n} \psi_{i,p}(x_p) \Big). \tag{7}$$

This formulation proved to be the most convenient from the standpoint of constructing the computational process.

Criticism of the Kolmogorov-Arnold theorem is based on three circumstances. First, the internal (Kolmogorov-Arnold and Sprecher formulations) or external (Hedberg, Doss, and Sprecher formulations) functions are everywhere continuous but nowhere differentiable (the "devil's staircase"). In particular, for a two-dimensional function $u(x_1, x_2) = \sin(\pi * x_1) \sin(\pi * x_2)$, the external function $\Phi(z)$ in the Sprecher representation (33) has the form (Fig.1).

Secondly, the original proof lacked a constructive algorithm for constructing both outer and inner functions. Thirdly, even the small number of functions (both inner and outer) that appear in the representation (6) in some cases appears too large and should be reduced.

The solution of the first of these problems is probably impossible in the general case. Here we can refer to the works of A. G. Vitushkin [21, 22], which show that on the cube I_3 one can define a k times continuously differentiable function of three variables that cannot be represented on this cube as a finite superposition of $\left[\frac{2}{3}k\right]$ times differentiable functions of two variables, with partial derivatives of order $\left[\frac{2}{3}k\right]$ that satisfy the Lipschitz condition, where k is an arbitrary positive integer. It should be emphasized, however, that since, unlike the general case considered in the Kolmogorov-Arnold theorem, we are seeking a solution to a partial differential equation, all necessary derivatives of the inner and outer functions exist and are differentiable on the desired solution. Which, of course, turns out to be far from the case as soon as we deviate even slightly from the solution. On the other hand, the algorithms for constructing inner functions proposed for various forms of the Kolmogorov-Arnold theorem are iterative algorithms that calculate the values of the specified functions at a finite number of points, the greater the iteration number. This means that at each specific iteration we have the value of the function at a finite number of points, and by approximating these values in one way or another (for example, with splines), we can consider them — for a finite iteration — to belong to any desired smoothness class. [3].

[3] In the limit in the number of iterations, they converge, according to Kolmogorov's theorem, to everywhere continuous, nowhere differentiable functions.

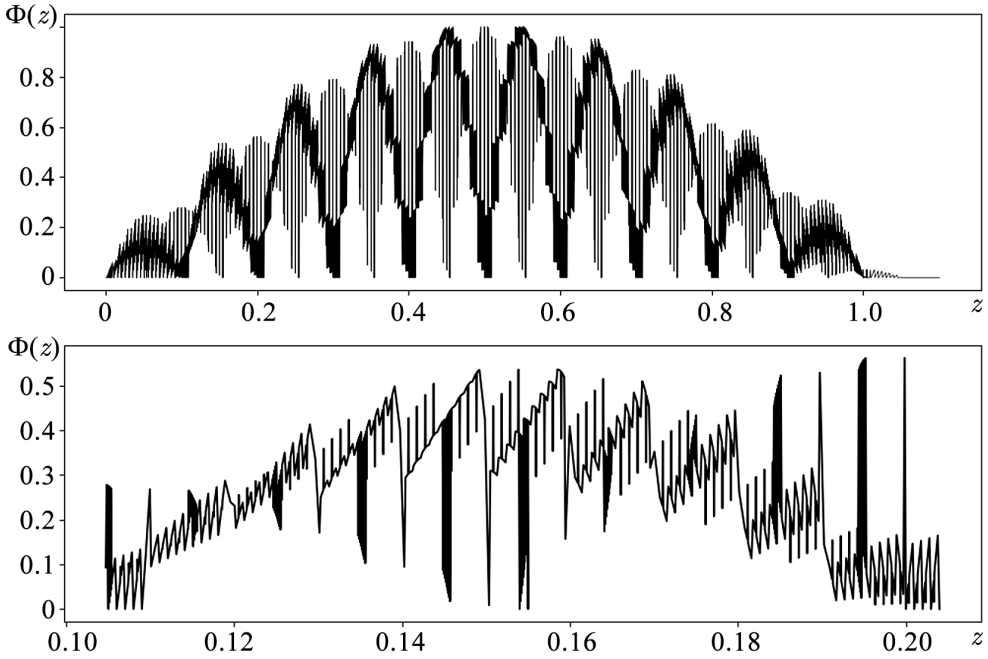

Fig. 1. Graph of the function $u(x_1, x_2) = \sin(\pi * x_1) \sin(\pi * x_2)$ in the Sprecher's representation (33)

Here we are "dangerously"approaching the question of the nature of mathematical modeling and differential equations. We will adhere to the point of view — admitting, however, the existence of the exact opposite — that we are modeling a real process, and differential relations are merely a convenient means of describing it. Consequently, our assumptions of infinite divisibility and similar ones, which underlie most differential equations, are merely assumptions convenient for deriving differential relations—it is obvious that merely halving the scale by a factor of a hundred brings us to the "world"of discrete atoms separated by gigantic distances, and further halving brings us to the quantum world. We must then assume that differential equation analysis algorithms, which allow us to describe the behavior of the corresponding system for an arbitrary but finite level of discretization, are a "legitimate"means of describing real processes.

A significant number of works are devoted to solving the second of these problems, which can be conditionally divided into two directions. Within the framework of the first direction, attempts are made to obtain constructive algorithms for constructing inner functions for formulations with fixed inner functions. Here, it is necessary to note, first of all, the works of D. Sprecher [17, 18], in which an iterative algorithm for constructing outer functions is proposed. M. Koppen [23] pointed out an error in Sprecher's algorithm for constructing an inner function, which leads to the non-monotonicity of this function — a violation of a necessary condition of the Kolmogorov–Arnold theorem. He also proposed a correct algorithm for constructing the inner function. J. Actor [24] proposes an algorithm for constructing Lipschitz-continuous inner functions, thereby solving a serious computational problem; In addition, the algorithm allows one to control the slope of functions. The paper also provides an overview of various formulations of the Kolmogorov-Arnold theorem. The second approach to solving the second of these problems involves constructing optimal approximations (for which the exact equality in formulations such as (4), (5), (6) is replaced by an approximate one), differentiable the required number of times, by functions of one variable. Thus, in the works of Mayorov-Pincus [16], an approximation using differentiable sigmoid functions is proposed. Related to this approach are the works of Tegmark [25], where

the Kolmogorov-Arnold theorem is used in the architecture of neural networks. In this approach, the inner functions are approximated by B-splines.

Among the works devoted to solving the third of these problems (reducing the number of functions), the articles [17, 26] are noteworthy. For example, in the article [17], D. Sprecher formulates a theorem that uses only one inner function (he uses linear shifts of its argument) and also proposes an algorithm for constructing this function. In the work by Lorentz [26], only two functions are used – one inner and one outer (he uses linear shifts of the inner and outer functions).

1.2. Algorithm for constructing a complete bifurcation structure. The algorithm for constructing a complete bifurcation structure for static problems for partial differential equations typically includes: an algorithm for identifying singular points (bifurcation criterion); Algorithm for classifying singular points with the identification of limit singular points and bifurcation points[4] and the order of their degeneracy; an algorithm for continuation by a parameter and a change in the continuation parameter to overcome limit singular points (path-tracing); an algorithm for constructing post-bifurcation branches of the solution emanating from a given bifurcation point. It should be emphasized here that the results of applying this algorithm should be consistent with the known theoretical results of bifurcation theory and catastrophe theory (for example, on the order of degeneracy of singular points – single or double) and should not depend on the number of approximation nodes. The latter requirement virtually excludes the possibility of applying grid methods for solving boundary value problems for partial differential equations, as well as a significant part of meshless methods (a review of the latter can be found in [27]). Within the framework of the presentation (3) for constructing a bifurcation picture of nonlinear partial differential equations, a new iterative approach to solving partial differential equations was proposed in the works [4,5,9,13,28] — the iterative generalized Kantorovich method (IGKM).

In this method, the desired functions of several variables are represented as a sum of products of one-dimensional functions (3), which allows one to reduce the boundary value problem of partial differential equations to a sequence of boundary value problems of ordinary differential equations. This approach also allows one to reduce (within the framework of the (3) representation) the bifurcation analysis of boundary value problems for nonlinear partial differential equations to the bifurcation analysis of boundary value problems for nonlinear ordinary differential equations. A significant advantage of this approach is that the results of the bifurcation analysis do not depend on the discretization method used, nor on the number of discretization points, but are determined only by the order of the equation itself, which allows one to fully apply the theorems of bifurcation theory to the analysis of partial differential equations. [5] In the works [4, 5, 8, 9, 28, 30] this method and associated algorithms are applied to the Karman equations of thin-walled shell theory to obtain solutions; Here, a complete bifurcation picture of the corresponding nonlinear boundary value problem is presented. It should be noted that, based on the proposed methodology of bifurcation analysis (solving the direct problem of bifurcation theory), the papers [5,6,31] consider the problem of predicting bifurcations and determining the state preceding a bifurcation (the inverse problem of bifurcation theory).

The representation (3) is not the most general representation of a function of several variables through a superposition of functions of one variable, which necessitates the use of different formulations of Kolmogorov's theorem to construct methods for constructing a complete

[4] In some cases, a combined option (hill-top bifurcation) is possible.

[5] Classical approaches to solving boundary value problems for partial differential equations (finite element method, finite difference method, etc.) usually allow one to solve the equation under study, but the results of the bifurcation analysis may differ qualitatively depending on the discretization order. [29].

bifurcation picture based on the IGKM representation.

In this study, a method for numerically constructing a complete bifurcation picture for boundary value problems of nonlinear partial differential equations is proposed, based on the representation of Kolmogorov's theorem in the Sprecher (4), Hedberg (5) and Doss (6) forms.

2. Problem Statement

We consider a boundary value problem for a nonlinear partial differential equation given by its variational statement:

$$\Xi(u) \longrightarrow \underset{u}{extr}, \tag{8}$$

where $\Omega \subseteq \mathbb{R}^n$ is a simply connected convex domain; $u \in W_2^k(\Omega)$ is an unknown function; $W_2^k(\Omega)$ is a Sobolev space, and F satisfies the necessary condition for a local extremum [32]:

$$\Xi = \int\limits_{\Omega}\int \ldots \int F\Big(x_1, \ldots, x_n, u(x_1, \ldots, x_n), \frac{\partial u(x_1, \ldots, x_n)}{\partial x_{i_1}}, \\ \frac{\partial^2 u(x_1, \ldots, x_n)}{\partial x_{i_1} \partial x_{i_2}}, \ldots, \frac{\partial^N u(x_1, \ldots, x_n)}{\partial x_{i_1} \ldots \partial x_{i_N}}\Big) dx_1 dx_2 \ldots dx_n \tag{9}$$

The constraints are boundary conditions of the form

$$S(u)\Big|_{\partial\Omega} = 0, \tag{10}$$

where $\partial\Omega$ is the boundary of a simply connected convex domain Ω in a finite-dimensional space $\mathbb{R}^n$, S is a linear operator that maps u to $\{u : u|_{\partial\Omega} = 0\}$.

It is necessary to propose a method for constructing a complete bifurcation structure for the nonlinear boundary value problem (8)–(10), which allows:

1. Construct solutions to a nonlinear boundary value problem and branches of a solution function depending on a parameter.
2. Identify singular points of the solution.
3. Determine their type and order of degeneracy[6].
4. Construct post-bifurcation branches of the solution.

Ultimately, this method should allow one to construct a complete bifurcation picture for a boundary value problem for nonlinear partial differential equations, including primary, secondary, tertiary, and other branches.

The Bratu equations are considered as model equations. [33, 34]:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \gamma e^{u(x_1, x_2)} = 0, \qquad (x_1, x_2) \in \Omega \tag{11}$$

with Dirichlet boundary conditions:

$$u(x_1, x_2) = 0, \qquad (x_1, x_2) \in \partial\Omega, \tag{12}$$

where $\Omega = [0, 1]^2$. The variational problem, equivalent to the differential one, is given by the formula (13):

$$\int_0^1 \int_0^1 \left[\Big[\frac{\partial u}{\partial x_1}\Big]^2 + \Big[\frac{\partial u}{\partial x_2}\Big]^2 - 2\gamma e^{u(x_1, x_2)} \right] dx_1 dx_2 \longrightarrow \underset{u(x_1, x_2)}{extr}. \tag{13}$$

The existence of a solution to the Bratu problem is proved, for example, in the works [35, 36].

[6] Here it is assumed that the specified characteristics will not depend on the approximation method, but only on the order of the equations themselves, and will be consistent with the theoretical results of bifurcation theory.

3. Method for numerically constructing a complete bifurcation structure of a nonlinear boundary value problem

3.1. Method for reducing a nonlinear boundary value problem to an equivalent Cauchy problem using the iterative formula of Newton's method (hereinafter — for brevity — Newton's method). Ultimately, any of the methods considered below for solving nonlinear boundary value problems for partial differential equations is reduced to a sequence of solving boundary value problems for ordinary differential equations. Newton's method was chosen to solve the boundary value problem for ordinary differential equations. A significant advantage of this method is that it allows one to easily formulate a bifurcation criterion for this type of problem (see Section 2.2). A detailed description of the method can be found in [5, 37].

Let us consider a boundary value problem for a system of ordinary differential equations (12):

$$\begin{cases} y'(x) = f\big(x, y(x), \gamma\big), & (12) \\ h\big(x_0, x_1, y(x_0), y(x_1)\big) = 0. & (13) \end{cases}$$

Here $x \in \mathbf{R}^n$, $y(x)$ is the vector function of the unknowns of the problem, $f\big(x, y(x), \gamma\big)$ is the vector function of the right-hand sides, γ is the parameter vector, and $h\big(x_0, x_1, y(x_0), y(x_1), \gamma\big)$ are the boundary conditions.

The method involves reducing the problem under consideration to an equivalent Cauchy problem. (14):

$$\begin{cases} y'(x) = f\big(x, y(x), \gamma\big), \\ y(x^*) = \xi, \end{cases} \tag{14}$$

where ξ is the vector of values of unknown functions of the problem $y(x)$ at an arbitrary but fixed point $x^* \in [x_0, x_1]$. Integration by any known method of numerical integration[7] from the point x^* to the boundaries of the integration interval allows us to calculate the residuals of the problem as functions of the vector (15):

$$\varphi(\xi, \gamma) = h\big(x_0, x_1, y(x_0, \xi), y(x_1, \xi), \gamma\big) = 0. \tag{15}$$

Here $\varphi(\xi, \gamma)$ is the vector of residuals implicitly defined through the solution of the Cauchy problem (14).

Therefore, the problem (12)–(13) is reduced to the problem of finding the vector ξ that makes the residual vector $\varphi(\xi, \gamma)$ vanish, that is, finding a solution to the system of transcendental equations (15).

To find a solution, Newton's method is used, the iterative formula of which is (16):

$$\xi^{(k+1)} = \xi^{(k)} - \mathcal{F}^{-1}\varphi(\xi^{(k)}, \gamma), \tag{16}$$

where $\mathcal{F}$ is the numerical analogue of the matrix of Frechet derivatives:

$$\mathcal{F}_{i,j} \approx \frac{\varphi_i(\xi_0, \ldots, \xi_j + \Delta, \ldots, \xi_n, \gamma) - \varphi_i(\xi_0, \ldots, \xi_j, \ldots, \xi_n, \gamma)}{\Delta}, \tag{17}$$

and Δ is a small but finite increment.

[7] We used the 4th-order Runge–Kutta method.

3.2. Variational formulation of the problem and its connection with catastrophe theory. The variational formulation of the boundary value problem (12)–(13) can be written as

$$J(y, y') \longrightarrow \min . \tag{18}$$

Adding all or part of the stationarity conditions of the functional to the variational formulation as constraints does not change the solution of the variational problem [10, 38]. Accordingly, adding conditions (13) to the functional (18) transforms it into a function of a finite number of variables ξ and a parameter vector γ, that is, into a mathematical object that can be studied within the framework of catastrophe theory:

$$H(y, y', \gamma; \xi) \longrightarrow \text{extr}. \tag{19}$$

The gradient of the function (19) yields the set of residuals (15), and its Hessian yields the Fréchet matrix (17). These circumstances mean that the problem (12)–(13), in the case of a variational formulation (18), can be studied not only by bifurcation theory methods, but also by catastrophe theory methods [1, 3, 29].

It should be emphasized that the order of the Fréchet matrix in this approach is determined not by the number of approximation nodes, but solely by the order of the problem itself (the number of equations in the system). This allows us to obtain numerical results that are in good agreement with those of catastrophe theory and bifurcation theory [3, 10, 39]: in particular, for most of the systems studied by the authors, the order of degeneracy of the matrix was one or two, with the observed solution surfaces corresponding to cuspoid and umbilical catastrophes, respectively, as predicted by catastrophe theory.

Within the above-formulated approach to solving nonlinear boundary value problems, a critical point of a potential function (19) is equivalent to the solution (12)–(13), and a degenerate critical point is equivalent to a singular point of the solution to this problem. This allowed Keller and Antman [10] to formulate a criterion for identifying and classifying singular points of a solution (critical points of a potential function (19)). A point (ξ, γ) is a singular point if, for a given value of the parameter vector, the Fréchet matrix $\mathcal{F}$ is singular:

$$\det(\mathcal{F}) = 0. \tag{20}$$

In practice, the approximation determinant (17) calculated at the last iteration of Newton's method is used.

Two important characteristics here are the order of singularity of the matrix $\mathcal{F}$ and its type. The order of singularity l (corank) of a matrix $\mathcal{F}$ of dimension $n \times n$ is defined as $l = n - r$, where r is the rank of this matrix. Since the matrix $\mathcal{F}$ is determined through the numerical solution of the problem of reducing a boundary value problem to an equivalent Cauchy problem, the corank of the matrix can also be numerically approximated. Within the framework of the proposed method, the numerical corank of the matrix is considered: to determine the numerical corank of the matrix $\mathcal{F}$, its singular values are calculated. Here it is convenient to follow the singular values of the matrix $\mathcal{F}$ as functions of the continuation parameter on the interval containing the singular point (see, for example, Fig. 2, Fig. 3): some singular values touch the point zero (vanish to zero), while some do not (although they may be close to zero, but do not touch).

The first case corresponds to a single degeneracy (cuspoid catastrophes), the second to a double degeneracy (umbilical catastrophes).[8] Here, a more subtle analysis is possible within

[8] On the graph of the dependence of the determinant of the matrix $\mathcal{F}$ on the parameter, these cases correspond to the intersection of the graph of the determinant of the x-axis and its tangency with this axis.

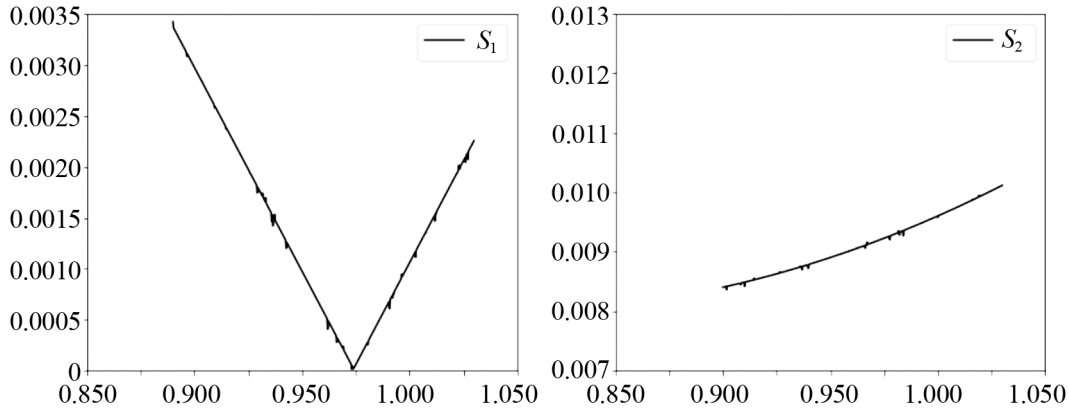

Fig. 2. From left to right: graphs of the behavior of the first and second singular values of the matrix $\mathcal{F}$ in the case of single degeneracy

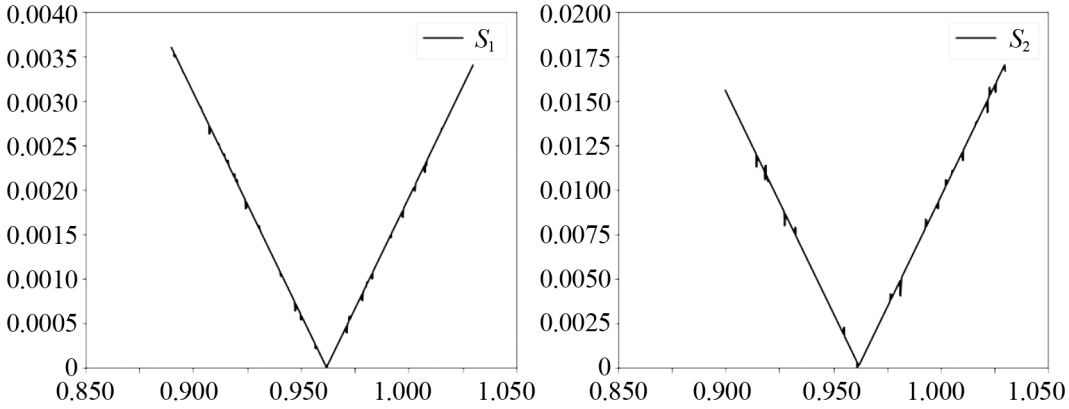

Fig. 3. From left to right: graphs of the behavior of the first and second singular values of the matrix $\mathcal{F}$ in the case of double degeneracy

the framework of catastrophe theory, allowing us to classify the observed diversity of solution branches as belonging to the type of one of the elementary catastrophes (see, for example, [37]), and the properties of elementary catastrophes are well known [2, 3, 29].

Another classification of singular points of the solution divides them into limit singular points and bifurcation points. To determine the type of singular point, an extended Fréchet matrix is constructed, which is obtained from the Fréchet matrix by adding a column of derivatives with respect to the parameter γ:

$$\mathcal{F}^*_{i,n+1} = \frac{\varphi_i(\xi, \gamma + \Delta) - \varphi_i(\xi, \gamma)}{\Delta}. \tag{21}$$

Let $\mathcal{F}^*_k$ be the matrix obtained from $\mathcal{F}^*$ by removing the k-th column. Then the type of the singular point is determined by the following criteria:

$$\begin{aligned} &\forall k = 1, \ldots, n+1 : \det(\mathcal{F}^*_k) = 0, \implies (\xi, \gamma) \text{ is the bifurcation point,} \\ &\exists k = 1, \ldots, n+1 : \det(\mathcal{F}^*_k) \neq 0, \det(\mathcal{F}) = 0, \implies (\xi, \gamma) \text{ is the limil point.} \end{aligned} \tag{22}$$

3.3. Overcoming Limit Singularities and Reaching Post-Bifurcation Branches of the Solution at Bifurcation Points. The convergence of Newton's method is largely determined by a "successful"choice of the initial approximation. To obtain the best possible initial approximation, the parameter continuation method [40] can be used, where the initial

approximation for the next parameter value γ_{i+1} is found from solutions corresponding to previous parameter values $(\gamma_i, \gamma_{i-1}, ...)$, using Newton interpolation polynomials. In particular, we used third-degree polynomials:

$$\begin{aligned}
&\xi(\gamma_{i+1}) = \xi(\gamma_{i-2}) + \big[\gamma_{i+1} - \gamma_{i-2}\big]\xi(\gamma_{i-2}, \gamma_{i-1}) + \big[\gamma_{i+1} - \gamma_{i-2}\big]\big[\gamma_{i+1} - \gamma_{i-1}\big]\xi(\gamma_{i-2}, \gamma_{i-1}, \gamma_i),\\
&\xi(\gamma_l, \gamma_j) = \frac{\xi(\gamma_j) - \xi(\gamma_l)}{\gamma_j - \gamma_l},\\
&\xi(\gamma_l, \gamma_j, \gamma_k) = \frac{\xi(\gamma_j, \gamma_k) - \xi(\gamma_l, \gamma_j)}{\gamma_k - \gamma_l}.
\end{aligned} \tag{23}$$

Here $\xi(\gamma_j), j \leqslant i$ are the values of the solution functions of the boundary value problem (12)–(13) at the point $x^* \in [x_0, x_1]$, corresponding to the solution for the value of the parameter γ_j.

If further movement along the parameter is impossible (for example, if a solution does not exist for large values of the parameter, as is the case for limit singular points — see Fig. 7), the method of changing the leading parameter is used: the fastest growing component ($\xi : i = \underset{i}{\operatorname{argmax}}|\xi_i(\gamma_{j+1}) - \xi_i(\gamma_j)|$) is derived from the initial approximation vector ξ, and then movement is performed along it as along the parameter. The old parameter takes the place of this component in the vector ξ. The iterative formula (16) is preserved, but the matrix of derivatives $\mathcal{F}$ is replaced by the matrix $\mathcal{F}_\gamma$, in which the column of derivatives with respect to the derived component is replaced by a column of derivatives with respect to the parameter γ. This approach helps overcome limiting singular points of the solution.

3.4. Representing a Function of Many Variables as a Superposition of a Function of One Variable Using the Sprecher Representation. To represent the unknown function of the boundary value problem, given by its variational formulation (8), (10), as a superposition of functions of one variable, we use Kolmogorov's theorem in Sprecher's formulation (4) and the fact that a is a small parameter.[9] We expand the representation (4) into a Taylor series in the parameter a:

$$\sum_{q=0}^{2n} \Phi_q\Big(\sum_{p=0}^{n} \alpha_p \psi(x_p + qa)\Big) = \sum_{m=0}^{M} \frac{a^m}{m!}\frac{d^m}{da^m}\left(\sum_{q=0}^{2n} \Phi_q\Big(\sum_{i=p}^{n} \alpha_p \psi(x_p + aq)\Big)\right)\Bigg|_{a=0}. \tag{24}$$

For a more compact notation, we'll use Bell polynomials and the Faa-di-Bruno formula.

The Bell polynomial is defined by the formula (25):

$$B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum\left(\frac{n!}{j_1! j_2! \ldots j_{n-k+1}!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{j_i}\right), \tag{25}$$

where the outer sum is taken over all non-negative $j_1, j_2, \ldots, j_{n-k+1}$ that satisfy two conditions (26), (27):

$$\sum_{i=1}^{n-k+1} j_i = k, \tag{26}$$

$$\sum_{i=1}^{n-k+1} i j_i = n. \tag{27}$$

[9] Thus, for $\gamma = 10$, which is the recommended value of the parameter for $n = 2$, we obtain $a = 1/90$. Since the only constraint on the parameter γ is $\gamma \geqslant 2n + 2$, we can set $\gamma \to \infty$, which yields $a \to 0$.

The Faa di Bruno formula (28) is used to represent the derivative of a complex function of arbitrary order:

$$\frac{d^m}{dx^m} f(g(x)) = \sum \left(\frac{m!}{j_1! j_2! \dots j_m!} \cdot f^{(j_1+\dots+j_m)}\big(g(x)\big) \prod_{i=1}^{m} \left(\frac{g^{(i)}(x)}{i!} \right)^{j_i} \right), \tag{28}$$

where the outer sum is taken over all non-negative integers that satisfy the condition (27). Using Bell polynomials, we write the Faa-di-Bruno formula as follows:

$$\frac{d^m}{dx^m} f(g(x)) = \sum_{k=0}^{m} f^{(k)}\big(g(x)\big) B_{m,k}\big(g'(x), g''(x), \dots, g^{(m-k-1)}(x)\big). \tag{29}$$

Then, using Bell polynomials, we obtain a compact form of the expansion of Kolmogorov's theorem in a Taylor series, as shown in the equation (30):

$$\begin{aligned} \frac{d^m}{da^m} \left(\sum_{q=0}^{2n} \Phi_q \Big(\sum_{p=1}^{n} \alpha_p \psi(x_p + aq) \Big) \right) \Bigg|_{a=0} = \sum_{k=0}^{m} \sum_{q=0}^{2n} \Phi_q^{(k)} \left(\sum_{p=1}^{n} \alpha_p \psi(x_p) \right) \times \\ \times q^m \tilde{B}_{m,k}(x_1, x_2 \dots, x_n), \end{aligned} \tag{30}$$

where

$$\begin{aligned} B_{m,k} \left(q \sum_{p=1}^{n} \alpha_p \psi'(x_p), q^2 \sum_{p=1}^{n} \alpha_p \psi''(x_p), \dots, q^{(m-k-1)} \sum_{p=1}^{n} \alpha_p \psi^{(m-k-1)}(x_p) \right) = \\ = q^m \tilde{B}_{m,k}(x_1, x_2 \dots, x_n). \end{aligned} \tag{31}$$

Then the representation of the Kolmogorov–Arnold theorem after expansion in a Taylor series in the neighborhood of $a = 0$ is given by the formula (32):

$$\sum_{q=0}^{2n} \Phi_q \left(\sum_{p=0}^{n} \alpha_p \psi(x_p + qa) \right) \approx \sum_{m=0}^{M} \sum_{k=0}^{m} \tilde{B}_{m,k}(x_1, \dots, x_n) \sum_{q=0}^{2n} \frac{a^m q^m}{m!} \Phi_q^{(k)} \left(\sum_{p=1}^{n} \alpha_p \psi(x_p) \right). \tag{32}$$

By making the substitution $z = \sum_{p=1}^{n} \alpha_p \psi(x_p)$, we obtain a representation of the function of several variables $u(x_1, \dots, x_n)$ as a sum of functions of one variable (33):

$$u(x_1, \dots, x_n) \approx \sum_{m=0}^{M} \sum_{k=0}^{m} \tilde{B}_{m,k}(x_1, x_2 \dots, x_n) \sum_{q=0}^{2n} \frac{a^m q^m}{m!} \Phi_q^{(k)}(z), \tag{33}$$

As a result, we obtain a representation of the unknown function of a nonlinear boundary value problem for a partial differential equation as a function of one variable.

3.5. Method for Reducing Partial Differential Equations to a System of Ordinary Differential Equations Using the Sprecher Representation [17]. To reduce a nonlinear boundary value problem for partial differential equations to a sequence of ordinary differential equation problems, we use the expansion (33) for the unknown functions of several variables included in the functional. We make a change of variables:

$$\begin{aligned} x_i &= \bar{x}_i, i = \overline{2, n}, \\ x_1 &= \psi^{-1} \left(\frac{z - \sum_{p=2}^{n} \alpha_p \psi(x_p)}{\alpha_1} \right), \end{aligned} \tag{34}$$

thereby turning $x_2, ..., x_n$ into parameters (their derivatives are no longer included in the functional). Then the integration boundaries change according to the formulas (35):

$$\begin{aligned} x_1^{\min} \mapsto z^{\min} &= \alpha_1\psi(x_1^{\min}) + \alpha_2\psi(\bar{x}_2) + \cdots + \alpha_n\psi(\bar{x}_n), \\ x_1^{\max} \mapsto z^{\max} &= \alpha_1\psi(x_1^{\max}) + \alpha_2\psi(\bar{x}_2) + \cdots + \alpha_n\psi(\bar{x}_n), \end{aligned} \tag{35}$$

and the Jacobian of the change of coordinates will have the form (36):

$$|J| = \begin{vmatrix} \alpha_1\psi'(x_1) & \alpha_2\psi'(x_2) & \dots & \alpha_n\psi'(x_n) \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix} = \alpha_1\psi'(x_1). \tag{36}$$

Let us express the partial derivatives of the function $u(x_1, \dots, x_n)$ through the ordinary derivatives of the functions $\Phi_i(z)$, differentiating the representation (33).

$$\frac{\partial^N u(x_1, \dots, x_n)}{\partial x_{i_1} \dots \partial x_{i_N}} = \sum_{m=0}^{M} \sum_{k=0}^{m} \left[\sum_{j=0}^{N} \left[\frac{\partial^N}{\partial x_{i_1} \dots \partial x_{i_N}} \left(\tilde{B}_{m,k}(x) \sum_{q=0}^{2n} \frac{a^m q^m}{m!} \Phi_q^{(k)}(z) \right) \right] \right]. \tag{37}$$

Note that as a result, any partial derivative of the function $u(x_1, \dots, x_n)$ is expressed in terms of ordinary derivatives of functions of one variable $\Phi_q(z)$. Thus, by successively applying the actions described above to the functional (9), we obtain a functional with respect to functions of one variable:

$$\begin{aligned} \tilde{\Xi}_S = \int_{z^{\min}}^{z^{\max}} L\Big(z, \Phi_0(z), \dots, \Phi_{2n}(z), \frac{d\Phi_0(z)}{dz}, \dots, \\ \frac{d\Phi_{2n}(z)}{dz}, \dots, \frac{d^N\Phi_0(z)}{dz^N}, \dots, \frac{d^N\Phi_{2n}(z)}{dz^N} \Big| \bar{x}_2, \dots, \bar{x}_n \Big) dz. \end{aligned} \tag{38}$$

Then the variational problem (8) is reduced to a parameterized variational problem for functions of one variable (39), which can be solved by methods of the calculus of variations.

$$\begin{gathered} \forall \{\bar{x}_2, \dots, \bar{x}_n\},\ \bar{x}_2 \in [\bar{x}_2^{\min}, \bar{x}_2^{\max}], \dots, \bar{x}_n \in [\bar{x}_n^{\min}, \bar{x}_n^{\max}] : \\ \tilde{\Xi}_S \to \underset{\Phi_q(z)}{\text{extr}}. \end{gathered} \tag{39}$$

By equating the variations of the functional with respect to the unknown functions to 0, we obtain a boundary value problem for ordinary differential equations. Methods for solving and performing bifurcation analysis for such problems are described above in sections 3.1, 3.2, 3.3.

3.6. A method for reducing partial differential equations to a system of ordinary differential equations using the Hedberg representation. Recall that the Hedberg representation (5) is valid for almost all inner functions $\psi_q(x)$. Similar to the previous approach based on the Sprecher representation, we make a change of variables:

$$z = \sum_{l=1}^{n} \lambda_l \psi_0(x_l) \implies x_1 = \psi_0^{-1}\Big(\frac{z - \sum_{l=2}^{n} \lambda_l \psi_0(x_l)}{\lambda_1} \Big), \tag{40}$$

thereby converting $x_2, ..., x_n$ into parameters (their derivatives are no longer included in the functional).

Then the integration boundaries change according to the formulas (41):

$$\begin{aligned} x_1^{\min} \mapsto z^{\min} &= \lambda_1\psi_0(x_1^{\min}) + \lambda_2\psi_0(\bar{x}_2) + \dots + \lambda_n\psi_0(\bar{x}_n), \\ x_1^{\max} \mapsto z^{\max} &= \lambda_1\psi_0(x_1^{\max}) + \lambda_2\psi_0(\bar{x}_2) + \dots + \lambda_n\psi_0(\bar{x}_n), \end{aligned} \tag{41}$$

and the Jacobian of the change of coordinates according to (42):

$$|J| = \begin{vmatrix} \lambda_1\psi_0'(x_1) & \lambda_2\psi_0'(x_2) & \dots & \lambda_n\psi_0'(x_n) \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix} = \lambda_1\psi_0'(x_1). \tag{42}$$

We define the functions $\theta_q(z)$ as follows:

$$\theta_q(z) = \lambda_1\psi_q\left(\psi_0^{-1}\left(\frac{z - \sum_{l=2}^{n}\lambda_l\psi_0(x_l)}{\lambda_1}\right)\right) + \sum_{l=2}^{n}\lambda_l\psi_q(x_l). \tag{43}$$

Then the formula (5) takes the form

$$u(x_1, ..., x_n) = \sum_{q=0}^{2n}\chi\big(\theta_q(z)\big), \tag{44}$$

and we express the partial derivatives of the function $u(x_1, ..., x_n)$ through functions of one variable, for example[10]:

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= \sum_{q=0}^{2n}\lambda_i\psi_q'(x_i)\chi'\big(\theta_q(z)\big), \\ \frac{\partial^2 u}{\partial x_i^2} &= \sum_{q=0}^{2n}\Big[\big[\lambda_i\psi_q'(x_i)\big]^2\chi''\big(\theta_q(z)\big) + \lambda_i\psi_q''(x_i)\chi'\big(\theta_q(z)\big)\Big]. \end{aligned} \tag{45}$$

The representation (45) involves ordinary derivatives with respect to different arguments. To reduce the derivatives to derivatives with respect to a single variable z, we use the Faa-di-Bruno formula for differentiating complex functions:

$$\begin{aligned} &\chi_z^{(n)}\big(\theta_q(z)\big) = \sum_{m=1}^{n}\chi^{(m)}\big(\theta_q(z)\big)B_{n,m}\big(\theta_q'(z), \dots, \theta_q^{(n-m+1)}(z)\big) \implies \\ &\chi^n\big(\theta_q(z)\big) = \frac{\chi_z^{(n)}\big(\theta_q(z)\big) - \sum_{m=1}^{n-1}\chi^{(m)}\big(\theta_q(z)\big)B_{n,m}\big(\theta_q'(z), \dots, \theta_q^{(n-m+1)}(z)\big)}{B_{n,n}\big(\theta_q'(z), \dots, \theta_q^{(n-m+1)}(z)\big)}. \end{aligned} \tag{46}$$

As a result, we obtain a functional (9) that depends on a function of one variable:

$$\tilde{\Xi}_H = \int_{\bar{x}}\int_{z^{\min}}^{z^{\max}} L\Big(z, \chi(\theta_q(z)), \frac{d\chi(\theta_q(z))}{dz}, \dots, \frac{d^N\chi(\theta_q(z))}{dz^N}, \Big|\bar{x}_2, \dots, \bar{x}_n\Big)dzd\bar{x}. \tag{47}$$

Then the variational problem (8) is reduced to a set of variational problems in ordinary derivatives, which are solved by varying the functional with respect to unknown functions and equating the

[10]Here the partial derivatives of the first and second order are presented. Obviously, this representation allows us to obtain expressions for derivatives of arbitrary order.

variations to 0. Methods for solving and bifurcation analysis of such problems are described above in the sections 3.1, 3.2, 3.3.

$$\begin{gathered}\forall\{\bar{x}_2,\dots,\bar{x}_n\},\ \bar{x}_2\in[\bar{x}_2^{\min},\bar{x}_2^{\max}],\dots,\bar{x}_n\in[\bar{x}_n^{\min},\bar{x}_n^{\max}]:\\ \tilde{\Xi}_H\to\underset{\chi(\theta_q(z))}{\operatorname{extr}}.\end{gathered}\tag{48}$$

3.7. A method for reducing partial differential equations to a system of ordinary differential equations using the Doss representation. We will use the representation (6) with the only exception that we will assume that all functions are known (here and below, given functions are marked with an asterisk), except $\varphi_i,\psi_{i,p},p=\overline{1,n}$:

$$\begin{aligned}u(x_1,...,x_n)=\varphi_i\Big(\prod_{p=1}^{n}\psi_{i,p}(x_p)\Big)+\sum_{q=0,q\neq i}^{2n}\varphi_q^*\Big(\prod_{p=1}^{n}\psi_{q,p}^*(x_p)\Big)=\\ =\varphi_i\Big(\prod_{p=1}^{n}\psi_{i,p}(x_p)\Big)+\Phi_i^*(x_1,...,x_n).\end{aligned}\tag{49}$$

Next, we introduce a new variable $z=\prod_{p=1}^{n}\psi_{i,p}(x_p)$ and express x_1 through $z,x_2,...,x_n$. For the inverse function to exist, $\psi_{i,1(x_1)}$ must be monotone, which is not the case in most cases. Therefore, first $\psi_{i,1}(x_1)$ is divided into monotonicity intervals $[x_1^{(k)},x_1^{(k+1)}]$ and on each interval the replacement is performed (50):

$$x_1=\psi_{i,1,k}^{(-1)}\Big(\frac{z}{\prod_{p=2}^{n}\psi_{i,p}(x_p)}\Big)=\bar{x}_1.\tag{50}$$

Then the integration boundaries change (51)

$$\begin{aligned}x_1^{(k)}\mapsto z^{(k)}=\psi_{i,1}(x_1^{(k)})\psi_{i,2}(\bar{x}_2)...\psi_{i,n}(\bar{x}_n),\\ x_1^{(k+1)}\mapsto z^{(k+1)}=\psi_{i,1}(x_1^{(k+1)})\psi_{i,2}(\bar{x}_2)...\psi_{i,n}(\bar{x}_n)\end{aligned}\tag{51}$$

and the Jacobian change of variables (52)

$$\begin{aligned}|J|=\begin{vmatrix}\psi'_{i,1}(x_1)\prod_{p=2}^{n}\psi_{i,p}(x_p) & \psi'_{i,2}(x_2)\prod_{p=1,p\neq2}^{n}\psi_{i,p}(x_p) & \dots & \psi'_{i,n}(x_n)\prod_{p=1}^{n-1}\psi_{i,p}(x_p)\\ 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1\end{vmatrix}=\\ =\psi'_{i,1}(x_1)\prod_{p=2}^{n}\psi_{i,p}(x_p).\end{aligned}\tag{52}$$

Using the new variable z, we express the function $u(x_1,...,x_n)$ through a function of z:

$$f(x_1,...,x_n)=\varphi_i(z)+\Phi_i^*(z,...,x_n).\tag{53}$$

Next we obtain two functionals: the first (54) by substituting the representation (49) into the functional (9)

$$\begin{aligned}\int\int_{\Omega}...\int L_1\Bigg(x_1,x_2,...,x_n,\varphi_i\Big(\prod_{p=1}^{n}\psi_{i,p}(x_p)\Big),...,\varphi_i^{(N)}\Big(\prod_{p=1}^{n}\psi_{i,p}(x_p)\Big),\\ \psi_{i,1}(x_1),...,\psi_{i,1}^{(N_1)}(x_1),...,\psi_{i,n}(x_n),...,\psi_{i,n}^{(N_n)}(x_n)\Bigg)dx_1dx_2...dx_n\end{aligned}\tag{54}$$

and the second (55) by replacing variables (50) and changing the order of integration in the functional (54).

$$\sum_k \int_{z^{(k)}}^{z^{(k+1)}} \int_{x_n^{\min}(z)}^{x_n^{\max}(z)} \dots \int_{x_2^{\min}(z)}^{x_2^{\max}(z)} L_2\Bigg(z, x_2, \dots, x_n, \varphi_i(z), \dots, \varphi_i^{(N)}(z), \\ \psi_{i,1}(\bar{x}_1), \dots, \psi_{i,1}^{(N_1)}(\bar{x}_1), \dots, \psi_{i,n}(x_n), \dots, \psi_{i,n}^{(N_n)}(x_n)\Bigg) dz dx_2 \dots dx_n. \tag{55}$$

Then, varying independently in $\psi_{i,p}(x_p)$ the functional (54), we obtain n ordinary integro-differential equations, one for each $\psi_{i,p}$.

$$\psi_{i,p}^{(N_p)}(x_p) = F_{i,p}\Big(x_p, \psi_{i,p}(x_p), \psi'_{i,p}(x_p), \dots, \psi_{i,p}^{(N_p-1)}(x_p)\Big), \quad p = \overline{1,n}. \tag{56}$$

By varying the functional (55) with respect to $\varphi_i(z)$, we obtain an ordinary integro-differential equation for the external function:

$$\varphi_i^{(N)}(z) = G_i\Big(z, \varphi_i(z), \varphi'_i(z), \dots, \varphi_i^{(N_p-1)}(z)\Big). \tag{57}$$

Then the entire method for finding a solution looks like this:

1. We initialize the functions $\varphi_i = 0, \psi_{q,p} = 0, q = \overline{1,2n}, p = \overline{1,n}$. We set $i = 0$.
2. While $i \leqslant 2n$:
   - 2.1. Let us set the outer function to be identical: $\varphi_i(z) = \varphi_i^{(0)}(z) = z$; and let the inner ones be arbitrary with the only condition that they satisfy the boundary conditions: $\psi_{i,p}(x_p) = \psi_{i,p}^{(0)}(x_p)$.
   - 2.2. While $\|\varphi_i^{(j+1)}(z) - \varphi_i^{(j)}(z)\| > \varepsilon_\varphi$:
     - 2.2.1. Let's find the inner functions using an iterative algorithm. For now $\| \prod_{p=1}^n \psi_{i,p}^{(k+1)}(x_p) - \prod_{p=1}^n \psi_{i,p}^{(k)}(x_p)\| > \varepsilon_\psi$:
       - 2.2.1.1. For each function $\psi_{i,p}^{(k+1)}(x_p)$ we solve the equation (56), considering other internal functions and the external one to be known (taken from the previous iteration).
     - 2.2.2. Let us find the outer function $\varphi_i^{(j+1)}(z)$ from (57), assuming that the inner functions are known.
   - 2.3. Functions with i^{th} iterations are fixed $\varphi_i(z) = \varphi_i^{(j)}(z), \psi_{i,p}(x_p) = \psi_{i,p}^{(k)}(x_p), p = \overline{1,n}$.
   - 2.4. $i = i + 1$.

3.8. Numerical implementation of Kolmogorov's theorem. When the reduction of a boundary value problem for partial differential equations to ordinary ones is based on Kolmogorov's superposition theorem as formulated by Sprecher, the inner functions are the same for all continuous functions of several variables and can be calculated using special algorithms. According to the theorem, an arbitrary function of n variables $f(x_1, \dots, x_n)$ can be represented as a sum of superpositions of functions of one variable (4). To calculate the function $\psi(x)$, we used M. Koppen's algorithm [23][11].

[11] The original algorithm proposed by D. Sprecher [17] contained a bug that caused $\psi(x)$ to be calculated incorrectly for some combinations of the algorithm's parameters.

In the remainder of this section, we provide a brief description of the algorithm. Let $\mathcal{D}_k = \{d_k\}$ be the set of convergent rational numbers given by the formula (58):

$$d_k = \sum_{j=1}^{k} \frac{i_j}{\gamma^k}, i_j = 0, \ldots, \gamma - 1, \tag{58}$$

where $k \in \mathbb{N}, \gamma \geqslant 2n + 2$ are the algorithm parameters. Next, we define the constants a, α_p according to the formulas (59):

$$\begin{aligned} a &= \frac{1}{\gamma(\gamma - 1)}, \\ \alpha_p &= \sum_{j=1}^{\infty} \gamma^{-(p-1)\frac{n^p - 1}{n-1}}. \end{aligned} \tag{59}$$

Then the function $\psi(x)$ is calculated recursively using the formula (60)

$$\psi^k(d_k) = \begin{cases} d_k, \text{ for } k = 1, \\ \psi^{k-1}\left(d_k - \dfrac{i_k}{\gamma_k}\right) + \dfrac{i_k}{\gamma^{(n^k-1)/(n-1)}} \quad \text{for} \quad k > 1, \quad i_k < \gamma - 1, \\ \dfrac{1}{2}\left(\psi_k\left(d_k - \dfrac{1}{\gamma^k}\right) + \psi_{k-1}\left(d_k + \dfrac{1}{\gamma_k}\right)\right) \quad \text{for} \quad k > 1, \quad i_k = \gamma - 1, \end{cases} \tag{60}$$

where $d_k = 0.i_1 i_2, \ldots, i_k$. For the case $x > 1$, the function $\psi(x)$ is calculated using the formula (61)

$$\psi(x) = \psi(x - [x]) + [x], \tag{61}$$

where $[x]$ is the integer part of x.

Figure 4 shows the graphs of the function $\psi(x)$ for the parameter values $\gamma = 10$ and $k = 2, 3, 4$, respectively.

The method of reducing partial differential equations to ordinary ones requires the ability to calculate the derivatives of the function $\psi(x)$, which is non-differentiable. However, it is possible to define a difference analogue of the derivative for it. (62)

$$\psi'(x) \approx \frac{\psi(x + \Delta) - \psi(x)}{\Delta}, \tag{62}$$

where Δ is an increment that depends on the parameters γ, k. Since the function $\psi(x)$ is defined on a finite number of points $\{d_k\}$, then $\Delta = \gamma^{-k}$ is the step size between two adjacent points

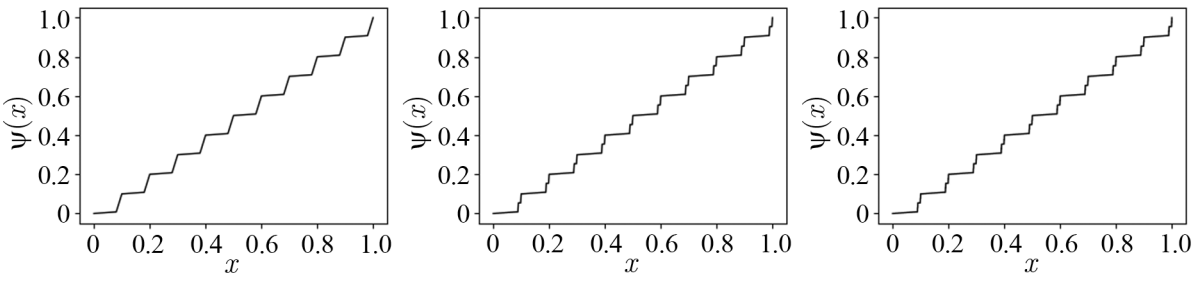

Fig. 4. Plots of $\psi(x)$ for $\gamma = 10$ и $k = 2, 3, 4$ respectively

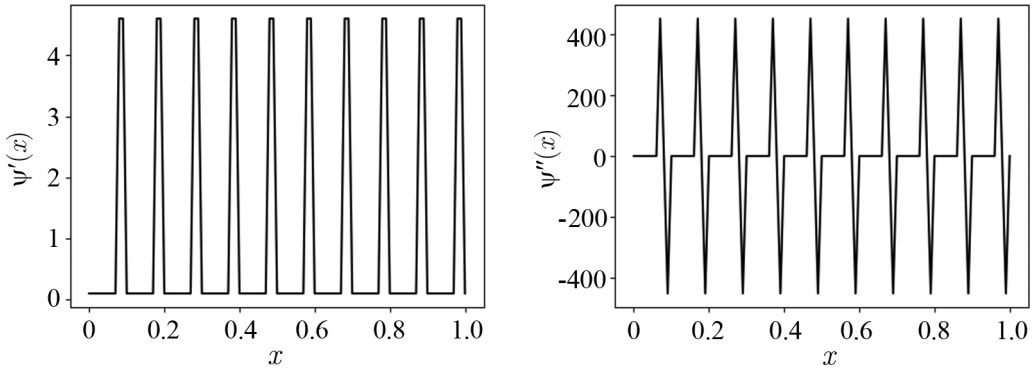

Fig. 5. Plots of first and second derivative of a function $\psi(x)$ for $\gamma = 10$ and $k = 2$

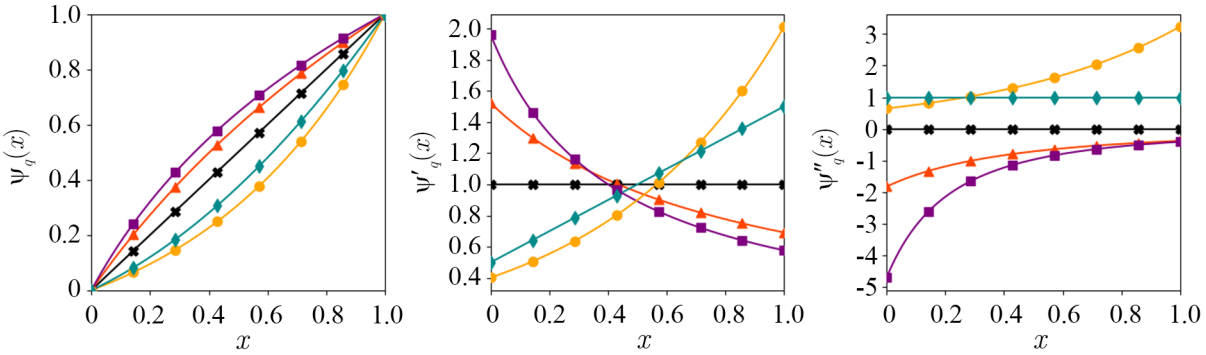

Fig. 6. From left to right: plots of functions $\psi_q(x), \psi'_q(x), \psi''_q(x)$ for Hedberg formulation, used in a computational experiment (color online)

from the set $\mathcal{D}_k$. Fig. 5 shows the graphs of the first and second derivatives of the function $\psi(x)$ for $\gamma = 10, k = 4$.

In the case where the formulation of Kolmogorov's theorem in the Hedberg representation is used, it is also necessary to determine the constants λ_p. In this case, according to the theorem [19], the inner functions $\psi_q(x)$ can be chosen almost arbitrarily with the only condition that they should not decrease on the interval $[0; 1]$ and take the values 0 and 1 at its ends, respectively: $\psi_q(0) = 0, \psi_q(1) = 1$. In the computational experiment, we used the options shown in Fig. 6. The constants λ_p are determined within the framework of the problem of finding the extremum of the functional (63).

$$\int_{\bar{x}} \tilde{\Xi}_S d\bar{x} \to \underset{\lambda_p}{\text{extr}}. \tag{63}$$

4. Results

The proposed method was applied to the two-dimensional Bratu problem. This problem is a nonlinear boundary value problem for partial differential equations. The bifurcation structure for this problem is well studied, allowing us to test the effectiveness of our bifurcation analysis method.

4.1. Derivation of resolving relations for the Bratu problem using Sprecher's formulation. The bifurcation analysis algorithm presented in the previous section was applied to the solution of the Bratu problem in a variational formulation (13). For this problem, $n = 2$.

We expand the representation of the function $u(x_1, x_2)$ in a Taylor series up to the zero term ($M = 0$). Then we obtain the representation (64):

$$u(x_1, x_2) = \sum_{q=0}^{4} \Phi_q(z) = \Phi(z), \tag{64}$$

where $z = \alpha_1\psi(x_1) + \alpha_2\psi(x_2)$. Let us perform a change of variables

$$\begin{aligned}
&x_2 = \bar{x}_2,\\
&x_1 = \psi^{-1}\Big(\frac{z - \alpha_2\psi(\bar{x}_2)}{\alpha_1}\Big),\\
&x_1 = 0 \to z^{\min} = \alpha_2\psi(\bar{x}_2),\\
&x_1 = 1 \to z^{\max} = \alpha_1 + \alpha_2\psi(\bar{x}_2),\\
&|J| = \begin{vmatrix} \alpha_1\psi'(x_1) & \alpha_2\psi'(x_2) \\ 0 & 1 \end{vmatrix} = \alpha_1\psi'(x_1) \implies dx_1 = \frac{dz}{\alpha_1\psi'\big(\frac{z-\alpha_2\psi(\bar{x}_2)}{\alpha_1}\big)}
\end{aligned} \tag{65}$$

and we will express the partial derivatives included in the functional of the Bratu problem through ordinary ones:

$$\begin{aligned}
&\frac{\partial u(x_1, x_2)}{\partial x_i} = \alpha_i\psi'(x_i)\sum_{q=0}^{4}\Phi_q'(z),\\
&\frac{\partial^2 u(x_1, x_2)}{\partial x_i^2} = \alpha_i\psi''(x_i)\sum_{q=0}^{4}\Phi_q'(z) + \alpha_i^2\psi'^2(x_i)\sum_{q=0}^{4}\Phi_q''(z).
\end{aligned} \tag{66}$$

Substituting the above substitutions into the functional, we vary it with respect to the unknown functions $\Phi(z)$. As a result, we obtain an ordinary differential equation (67), which was solved by the method of reducing PDEs to ODEs presented in section 3.1.

$$\begin{aligned}
\Phi''(z)\left[-\frac{2\alpha_2^2\psi'(x_2)^2}{\alpha_1\psi'\left(\psi^{(-1)}\left(\frac{z-\alpha_2\psi(x_2)}{\alpha_1}\right)\right)} - 2\alpha_1\psi'\left(\psi^{(-1)}\left(\frac{z-\alpha_2\psi(x_2)}{\alpha_1}\right)\right)\right] +\\
+\Phi'(z)\left[\frac{2\alpha_2^2\psi'(x_2)^2\psi''\left(\psi^{(-1)}\left(\frac{z-\alpha_2\psi(x_2)}{\alpha_1}\right)\right)}{\alpha_1^2\psi'\left(\psi^{(-1)}\left(\frac{z-\alpha_2\psi(x_2)}{\alpha_1}\right)\right)^3} - \frac{2\psi''\left(\psi^{(-1)}\left(\frac{z-\alpha_2\psi(x_2)}{\alpha_1}\right)\right)}{\psi'\left(\psi^{(-1)}\left(\frac{z-\alpha_2\psi(x_2)}{\alpha_1}\right)\right)}\right] -\\
-\frac{2\gamma e^{\Phi(z)}}{\alpha_1\psi'\left(\psi^{(-1)}\left(\frac{z-\alpha_2\psi(x_2)}{\alpha_1}\right)\right)} = 0.
\end{aligned} \tag{67}$$

4.2. Derivation of resolution relations for the Bratu problem using Hedberg's formulation. A method using Hedberg's representation was also applied to the Bratu problem. For the case of two variables, we obtain

$$u(x_1, x_2) = \sum_{q=0}^{4}\chi(\theta_q(z)), \tag{68}$$

where $z = \lambda_1\psi_0(x_1) + \lambda_2\psi_1(x_2)$, $\theta_q(z) = \lambda_1\psi_q\Big(\psi_0^{-1}\Big(\frac{z-\lambda_2\psi_0(x_2)}{\lambda_1}\Big)\Big) + \lambda_2\psi_q(x_2)$. Let us make a change of variables

$$\begin{aligned}
&x_2 = \bar{x}_2,\\
&x_1 = \psi_0^{-1}\Big(\frac{z-\lambda_2\psi_0(x_2)}{\lambda_1}\Big),\\
&x_1 = 0 \to z^{\min} = \lambda_2\psi_0(\bar{x}_2),\\
&x_1 = 1 \to z^{\max} = \lambda_1 + \lambda_2\psi_0(\bar{x}_2),\\
&|J| = \begin{vmatrix}\lambda_1\psi_0'(x_1) & \lambda_2\psi_0'(x_2)\\ 0 & 1\end{vmatrix} = \lambda_1\psi_0'(x_1) \implies dx_1 = \frac{dz}{\lambda_1\psi_0'\big(\frac{z-\lambda_2\psi_0(\bar{x}_2)}{\lambda_1}\big)}
\end{aligned} \tag{69}$$

and we define partial derivatives of the first order in terms of derivatives of a function of one variable $\chi(\theta_q(z))$:

$$\frac{\partial u}{\partial x_i} = \sum_{q=0}^{4}\lambda_i\psi_q'(x_i)\chi'\big(\theta_q(z)\big) \tag{70}$$

Variation of the problem functional by the function $\chi(\theta_q(z))$ gives $\forall\bar{x}_2 \in [0,1]$, we get:

$$\sum_{j=0}^{4}\int_{z^{\min}}^{z^{\max}}\left[\sum_{q=0}^{4}a_{q,j}(z,\bar{x}_2)\chi'(\theta_q(z))\delta\chi'(\theta_j(z)) - \gamma b(z,\bar{x}_2)e^{\sum_{q=0}^{4}\chi(\theta_q(z))}\delta\chi(\theta_j(z))\right]dz = 0, \tag{71}$$

where

$$\begin{aligned}
a_{q,j}(z,x_2) &= \frac{\lambda_1^2\psi_q(\bar{x}_1)\psi_j(\bar{x}_1)}{\lambda_1\psi_0'(\bar{x}_1)} + \frac{\lambda_2^2\psi_q(\bar{x}_2)\psi_j(\bar{x}_2)}{\lambda_1\psi_0'(\bar{x}_1)},\\
b(z,\bar{x}_2) &= \frac{1}{\psi_0(\bar{x}_1)}.
\end{aligned} \tag{72}$$

For the j-th element of the sum, we replace the integration variable from z to $\theta_j(z)$ (for brevity, denoting the new integration variable as z); we obtain

$$\begin{aligned}
\sum_{j=0}^{4}\int_{\theta_j(z^{\min})}^{\theta_j(z^{\max})}\frac{1}{\theta_j'\big(\theta_j^{-1}(z)\big)}\Bigg[&\sum_{q=0}^{4}a_{q,j}\big(\theta_j^{-1}(z),\bar{x}_2\big)\chi'\big(\theta_q(\theta_j^{-1}(z))\big)\delta\chi'(z) -\\
&-\gamma b\big(\theta_j^{-1}(z),\bar{x}_2\big)e^{\sum_{q=0}^{4}\chi\left(\theta_q(\theta_j^{-1}(z))\right)}\delta\chi(z)\Bigg]dz = 0.
\end{aligned} \tag{73}$$

Let us reduce all derivatives to derivatives with respect to the variable z, using the formula (46):

$$\chi'\big(\theta_q(\theta_j^{(-1)}(z))\big) = D_{1,q,j}(z)\chi_z'\big(\theta_q(\theta_j^{(-1)}(z))\big), \tag{74}$$

where

$$D_{q,j}(z) = \frac{\theta_j'(\theta_j^{(-1)}(z))}{\theta_q'(\theta_j^{(-1)}(z))}. \tag{75}$$

Then, integrating the terms with the derivatives of the variations by parts, we obtain

$$\sum_{j=0}^{4}\int_{\theta_j(z^{\min})}^{\theta_j(z^{\max})}\Bigg[-\sum_{q=0}^{4}\bar{a}_{q,j}\big(z,\bar{x}_2\big)\chi_z''\big(\theta_q(\theta_j^{-1}(z))\big)-\sum_{q=0}^{4}\frac{d\bar{a}_{q,j}\big(z,\bar{x}_2\big)}{dz}\chi_z'\big(\theta_q(\theta_j^{-1}(z))\big)-$$
$$-\gamma\bar{b}_j\big(z,\bar{x}_2\big)e^{\sum_{q=0}^{4}\chi\left(\theta_q(\theta_j^{-1}(z))\right)}\Bigg]\delta\chi(z)dz+ \qquad (76)$$
$$+\left[\sum_{q=0}^{4}\bar{a}_{q,j}\big(z,\bar{x}_2\big)\chi_z'\big(\theta_q(\theta_j^{-1}(z))\big)\right]\delta\chi'(z)\Bigg|_{\theta_j(z^{\min})}^{\theta_j(z^{\max})}=0,$$

where

$$\bar{a}_{q,j}(z,x_2)=\frac{D_{q,j}(z)a_{q,j}(\theta_j^{-1}(z),x_2)}{\theta_j'\big(\theta_j^{-1}(z)\big)},$$
$$\bar{b}_j(z,x_2)=\frac{b(\theta_j^{-1}(z),x_2)}{\theta_j'\big(\theta_j^{-1}(z)\big)}, \qquad (77)$$

which gives the following differential equation:

$$\forall x_2\in[0,1]:$$
$$\sum_{j=0}^{4}\mathbf{I}(z\in[\theta_j(z^{\min}),\theta_j(z^{\max})])\Bigg[\sum_{q=0}^{4}\bar{a}_{q,j}\big(z,\bar{x}_2\big)\chi_z''\big(\theta_q(\theta_j^{-1}(z))\big)+ \qquad (78)$$
$$+\sum_{q=0}^{4}\frac{d\bar{a}_{q,j}\big(z,\bar{x}_2\big)}{dz}\chi_z'\big(\theta_q(\theta_j^{-1}(z))\big)+\gamma\bar{b}_j\big(z,\bar{x}_2\big)e^{\sum_{q=0}^{4}\chi\left(\theta_q(\theta_j^{-1}(z))\right)}\Bigg]=0,$$

where $\mathbf{I}$ is the indicator function.

4.3. Derivation of resolving relations for the Bratu problem using the Doss formulation. A method using the Doss representation was applied to the Bratu problem. For the case of two variables, we obtain

$$u(x_1,x_2)=\varphi_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))+\Phi_i^*(x_1,x_2). \qquad (79)$$

Let us make a change of variables

$$z=\psi_{i,1}(x_1)\psi_{i,2}(x_2),$$
$$x_1=\psi_{i,1,k}^{(-1)}\Big(\frac{z}{\psi_{i,2}(x_2)}\Big)=\bar{x}_{1,k}, \qquad (80)$$

where $[x_k,x_{k+1}]$ are the monotonicity intervals of the function $\psi_{i,1}(x_1)$. The Jacobian of the change of variables is given by the formula (81):

$$|J|=\psi_{i,1}'(x_1)\psi_{i,2}(x_2)\implies dx_1=\frac{dz}{\psi_{i,1}'(\bar{x}_{1,k})\psi_{i,2}(x_2)}. \qquad (81)$$

After the change of variables, we obtain

$$\begin{aligned} u(x_1,x_2) &= \varphi_i(z) + \Phi_i^*(\bar{x}_{1,k},x_2),\\ \frac{\partial u}{\partial x_1} &= \psi'_{i,1}(\bar{x}_{1,k})\psi_{i,2}(x_2)\varphi'_i(z) + \frac{\partial \Phi_i^*}{\partial \bar{x}_{1,k}}(\bar{x}_{1,k},x_2),\\ \frac{\partial u}{\partial x_2} &= \psi_{i,1}(\bar{x}_{1,k})\psi'_{i,2}(x_2)\varphi'_i(z) + \frac{\partial \Phi_i^*}{\partial x_2}(\bar{x}_{1,k},x_2). \end{aligned} \tag{82}$$

Let us substitute the substitutions into the functional. Before the change of variables, we get (83):

$$\begin{aligned} L_1 = \int_0^1\int_0^1 &\left[\psi'_{i,1}(x_1)\psi_{i,2}(x_2)\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) + \frac{\partial \Phi_i^*}{\partial x_1}(x_1,x_2)\right]^2 +\\ &+\left[\psi_{i,1}(x_1)\psi'_{i,2}(x_2)\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) + \frac{\partial \Phi_i^*}{\partial x_2}(x_1,x_2)\right]^2 +\\ &- 2\gamma e^{\varphi_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))+\Phi_i^*(x_1,x_2)}dx_1dx_2. \end{aligned} \tag{83}$$

After replacing variables and changing the order of integration, we obtain (84):

$$\begin{aligned} L_2 = \sum_k \int_{z_k}^{z_{k+1}}\int_0^1 &\left[\left[\psi'_{i,1}(\bar{x}_{1,k})\psi_{i,2}(x_2)\varphi'_i(z) + \frac{\partial \Phi_i^*}{\partial \bar{x}_{1,k}}(\bar{x}_{1,k},x_2)\right]^2 +\right.\\ &+\left[\psi_{i,1}(\bar{x}_{1,k})\psi'_{i,2}(x_2)\varphi'_i(z) + \frac{\partial \Phi_i^*}{\partial x_2}(x_1,x_2)\right]^2 +\\ &\left.- 2\gamma e^{\varphi_i(z)+\Phi_i^*(\bar{x}_{1,k},x_2)}\right]\frac{1}{\psi'_{i,1}(\bar{x}_{1,k})\psi_{i,2}(x_2)}dx_2dz. \end{aligned} \tag{84}$$

We vary the functional (83) with respect to $\psi_{i,1}(x_1)$ and $\psi_{i,2}(x_2)$ to obtain the equations (85, 86), respectively:

$$\begin{aligned} \psi''_{i,1}(x_1) = &-\frac{1}{\int_0^1 \psi_{i,2}(x_2)^2\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))^2dx_2}\times\\ &\times\int_0^1\left[\gamma\psi_{i,2}(x_2)\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))e^{\varphi_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))+\Phi(x_1,x_2)} -\right.\\ &-\Phi^{(0,1)}(x_1,x_2)\psi_{i,1}(x_1)\psi_{i,2}(x_2)\psi'_{i,2}(x_2)\varphi''_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) +\\ &+\Phi^{(2,0)}(x_1,x_2)\psi_{i,2}(x_2)\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) -\\ &-\Phi^{(0,1)}(x_1,x_2)\psi'_{i,2}(x_2)\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) +\\ &+\psi_{i,2}(x_2)^3\psi'_{i,1}(x_1)^2\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))\varphi''_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) -\\ &-\psi_{i,1}(x_1)^2\psi_{i,2}(x_2)\psi'_{i,2}(x_2)^2\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))\varphi''_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) -\\ &\left.-\psi_{i,1}(x_1)\psi'_{i,2}(x_2)^2\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))^2\right]dx_2, \end{aligned} \tag{85}$$

$$\psi''_{i,2}(x_2) = -\frac{1}{\int_0^1 \psi_{i,1}(x_1)^2 \varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))^2 dx_1} \times$$
$$\times \int_0^1 \Big[\gamma \psi_{i,1}(x_1) \varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) e^{\varphi_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))+\Phi(x_1,x_2)} -$$
$$- \Phi^{(1,0)}(x_1,x_2)\psi_{i,2}(x_2)\psi_{i,1}(x_1)\psi'_{i,1}(x_1)\varphi''_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) +$$
$$+ \Phi^{(0,2)}(x_1,x_2)\psi_{i,1}(x_1)\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) -$$
$$- \Phi^{(1,0)}(x_1,x_2)\psi'_{i,1}(x_1)\varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) +$$
$$+ \psi_{i,1}(x_1)^3 \psi'_{i,2}(x_2)^2 \varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))\varphi''_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) -$$
$$- \psi_{i,2}(x_2)^2 \psi_{i,1}(x_1)\psi'_{i,1}(x_1)^2 \varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))\varphi''_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2)) -$$
$$- \psi_{i,2}(x_2)\psi'_{i,1}(x_1)^2 \varphi'_i(\psi_{i,1}(x_1)\psi_{i,2}(x_2))^2 \Big] dx_1. \tag{86}$$

As boundary conditions for the first and second equations we take the Dirichlet boundary conditions:

$$\begin{cases} \psi_{i,j}(0) = 0, \\ \psi_{i,j}(1) = 0. \end{cases} \tag{87}$$

Next, we vary the functional (84) with respect to the function $\varphi_i(z)$, and obtain

$$\varphi''_i(z) = \sum_k \int_0^1 -\gamma \psi'_{i,1}(\bar{x}_{1,k}) e^{\Phi(\bar{x}_{1,k},x_2)+\varphi_i(z)} +$$
$$+ \bar{x}'_{1,k}\Phi^{(0,1)}(\bar{x}_{1,k},x_2)\psi'_{i,2}(x_2)\psi_{i,1}(\bar{x}_{1,k})\psi''_{i,1}(\bar{x}_{1,k}) -$$
$$- \bar{x}'_{1,k}\Phi^{(0,1)}(\bar{x}_{1,k},x_2)\psi'_{i,2}(x_2)\psi'_{i,1}(\bar{x}_{1,k})^2 -$$
$$- \bar{x}'_{1,k}\psi_{i,2}(x_2)\Phi^{(2,0)}(\bar{x}_{1,k},x_2)\psi'_{i,1}(\bar{x}_{1,k})^2 -$$
$$- \bar{x}'_{1,k}\Phi^{(1,1)}(\bar{x}_{1,k},x_2)\psi'_{i,2}(x_2)\psi_{i,1}(\bar{x}_{1,k})\psi'_{i,1}(\bar{x}_{1,k}) -$$
$$- \bar{x}'_{1,k}\psi_{i,2}(x_2)^2 \varphi'_i(z)\psi'_{i,1}(\bar{x}_{1,k})^2 \psi''_{i,1}(\bar{x}_{1,k}) +$$
$$+ \bar{x}'_{1,k}\varphi'_i(z)\psi'_{i,2}(x_2)^2 \psi_{i,1}(\bar{x}_{1,k})^2 \psi''_{i,1}(\bar{x}_{1,k}) -$$
$$- 2\bar{x}'_{1,k}\varphi'_i(z)\psi'_{i,2}(x_2)^2 \psi_{i,1}(\bar{x}_{1,k})\psi'_{i,1}(\bar{x}_{1,k})^2 dx_2 \Big|_{z\in[z_k,z_{k+1}]} \times$$
$$\times \left[\sum_k \int_0^1 \psi'_{i,1}(\bar{x}_{1,k}) \left(\psi_{i,2}(x_2)^2 \psi'_{i,1}(\bar{x}_{1,k})^2 + \psi'_{i,2}(x_2)^2 \psi_{i,1}(\bar{x}_{1,k})^2 \right) dx_2 \Big|_{z\in[z_k,z_{k+1}]} \right]^{-1}. \tag{88}$$

The initial Dirichlet conditions, after finding the functions $\psi_{i,1}(x_1)$ and $\psi_{i,2}(x_2)$, collapse into a single condition (89):

$$\varphi_i(0) = 0. \tag{89}$$

Since the equation is second-order, a second condition is also necessary, for which we take the natural boundary condition obtained by varying the functional at the other end of the integration interval. (90):

$$\frac{1}{\psi_{i,2}(x_2)\,\psi'_{i,1}(\bar{x}_{1,k})} 2\psi'_{i,2}(x_2)\,\psi_{i,1}(\bar{x}_{1,k})\Big(\Phi^{(0,1)}(\bar{x}_{1,k},x_2)+ \\ +\psi'_{i,2}(x_2)\,\psi_{i,1}(\bar{x}_{1,k})\,\varphi'_i(z)\Big)+2\psi_{i,2}(x_2)\,\psi'_{i,1}(\bar{x}_{1,k})\Big(\Phi^{(1,0)}(\bar{x}_{1,k},x_2)+ \\ +\psi_{i,2}(x_2)\,\varphi'_i(z)\psi'_{i,1}(\bar{x}_{1,k})\Big)\Big|_{z=z_{\max}}=0. \tag{90}$$

Next, we will solve equations (85), (86), (88) using the iterative approach described above.

4.4. Results of the computational experiment. For the approach using Sprecher's formulation, we use the algorithm parameter values $\gamma = 10, k = 1$, then $a = 1/90$, $\alpha_1 = 1$, $\alpha_2 = 0.10100010000000001$. Fig. 7, *a* shows the plot of the matrix determinant Frechet at the

Table. Comparison of methods for detecting bifurcation points for a two-dimensional Bratu problem. The results for other methods are taken from the article [34]

Method	γ
Finite difference method	7.122
Collocation method	6.808
Weighted residual method	6.780
Almost exact solution	7.028
Iterative differential-quadrature method	7.028
Our approach	7.031

last iteration of Newton's method for the function $\Phi(z)$ depending on the values of the parameter γ, and in Fig. 7, *b* L_∞ — the norm of the solution depending on the values of the parameter.

As a result of the computational experiment, one singular point (the limiting one) was found at the parameter value $\gamma \approx 7.03$, which is in good agreement with the result known from the literature [34] (see Table). For the Hedberg and Doss formulations, the obtained results were

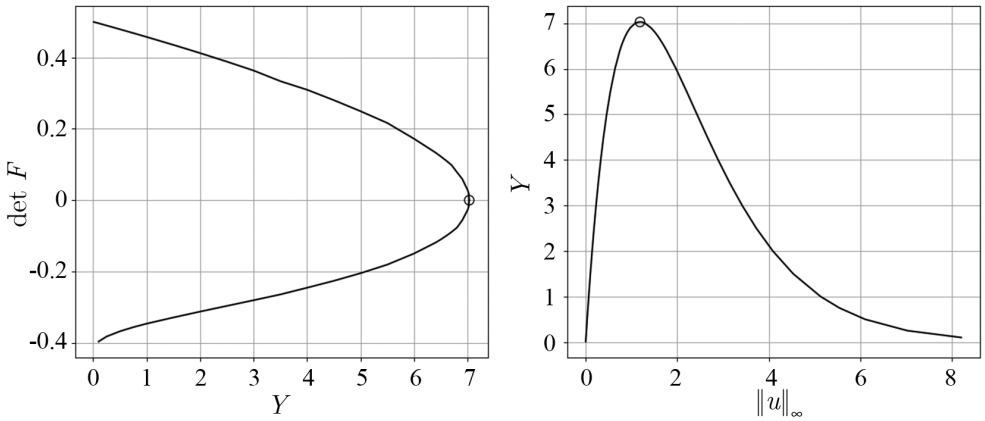

Fig. 7. *a* — Graph of the determinant of the Fréchet matrix on the last iteration of the Newton method for different value of the parameter γ; circle denotes critical point. *b* — L_∞ norm of the solution for different values of the parameter γ; circle denotes critical point

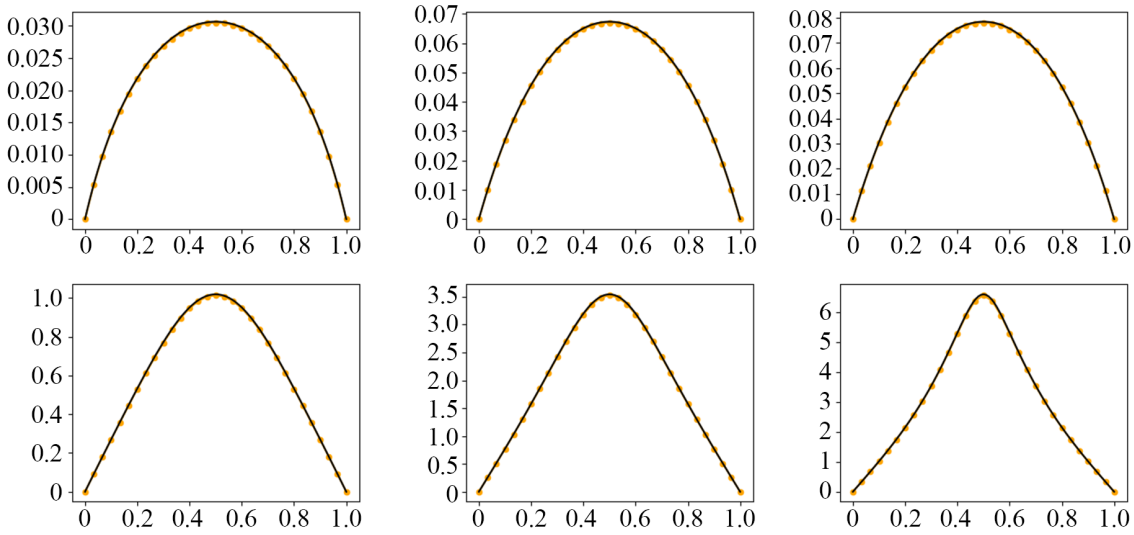

Fig. 8. x_2 cross sections of the solution to the Bratu problem for $\gamma = 1$. The solution obtained using the Kolmogorov–Arnold theorem (black line). The solution obtained using the finite difference method (orange dots). Left: cross section $x_1 = 0.1$; center: cross section $x_1 = 0.3$; right: cross section $x_1 = 0.5$. The upper graphs are the left branch of the solutions. The lower graphs are the right branch of the solutions (color online)

indistinguishable from those obtained using Sprecher's formulation. Figure 8 shows solutions to the Bratu problem obtained using the approach proposed in the article (orange) and solutions calculated using the finite difference method (black). We emphasize that the resulting solution functions are quite smooth, despite the potential nonsmoothness associated with the properties of the functions in the Kolmogorov-Arnold theorem.

Conclusions

1. The paper presents an algorithm for constructing a complete bifurcation structure (branching pattern) for boundary value problems for nonlinear partial differential equations given their variational formulations, based on Kolmogorov's superposition theorem.
2. Kolmogorov's superposition theorem in Sprecher form leads to a formulation in which the inner functions are calculated using a special algorithm, in Hedberg form, where the inner functions are specified by the user, and in Doss form, where the inner functions are found using an iterative algorithm. The resulting solution functions are quite smooth, despite the potential nonsmoothness associated with the properties of the Kolmogorov-Arnold theorem functions.
3. All three versions of the algorithm were applied to the Bratu equations and demonstrated good agreement with solutions known from the literature both in the form of the solutions and in the bifurcation structure.

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