Applied Problems of Nonlinear Oscillation and Wave Theory

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# Free linear vibrations of a viscoelastic spherical shell with filler

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***Abstract.*** *Purpose.* Thin multilayered shells are widely used in aviation, shipbuilding, and mechanical engineering. Recently, interest in the dynamic analysis of shell structures under various load effects has increased. This study examines the effect of moving normal internal pressure on a viscoelastic cylindrical shell. *Methods.* The viscoelastic medium filling the spherical shell has a significantly lower instantaneous modulus of elasticity than the shell itself. The solution is presented for the free vibrations of the viscoelastic system "shell–filler". An analytical frequency equation in the form of a transcendental equation is derived and solved numerically using the Müller method. *Results.* It has been found that, at certain values of viscoelastic and density parameters, low-frequency natural vibrations occur. These vibrations represent an aperiodic motion since the imaginary part of the natural frequency is large. For viscoelastic mechanical systems, the dependence of damping coefficients on physical and mechanical parameters has been identified. *Conclusion.* A theory and methods for calculating the complex natural frequencies of vibrations of an elastic spherical inhomogeneity in an elastic medium have been developed. A classification of such vibrations into radial, torsional, and spheroidal modes has been carried out. The problem is reduced to finding those frequencies at which the system of motion equations has nonzero solutions in the class of infinitely differentiable functions.

***Keywords***: spherical shell, filler, oscillations, frequency equation, damping coefficient.

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## Introduction

Thin layered shells are widely used in aircraft manufacturing, shipbuilding, mechanical engineering, and other fields. Therefore, interest in the dynamic analysis of shell structures under various loads has recently increased significantly [1, 2]. Spherical shells with a filler are used in industry to store various oil and gas substances [3, 4]. The Earth can also be considered a spherical shell with a multilayer filler [5].

In the works [6, 7], asymptotic methods are used to study the frequencies and modes of natural vibrations of spherical and cylindrical shells interacting with elastic and fluid media. In these works, simple approximate formulas for calculating the frequency are obtained. Based on these formulas, the modes of natural vibrations are obtained. In the work [8], the natural vibrations of a thin-walled spherical shell with a compressible fluid are considered. This work also examines the influence of physical and geometric parameters on the dependence of the shell's natural frequencies, and a qualitative analysis of the frequency as a function of the parameters is performed.

[9, 10] investigated the resonant frequencies of axisymmetric vibrations of a hollow sphere with a core. Three-dimensional equations of elasticity theory are used to describe the equations for the vibrations of the sphere and the core. The problem is solved in the axisymmetric case using asymptotic expressions for special Bessel and Neumann functions.

This paper examines the natural vibrations of a layered spherical shell containing a viscoelastic medium. For spherical bodies with a core, the researchers obtained the natural frequencies and corresponding natural modes using numerical methods. An exact analytical solution to the problem is constructed. A transcendental frequency equation is derived, representing the dependence of the complex frequency of the shell with the core on various geometric and physical-mechanical parameters.

## 1. Methodology

**1.1. Statement of the problem and solution methods. Systems of integro-differential equations describing the natural oscillations of a spherical shell in contact with a medium.** The system of equations of a spherical shell is represented as

$$\frac{\partial \Delta_0}{\partial \psi} - b^2 \frac{\partial}{\partial \psi}(\nabla^2+2)w - (1-\nu)\left(\frac{1}{\sin\psi}\frac{\partial \Delta_1}{\partial \phi} - u + \frac{\partial w}{\partial \psi}\right) - \\ - \int_0^t R_E(t-\tau)\bar{V}(\psi,\phi,\tau)d\tau = -\frac{1-\nu^2}{h}R^2\frac{X_1}{E_0},$$

$$\frac{1}{\sin\phi}\left[\frac{\partial \Delta_0}{\partial \phi} - b^2\frac{\partial}{\partial \psi}(\nabla^2+2)w\right] + (1-\nu)\left[\frac{\partial \Delta_1}{\partial \psi} + \vartheta - \frac{1}{\sin\psi}\frac{\partial w}{\partial \phi}\right]\frac{\partial \vartheta}{\partial \phi} - \\ - \int_0^t R_E(t-\tau)\bar{W}(\psi,\phi,\tau)d\tau = -\frac{1-\nu^2}{h}R^2\frac{Y}{E_0},$$

$$b^2\left[\nabla^2+1-\nu\right]\left[\Delta_0 - (\nabla^2+2)w\right] - (1+\nu)\Delta_0 - \\ - \int_0^t R_E(t-\tau)\bar{U}(\psi,\phi,\tau)d\tau = -\frac{1-\nu^2}{h}R^2\frac{Z}{E_0}, \tag{1}$$

where

$$\Delta_0 = \frac{1}{\sin\psi}\left[\frac{\partial}{\partial\psi}(u\sin\psi) + \frac{\partial\vartheta}{\partial\phi}\right] + 2w, \Delta_1 = \frac{1}{2\sin\psi}\left[\frac{\partial}{\partial\psi}(\vartheta\sin\psi) - \frac{\partial u}{\partial\phi}\right],$$

$$\nabla^2 = \frac{1}{\sin\psi}\left[\frac{\partial}{\partial\psi}(\sin\psi\frac{\partial}{\partial\psi}) + \frac{1}{\sin\psi}\frac{\partial^2}{\partial\phi^2}\right].$$

$$\bar{V}(\psi,\phi,\tau) = \frac{\partial \Delta_0}{\partial \psi} - b^2 \frac{\partial}{\partial \psi}(\nabla^2+2)w - (1-\nu)\left(\frac{1}{\sin\psi}\frac{\partial \Delta_1}{\partial \phi} - u + \frac{\partial w}{\partial \psi}\right),$$

$$\bar{W}(\psi,\phi,\tau) = \frac{1}{\sin\phi}\left[\frac{\partial \Delta_0}{\partial \phi} - b^2\frac{\partial}{\partial \psi}(\nabla^2+2)w\right] + (1-\nu)\left[\frac{\partial \Delta_1}{\partial \psi} + \vartheta - \frac{1}{\sin\psi}\frac{\partial w}{\partial \phi}\right]\frac{\partial \vartheta}{\partial \phi},$$

$$\bar{U}(\psi,\phi,\tau) = b^2\left[\nabla^2+1-\nu\right]\left[\Delta_0 - (\nabla^2+2)w\right] - (1+\nu)\Delta_0.$$

Here $X$, $Y$, $Z$ are projections of inertial forces and surface stress onto directions $r, \phi, \psi$ of the spherical coordinate system, $E_0$ and $\nu$ are instantaneous modulus of elasticity and Poisson's ratio, $b^2 = \frac{h^2}{12R^2}$, $R_E(t-\tau)$ is the relaxation kernel.

The equation of motion of a viscoelastic filler is [4, 11–13]:

$$(\lambda_{0c} + 2\mu_{0c})\,\mathrm{grad\,div}\,\vec{u}_c - \mu_{0c}\,\mathrm{rot\,rot}\,\vec{u}_c - \int\limits_0^t R_{Ec}(t-\tau)\vec{\Delta}_c(r,\theta,\beta,\tau)d\tau = \rho_c \frac{\partial^2 \vec{u}_c}{\partial t^2}, \tag{2}$$

where $\rho_c$ is the filler density, $R_{Ec}(t-\tau)$ is the filler relaxation kernel, $\lambda_{0c}, \mu_{0c}$ are the instantaneous elastic moduli, $\vec{u}_c(u_r, u_\phi, u_\psi)$ is the filler displacement vector, $\vec{\Delta}_c(r,\phi,\psi,\tau) = (\lambda_0+2\mu_0)\,\mathrm{grad\,div}\,\vec{u}_c - \mu_{0c}\,\mathrm{rot\,rot}\,\vec{u}_c$. In the spherical coordinate system $\beta, \theta, r$, the equations (2) have the form:

$$(\lambda_c + 2\mu_c) r \sin\psi \frac{\partial \theta_1}{\partial r} - 2\mu_c\Big(\frac{\partial \omega_\phi}{\partial \phi} - \frac{\partial}{\partial \phi}(\omega_\phi \sin\psi)\Big) - $$
$$-E_{c0}\int\limits_0^t R_{Ec}(t-\tau)\Delta_r(\vec{r},\tau)d\tau - \rho_c r \sin\psi \frac{\partial^2 u_r}{\partial t^2} = 0,$$

$$(\lambda_c + 2\mu_c) r \sin\psi \frac{\partial \theta_1}{\partial \phi} - 2\mu_c\Big(\frac{\partial \omega_r}{\partial \psi} - \frac{\partial}{\partial r}(r\omega_\phi)\Big) - $$
$$-E_{c0}\int\limits_0^t R_{Ec}(t-\tau)\Delta_\phi(\vec{r},\tau)d\tau - \rho_c r \sin\psi \frac{\partial^2 u_\phi}{\partial t^2} = 0,$$

$$(\lambda_c + 2\mu_c) r \sin\psi \frac{\partial \theta_1}{\partial \psi} - 2\mu_c\Big(\frac{\partial}{\partial r}(r\omega_\phi \sin\psi) - \frac{\partial \omega_r}{\partial \phi} - $$
$$-E_{c0}\int\limits_0^t R_{Ec}(t-\tau)\Delta_\psi(\vec{r},\tau)d\tau - \rho_c r \sin\psi \frac{\partial^2 u_\psi}{\partial t^2} = 0, \tag{3}$$

where

$$\theta_1 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\psi}\Big(\frac{\partial}{\partial r}(u_\psi \sin\psi) + \frac{\partial u_\phi}{\partial \phi}\Big),$$

$$\omega_\phi = \frac{1}{2r}\Big(\frac{\partial u_r}{\partial \psi} - \frac{\partial}{\partial r}(r u_\psi)\Big),$$

$$\omega_r = \frac{1}{2r\sin\psi}\Big(\frac{\partial u_\psi}{\partial \psi} - \frac{\partial (u_\phi \sin\psi)}{\partial \phi}\Big),$$

$$\omega_\psi = \frac{1}{2r\sin\psi}\Big(\frac{\partial}{\partial r}(u_\phi \sin\psi) + \frac{\partial u_\phi}{\partial \phi}\Big).$$

The equations of motion of the shell (1) and the filler (3) are supplemented by contact and boundary conditions. At the junction ($r = R$) of the filler and the shell, conditions of equality of the displacement components are imposed:

$$u_r = w, \quad u_\phi = \vartheta, \quad u_\psi = u \tag{4}$$

and equality of pressures

$$X = -\sigma_1 - \rho h \frac{\partial^2 u}{\partial t^2}, \quad Y = -\sigma_2 - \rho h \frac{\partial^2 \vartheta}{\partial t^2}, \quad Z = -\sigma_3 - \rho h \frac{\partial^2 w}{\partial t^2}, \tag{5}$$

where $X$, $Y$, $Z$ is pressure from the shell to the filler.

The freezing method is used to solve integro-differential equations [14, 15]. Then, from equations (1) and (3), a system of differential equations with complex coefficients is obtained. The solution to the equations of oscillation of a viscoelastic medium (partial differential equations with complex coefficients) is obtained

$$u_r = \frac{1}{\mu_e}\frac{\partial\varphi}{\partial r} + \frac{1}{\mu_l}\Big(\mu_l^2 r\chi + \frac{\partial^2(r\chi)}{\partial r^2}\Big),$$
$$u_\psi = \frac{1}{\mu_e r}\frac{\partial\varphi}{\partial\psi} + \frac{1}{\sin\psi}\frac{\partial\bar\psi}{\partial\phi} + \frac{1}{\mu_l r}\frac{\partial^2(r\chi)}{\partial r\partial\psi},$$
$$u_\phi = \frac{1}{\mu_e r\sin\psi}\frac{\partial\varphi}{\partial\psi} - \frac{\partial\bar\psi}{\partial\phi} + \frac{1}{\mu_l r\sin\psi}\frac{\partial^2(r\chi)}{\partial r\partial\psi}, \tag{6}$$

where $\varphi, \bar\psi, \chi$ are solutions of the scalar wave equations [13].

$$\nabla^2\varphi - \frac{1}{{}_p^2\Gamma_{ep}}\frac{\partial^2\varphi}{\partial t^2} = 0, \quad \nabla^2\bar\psi - \frac{1}{{}_s^2\Gamma_{es}}\frac{\partial^2\bar\psi}{\partial t^2} = 0, \quad \nabla^2\chi - \frac{1}{{}_s^2\Gamma_{es}}\frac{\partial^2\chi}{\partial t^2} = 0, \tag{7}$$

$\nabla^2$ is the Laplace operator in a spherical coordinate system,

$$\Gamma_{ep} = 1 - \Gamma_{ep}^c(\omega_R) - i\Gamma_{ep}^c(\omega_R), \quad \Gamma_{es} = 1 - \Gamma_{es}^c(\omega_R) - i\Gamma_{es}^c(\omega_R),$$

$\Gamma_{ep}^c(\omega_R) = \int_0^\infty R_E(\tau)\cos\omega_R\tau\, d\tau$, $\Gamma_{ep}^s(\omega_R) = \int_0^\infty R_E(\tau)\sin\omega_R\tau\, d\tau$ are the cosine and sine Fourier transforms of the material relaxation kernel, respectively. The solution of the scalar wave equation (7) with respect to the potential $\phi$ is sought in the form

$$\varphi = \Gamma(r)\Upsilon(\psi,\phi)e^{i\omega t}. \tag{8}$$

Substituting (8) into the first wave equation (7), we obtain the following partial differential equations:

$$\frac{d^2\Gamma}{dr^2} + \frac{2}{r}\frac{d\Gamma}{dr} + (\bar\mu_e^2 - \frac{\lambda_l}{r^2})\Gamma = 0,$$
$$\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}\Big(\sin\psi\frac{\partial Y}{\partial\psi}\Big) + \frac{1}{\sin^2\psi}\frac{\partial^2 Y}{\partial\phi^2} + \lambda_l Y = 0, \tag{9}$$

where $\mu_e = \omega/c_p$, $\mu_l = \omega/c_s$, $\lambda_l$ is separation constant.

The second differential equation (9) represents the Legendre equation; its solution exists only under the condition $\lambda_l = n(n+1)$, $(n = 1, 2..., )$. Particular solutions are formed from functions of the type of associated spherical harmonics of the first kind:

$$\Upsilon_{kn}^c = P_n^k(\cos\psi)\cos(k\phi), \quad \Upsilon_{kn}^s = P_n^k(\cos\psi)\sin(k\phi),$$

where $k$ is half the number of nodal meridians or the number of nodal interradial planes.

The first equation (9) represents the Bessel equation of a complex argument, the solution of which has the following form [3, 4, 16–18]:

$$\Gamma(r) = A_n j_n(\mu_e r) + B_n h_n(\mu_e r), \tag{10}$$

where $j_n(\mu_e r)$ и $h_n(\mu_e r)$ are spherical Bessel and Neumann functions.

From the condition of finiteness of the force and kinematic parameters at $r = 0$, the Neumann function vanishes. Then the solution (10) takes the form $\Gamma(r) = A_n j_n(\mu_e r)$. The spherical Bessel functions are related to the cylindrical functions by the formula

$$j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+1/2}(z). \tag{11}$$

Based on the above solution, the remaining equations (7) take the following form:

$$\varphi = [A_n j_n(\mu_e r)]\, \Upsilon^c(\psi,\phi)e^{i\omega t}, \quad \bar{\psi} = [B_n j_n(\mu_l r)]\, \Upsilon^c(\psi,\phi)e^{i\omega t}, \quad \chi = [C_n j_n(\mu_l r)]\, \Upsilon^c(\psi,\phi)e^{i\omega t}. \tag{12}$$

Taking into account (4) and (12), we obtain expressions for the components of the displacement vector:

$$u_r = a_0 Y^c, \quad u_\psi = a_1\frac{\partial}{\partial\psi}Y^c - a_2\frac{k}{\sin\psi}Y^s, \quad u_\phi = a_2\frac{\partial}{\partial\psi}Y^c - a_1\frac{k}{\sin\psi}Y^s. \tag{13}$$

The following notations are used here:

$$\begin{aligned}
a_0 &= \frac{1}{\bar{\mu}_e}\left[A_n\frac{\partial}{\partial r}j_n(\mu_e r)\right] + \frac{N}{\bar{\mu}_l r}\left[C_n j_n(\mu_l r)\right],\\
a_1 &= \frac{1}{\bar{\mu}_e r}\left[A_n j_n(\mu_e r)\right] + \frac{1}{\bar{\mu}_l r}\left[C_n\frac{\partial}{\partial r}[r j_n(\mu_l r)]\right],\\
a_2 &= \left[B_n j_n(\mu_l r)\right], \quad Y^s = \sin k\phi\, P_n^k(\cos\psi), \quad N = n(n+1).
\end{aligned} \tag{14}$$

To find the natural frequency of the shell-filler system under consideration, we will formulate a frequency equation. To satisfy the boundary conditions (5), we will need expressions on a spherical surface. According to Hooke's law,

$$\begin{aligned}
&\sigma_{rr} = b_0\Upsilon^c(z), \quad \sigma_{r\psi} = b_1\frac{\partial}{\partial\psi}\Upsilon^c(z) - b_2\frac{k}{\sin\psi}\Upsilon^s(z),\\
&\sigma_{r\phi} = -b_2\frac{\partial}{\partial\psi}\Upsilon^c(z) - b_1\frac{k}{\sin\psi}\Upsilon^s(z),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
b_0 &= \lambda_{0c}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_0) - N\frac{a_1}{r}\right] + 2\mu_{0c}\frac{\partial a_0}{\partial r},\\
b_1 &= \mu_{0c}\left[r\frac{\partial}{\partial r}(\frac{a_1}{r}) + \frac{a_0}{r}\right],\\
b_2 &= \mu_{0c} r\frac{\partial}{\partial r}(\frac{a_1}{r}).
\end{aligned}$$

Thus, we substitute the found displacements and stresses into the boundary conditions (4) and (5) and obtain the following system of homogeneous algebraic equations:

$$\begin{aligned}
&(N-1+\nu-K(\omega_R)\,\omega^2)a_1(R) - \left[b^2(N-2)+\nu+1\right]a_0(R) + \frac{K(\omega_R)}{\rho h}b_1(R) = 0,\\
&\left[\frac{1}{2}(N-2)(1-\nu) - K(\omega_R)\right] + \frac{K(\omega_R)}{\rho h}b_2(R) = 0,\\
&N[b^2(N-1+\nu)+1+\nu]a_1(R) - [b^2(N-1+\nu)N + 2(1+\nu) - K(\omega_R)\omega^2] -\\
&\qquad\qquad\qquad\qquad\qquad\qquad - \frac{K(\omega_R)}{\rho h}b_0(R) = 0,\\
&b_1(R_0) = 0,\\
&b_2(R_0) = 0,\\
&b_0(R_0) + \rho_0 R_0\omega^2\Delta_n(R_0)\,a_0(R_0) = 0,
\end{aligned} \tag{16}$$

where $K(\omega_R) = (1-\nu)\rho R/\bar{E}$, $\Delta_n(R_0) = [n - \bar{\mu}_c R_0 j_{n+1}(\bar{\mu}R_0)/j_n(\bar{\mu}R_0)]$.
Homogeneous algebraic equations (16) allow us to determine the complex natural frequencies of a spherical shell with a filler. Two types of oscillations can be distinguished: $(a_0, a_1, b_0, b_1)$ and

$(a_2, b_2)$. The first of them is characterized by displacements without rotation components: $A_1$, $A_2$, $C_1$, $C_{20} \neq 0$, $A_1$, $A_2$, $C_1$, $C_{20} \neq 0$. The corresponding frequency equation is obtained from the 1st, 3rd, 4th, and 6th equations (16). Purely radial oscillations of the sphere correspond to the values $k = n = 0$. For $k = 0$ and $n = 1$, the system moves as a rigid body. The case $k = 0$, $n = 2$ corresponds to a change in shape from a prolate spheroid. The frequency equation for free oscillations of a sphere with a viscoelastic filler is obtained from (16):

$$[(N - \rho_c^\bullet + \rho_c^{\bullet 2}) - \rho_c^\bullet \mu_l^\bullet \alpha - (1 + \rho_c^\bullet)\mu_e^\bullet \beta - \mu_e^\bullet \mu_l^\bullet \alpha\beta]\lambda^2 +$$
$$+ [NL_2 + \rho_c^\bullet (L_3(2N-1)\bar{\chi} E_c^\bullet) - (r_1 + \bar{\chi} E_c^\bullet)\rho_c^\bullet \mu_l^\bullet \alpha + (-L_2 + \rho_c^\bullet (r_n - 2\bar{\chi} E_c^\bullet))\mu_e^\bullet \beta - L\mu_l^\bullet \alpha\beta]\,\lambda +$$
$$+ NL_1 - L_1 \mu_e^\bullet \beta - L_1 \mu_e^\bullet \mu_l^\bullet \alpha\beta = 0. \tag{17}$$

Here

$$\alpha = \frac{j'_n(\mu_l^\bullet)}{j_n(\mu_l^\bullet)}, \quad \beta = \frac{j'_n(\mu_e^\bullet)}{j_n(\mu_e^\bullet)}, \quad \mu_l^\bullet = \mu_l R, \quad \mu_e^\bullet = \mu_e R, \quad \lambda = k\omega^2,$$

$$\mu_l = \frac{\omega}{\bar{c}_l}, \quad \mu_e = \frac{\omega}{\bar{c}_e}, \quad k = (1 - \nu^2)\rho R^2/\bar{E}, \bar{c}_l = \sqrt{\frac{E_{0c}}{2(1+\nu_c)\rho_c}}\Gamma_k,$$

$$\bar{c}_l = \sqrt{\frac{E_{0c}(1-\nu_c)}{2(1+\nu_c)\rho_c}}\Gamma_k, \quad E_c^\bullet = \frac{\bar{E}_c}{\bar{E} h_\bullet}, \quad \rho_c^\bullet = \frac{\rho_c}{\rho h_\bullet}, \quad h_\bullet = \frac{h}{R}, \quad \bar{\chi} = \frac{1-\nu^2}{1+\nu^2},$$

$$L_1 = r_1 r_n - N r_2 r_n + \bar{\chi}\bar{E}_c^\bullet(-r_n - 2r_1 + Nr_3 + Nr_2) + \bar{\chi}^2 E_c^{\bullet 2}(-2 - N),$$

$$L_2 = -r_n - r_1 + 3\bar{\chi}\, E_c^\bullet, \quad L_3 = -r_1 - Nr_2 - Nr_3, \quad r_1 = N - 1 + \nu,$$

$$r_2 = b^2(N-2) + 1 + \nu, \quad r_3 = b^2 r_1 + 1 + \nu, \quad r_n = b^2 N r_1 + 2(1+\nu).$$

$\rho$ is the shell density, $\rho_c$ is the filler density, $h$ is the shell thickness, $\bar{E} = E_0\Gamma_k, \nu$ is the elastic modulus and Poisson's ratio of the shell.

In the absence of a filler, the equation (17) takes the following form:

$$(N - \rho_c^\bullet + \rho_c^{\bullet 2})\lambda_0^2 + (NL_2^0 - L_2^0)\lambda_0 + NL_1^0 = 0, \quad \lambda_0 = k\omega^2 \tag{18}$$

In equation (18), the coefficients depend on the real part of the frequency. Therefore, the algebraic equation (18) for viscoelastic shells and fillers becomes a transcendental equation.

Equation (17) is solved numerically using Muller's method.

## 2. Results and analysis

The transcendental equation is solved numerically using the Muller method [15, 16], as well as using the MAPLE-18 software package. The viscoelastic properties of the material are described using a three-parameter relaxation kernel [17]:

$$R_{\mu j}(t) = R_{\lambda j} = \frac{A_j e^{-\beta_j t}}{t^{1-\alpha_j}}, \quad j = 1, 2.$$

For specific calculations, the following parameters of the material of an infinite viscoelastic medium were adopted: [13, 14, 18]: $n = 2$, $h = 10^4$ m, $\nu_c = 0.40$, $A = 0.048$; $\beta = 0.05$; $\alpha = 0.10$, $h = 0.00165$, $\lambda_2 = 6$, $E_{0c} = 4$, $\rho = 7.8 \cdot 10^3$, $\rho = 3.2 \cdot 10^3$, $E_0 = 40$, $\nu = 0.25$, $R = 2.10^{-4}$. The roots of

Table. Roots of the frequency equation at different values of $\nu_c$

| Root number | 0 | 0.25 | 0.35 | 0.40 |
|---|---|---|---|---|
| 1 | 0.662 | 0.816 | 0.85 | 0.8733 |
| 2 | 1.8909 | 1.9285 | 1.9391 | 1.9479 |
| 3 | 2.9303 | 2.9539 | 2.9606 | 2.9656 |
| 4 | 3.0485 | 3.9658 | 3.9707 | 3.9744 |
| 5 | 4.959 | 4.9728 | 4.9607 | 4.9776 |
| 6 | 5.966 | 5.9774 | 5.0806 | 5.983 |

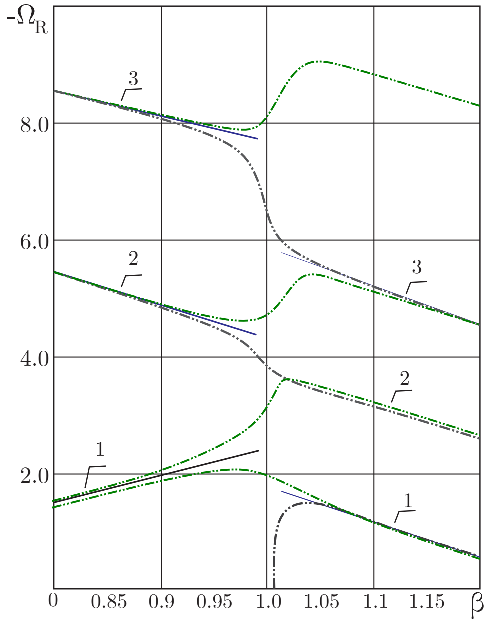


Fig. 1. Variation of the natural frequency of torsional oscillations from the ratio of transverse wave velocities $\beta = \frac{c_{s2}}{c_{s1}}$ at: 1 — $\rho^\bullet = 0.8$, 2 — $\rho^\bullet = 0.85$, 3 — $\rho^\bullet = 0.98$

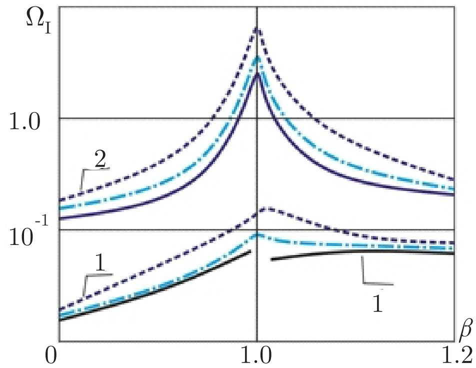


Fig. 2. Dependence of damping coefficients on transverse wave velocity ratios $\beta = c_{s2}/c_{s1}$

the frequency equation (18) describing the vibrations of a viscoelastic spherical shell are given in the Table. When determining the real and imaginary parts of the frequency, the accuracy of the method is $10^{-10}$.

The obtained numerical results are also shown in Fig. 1 and Fig. 2. Fig. 1 shows the change in the natural frequency of torsional vibrations from the ratios of the shear wave velocities $\beta = c_{s2}/c_{s1}$ for: 1 — $\rho^\bullet = 0.8$, 2 — $\rho^\bullet = 0.85$, 3 — $\rho^\bullet = 0.98$. It is evident that at $\beta = c_{s2}/c_{s1} = 1$ a discontinuity in the natural frequencies occurs. In Fig. 2 shows the dependence of damping coefficients on the ratio of transverse wave velocities $\beta = c_{s2}/c_{s1}$. It was substantiated that control of resonance phenomena during vibration can be achieved through energy dissipation.

It was found that energy dissipation is fundamentally different for dissipatively homogeneous and inhomogeneous mechanical systems. When calculating stresses and displacements using formulas (13) and (15), summation was performed until the ratio of the current term to the current partial sum became less than $10^{-10}$. To ensure convergence of the series (13) and (15), a numerical experiment was conducted each time.

It has been established that to ensure calculation accuracy when calculating displacements and stresses, it is necessary to retain 11–16 terms of the series. In this case, the rounding error is up to 1% : 1 — $\rho_2/\rho_1 = 0.02$, $\bar{C} = 0.5$, $a = 1$; 2 — $\rho_2/\rho_1 = 50$, $\bar{C} = 0.5$, $A1 = 0.01$, $\beta = 0.05$. $\alpha = 0.1$.

Low-frequency aperiodic oscillations decay rapidly over time. This process requires in-depth analysis, as low-frequency oscillations (where the real and imaginary parts of the natural frequency differ from zero) can lead to resonance phenomena. It is assumed that the imaginary parts of the natural frequencies for aperiodic oscillations are close to zero. In this case, very large displacement and stress amplitudes arise. If the real parts of the natural frequency are zero and the imaginary parts are nonzero and negative, the oscillations decay exponentially.

The discovered real parts of the natural frequency describe the resonant frequencies of the mechanical system under consideration, while the imaginary parts characterize the damping of the mechanical system's oscillations as a whole. This discovered phenomenon is of great importance for studying energy dissipation in vibration-protection systems.

## Conclusion

1. A theory and methods for calculating the complex natural frequencies of spherical elastic inhomogeneities in an elastic medium are developed. These oscillations are classified into radial, torsional, and spheroidal. The problem reduces to finding the frequencies $\Omega = \Omega_R + i\Omega_i$, where $\Omega_R$ is the real part and $\Omega_i$ is the imaginary part of the complex natural frequencies for which the systems of equations of motion have a nonzero solution in the class of infinitely differentiable functions.
2. It is found that low-frequency natural oscillations arise for certain values of viscoelastic density parameters. These oscillations represent a certain aperiodic motion, since the imaginary part of the natural frequency is significant.

## References


1. Pozhuev VI. Action of a mobile load on a cylindrical shell in an elastic medium. Construction Mechanics and Design of Structures. 1978;(1):44–48.
2. Safarov II, Teshaev MKh. Control of resonant oscillations of viscoelastic systems. Theor. Appl. Mech. 2024;51(1):1–12. DOI: 10.2298/TAM220510007S.
3. Durdiyev DQ, Safarov II, Teshaev MKh. Propagation of waves in a fluid in a thin elastic cylindrical shell. WSEAS Transactions on Fluid Mechanics. 2024;19:113–119. DOI: 10.37394/232013.2024.19.11.
4. Safarov II, Teshaev MKh. Nonstationary motions of spherical shells in a viscoelastic medium. Tomsk State University Journal of Mathematics and Mechanics. 2023;83:166–179.
5. Gontkevich VS. Natural Oscillations of Shells and Fluids. Kyiv: Naukova Dumka; 1964. 103 p.
6. Ukrainets VN, Girnis SN. Mathematical Modeling of the Dynamics of Tunnels Reinforced with Two-Layer Shells under the Action of Transport Loads. Pavlodar: Kereku; 2018. 116 p.
7. Alekseyeva LA, Ukrainets VN. Dynamics of an elastic half-space with a reinforced cylindrical cavity under moving loads. Int. Appl. Mech. 2009;45:981–990. DOI: 10.1007/s10778-010-0238-z.
8. Bykov AA, Matveenko VP, Shardakov IN, Shestakov AP. Shock wave method for monitoring crack repair processes in reinforced concrete structures. Mech. Solids. 2017;52:378–383. DOI: 10.3103/ S0025654417040033.
9. Safarov II, Teshaev MK. Dynamic damping of vibrations of a solid body mounted on viscoelastic supports. Izvestiya VUZ. Applied Nonlinear Dynamics. 2023;31(1):63–74.

DOI: 10.18500/0869-6632-003021.
10. Adamov AA, Matveenko VP, Trufanov NA, Shardakov IN. Methods of Applied Viscoelasticity. Ekaterinburg: UB RAS; 2003. 411 p.
11. Fedorov AYu, Matveenko VP, Shardakov IN. Numerical analysis of stresses in the vicinity of internal singular points in polymer composite materials. International Journal of Civil Engineering and Technology. 2018;9(8):1062–1075.
12. Amabili M. Free vibration of partially filled, horizontal cylindrical shells. J. Sound Vib. 1996;191(5): 757–780. DOI: 10.1006/jsvi.1996.0154.
13. Karimov K, Turakhodjaev N, Akhmedov A, Chorshanbiev Sh. Mathematical model for producing machine parts. E3S Web of Conferences. 2021;264(7):04078. DOI: 10.1051/e3sconf/202126404078.
14. Mirsaidov MM, Sultanov TZ. Use of linear heredity theory of viscoelasticity for dynamic analysis of earthen structures. Soil Mech. Found. Eng. 2013;49:250–256. DOI: 10.1007/s11204-013-9198-8.
15. Adamov AA, Matveenko VP, Trufanov NA, Shardakov IN. Methods of Applied Viscoelasticity. Yekaterinburg: Ural Branch of the Russian Academy of Sciences; 2003. 411 p.
16. Fedorov AYu, Matveenko VP, Shardakov IN. Numerical analysis of stresses in the vicinity of internal singular points in polymer composite materials. International Journal of Civil Engineering and Technology. 2018;9(8):1062–1075.
17. Bykov AA, Matveenko VP, Shardakov IN, Shestakov AP. Shock wave method for monitoring crack repair processes in reinforced concrete structures. Mech. Solids. 2017;52(4):378–383. DOI: 10.18500/0869-6632-00316210.3103/S0025654417040033.
18. Karimov K, Akhmedov A, Karimova A. Development of mathematical model, classification, and structures of controlled friction and vibration mechanisms. AIP Conference Proceedings. 2023;2612:030014. DOI: 10.1063/5.0116891.