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Investigation of the stability of the oscillatory zone boundary mode in the perturbed one-dimensional Toda lattice

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Abstract. The *goal* of this paper is to investigate the stability of a dynamical regime corresponding to the vibrational mode with the shortest wavelength (known as the π-mode) in the Toda lattice with a cubic perturbation of the original potential. *Methods.* The study is based on the standard Floquet method. The variational system for the corresponding dynamical regime is decomposed into a set of independent two-dimensional subsystems. This allows us to determine the π-mode stability for a chain with an arbitrary number of particles. The decomposition is carried out both by a general group-theoretic approach and by a new method proposed in this work, which is based on the discrete Fourier transform. *Results.* The resulting stability diagrams provide information about the stability of the regime for various oscillation amplitudes and numbers of particles. A correspondence between the perturbed Toda lattice and the Fermi–Pasta–Ulam–Tsingou model is established for large magnitude of the perturbation. For the original (unperturbed) Toda lattice, it is observed that its integrals of motion are functionally dependent in the vicinity of the considered dynamical regime. Therefore, the observed trajectory does not satisfy the conditions of Poincaré's theorem, which states that the Floquet multipliers of fully integrable systems are equal to one. Despite this fact, the considered regime in the original Toda lattice is shown to be stable for any number of particles and any oscillational amplitude. *Conclusion.* We have investigated the stability of the zone boundary mode (π-mode) in the Toda lattice with a cubic perturbation in the interaction potential. The study has been carried out for the decomposed variational system consisting of independent two-dimensional subsystems. The independent subsystems are obtained by the general group-theoretic method. In addition, a new decomposition method is proposed based on the discrete Fourier transform. The proposed approach can be further applied to investigate the stability of any nonlinear regimes possessing temporal and spatial periodicity.

Keywords: nonlinear dynamics, Toda lattice, stability, zone boundary mode, group-theoretics methods.

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Introduction

Nonlinear oscillatory modes in discrete systems of various physical nature have traditionally attracted the attention of researchers [1–10]. There are two classes of such modes: localized and delocalized. Localized modes (discrete breathers) can contribute to energy transfer processes [6, 8, 11], affect the macroscopic properties of crystals [6,12–14], as well as change the structure of defects [6,15–17]. Delocalized modes, in addition to the fact that they can also affect the macroscopic parameters of crystals [18, 19], are closely related to localized ones. Thus, the initial profiles of discrete breathers can be obtained from patterns of atomic displacements of delocalized modes by superimposing a spatial localization function [20–22]. In addition, the modulation instability of delocalized nonlinear modes can lead to the formation of so-called chaotic breathers [4, 10, 23, 24]. The formation of breathers as a result of modulation instability was first detected in [23] for the π-mode, which is the shortest wavelength mode in the monatomic chain. In this mode, the displacements of neighboring particles from the equilibrium position at each moment of time have the opposite sign. This oscillatory mode is, in a certain sense, the antipode of the mode for which the famous return phenomenon [1] was observed in the Fermi–Pasta–Ulam–Tsingou chain (FPUT) therefore, the study of the temporal evolution of the π-mode can be called an «anti-FPU» problem [4].

A significant number of papers have been devoted to the study of the π-mode in monatomic chains of the FPUT type [4, 25–28]. On the other hand, such a monatomic nonlinear system as the Toda lattice is widely known, which is one of the few known multiparticle complete integrable Hamiltonian systems in the sense of Liouville. It was proposed by M. Toda [29] as an example of a system that, unlike the FPUT, has analytical solutions. Because the Toda lattice is completely integrable, it is non-ergodic. Consequently, thermalization and the resulting energy equipartition among the system's modes do not occur. At the same time, in the approximation of small amplitudes, the dynamics of the Toda lattice is close to the dynamics of the α-FPUT.

Therefore, the natural question is how the stability of periodic regimes, in particular the π-mode, affect the full integrability of the system and what happens to stability when this integrability is violated. The present work is intended to shed light on this issue by studying the stability of the π-mode in the Toda lattice with introduced perturbation into the interaction potential.

The behavior of such a perturbed Toda lattice was numerically investigated in [30], where the interaction potential between particles was given in the form

$$V = V_T(\varepsilon) + \theta_n \varepsilon^n, \quad n \geqslant 3. \tag{1}$$

Here ε is the dimensionless deviation of the distance between neighboring particles from the equilibrium position, V_T is the interaction potential in the classical Toda lattice, determined by

$$V_T = \exp(-\varepsilon) + \varepsilon - 1.$$

Specifically, in [30] it is shown that energy thermalization occurs in the perturbed Toda lattice, and without the recurrence phenomenon observed in the FPUT lattice [1].

In this paper, we used a perturbation of the form (1) at $n = 3$, the strength of which was regulated by the value of the coefficient θ_3. This made it possible to investigate how the stability of the studied regime changes over a wide range of amplitudes, depending on the magnitude of the introduced disturbance.

The Floquet method is a standard tool for studying the stability of periodic regimes, but using this method for a chain of N particles requires the construction of a monodromy matrix of size $2N \times 2N$. It is clear that for $N \gg 1$, the study of the stability of the periodic regime by this method turns out to be extremely difficult.

On the other hand, in [3, 31] a universal method has been proposed based on the theory of representations of symmetry groups, which makes it possible to study the stability of periodic regimes by calculating a monodromy matrix of finite size, independent of the number of particles in the system N. Previously, this method was used to study the stability of periodic modes in the FPUT chains [3] and the so-called LC-chain [32, 33].

This method was also used in the present work to study the stability of the π-mode in the classical and perturbed Toda lattice. In addition, a similar approach based on the discrete Fourier transform has been proposed.

The work is organized as follows. Section 1 describes the analytical form of the solution corresponding to the π-mode in the system under consideration. Section 2 describes a method for studying stability. Section 3 presents the results of a study of the stability of the classical Toda lattice, and Section 4 presents the results of the stability analysis of the perturbed Toda lattice. Conclusion summarizes the results of this work.

1. Oscillatory π-mode in a perturbed Toda lattice

A nonlinear homogeneous chain of pairwise interacting particles is described by the following equations of motion:

$$\ddot{u}_n(\tau) = \exp\left(u_{n-1} - u_n\right) - \exp\left(u_n - u_{n+1}\right) + \alpha\left(\left(u_{n+1} - u_n\right)^2 - \left(u_{n-1} - u_n\right)^2\right). \tag{2}$$

Here u_n is the dimensionless displacement of the n-th particle from its equilibrium position, τ is the dimensionless time, the dot denotes the time derivative, α is a coefficient that allows us to control the cubic component of the interaction potential between particles, $n \in \{1 \dots N\}$ and N is the total number of particles in the chain. For $\alpha = 0$, system (2) takes the form corresponding to the classical Toda lattice. Further we refer to the system with a nonzero coefficient α as the perturbed Toda lattice. The boundary conditions are assumed to be periodic, i.e., $u_{N+i} = u_i$.

In this work, we study the stability of the π-mode, which has the form

$$u_n(\tau) = (-1)^n U(\tau). \tag{3}$$

Substituting (3) into (2) reduces the system of second-order ordinary differential equations to a single ordinary differential equation for $U(\tau)$:

$$\ddot{U} = -2\sinh\left(2U\right). \tag{4}$$

For this equation, an exact solution can be found in the form:

$$U(\tau) = \operatorname{arctanh}\left(\frac{\operatorname{sn}\left(\tau\sqrt{v_0^2+4}\,\middle|\,\frac{1}{1+4/v_0^2}\right)}{\sqrt{1+4/v_0^2}}\right), \quad U(0) = 0, \quad \dot{U}(0) = v_0. \tag{5}$$

Here, arctanh is the hyperbolic arctangent, and sn is the Jacobi elliptic sine. Solution (5) is periodic:

$$U\left(\tau + T\right) = U\left(\tau\right), \qquad T = \frac{4K\left(\frac{v_0^2}{4+v_0^2}\right)}{\sqrt{4+v_0^2}}, \tag{6}$$

where K is the complete elliptic integral of the first kind. The derivation of the solution (5) is given in the *Appendix*. Note that after substituting given in the (3) into (2), resulting equation (4) and its solution (5) do not depend on the coefficient α. Thus, introducing a cubic term to the Toda lattice potential does not affect the form of the solution corresponding to the π-mode oscillation. However, as will be shown later, the value of α affects the stability of this periodic mode.

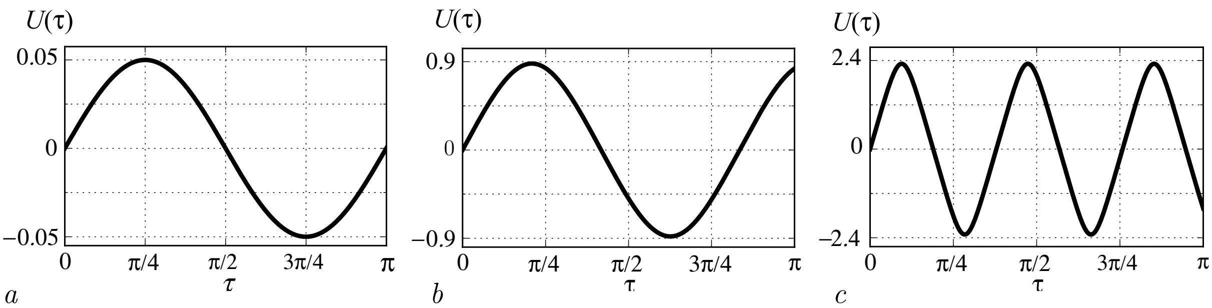

Fig. 1. Plot of the solutions (5) for various values of v_0: $v_0 = 0.1$ (a); $v_0 = 2$ (b); $v_0 = 10$ (c)

2. Study of the stability of the oscillatory π-mode

2.1. Variational system and Floquet multipliers. A classical method for determining the stability of a nonlinear periodic mode is to study the stability of the zero solution of the corresponding variational system of linear equations obtained by linearizing the original system around the oscillatory mode under consideration. If u_n^0, $n\in\{1\dots N\}$ is a solution of the system $\ddot{u}_n=F_n(u_1,u_2\dots u_N)$, $n\in\{1\dots N\}$, then the corresponding variational system of equations linearized around this mode will be defined by the following equations:

$$u_n = u_n^0 + \varepsilon_n, \quad A_{nm}(\tau) = \frac{\partial}{\partial u_m} F_n(u_1, u_2 \dots u_l)\bigg|_{u_i=u_i^0(\tau),\, i=1\dots N}, \qquad \ddot{\varepsilon}_n = \sum_m A_{nm}(\tau)\varepsilon_m, \qquad n \in \{1\dots N\}, \tag{7}$$

where $\varepsilon_n = \varepsilon_n(\tau)$ are small quantities compared to u_n^0 that depend on time.

The stability of the zero solution of such a variational system is determined by calculating the Floquet multipliers μ, which are the eigenvalues of the monodromy matrix [34] of this system. For the stability of the solution u_n^0, it is necessary that are not greater than one: $|\mu| \leqslant 1$ (for autonomous Hamiltonian systems, only $|\mu| = 1$ is possible in the case of stability). More information about the monodromy matrix and Floquet multipliers can be found in [34, 35].

2.2. Variational system for the perturbed Toda lattice. Variational system obtained by linearizing (2) about solution 3, has the form

$$\begin{cases} \ddot{\varepsilon}_{2n} = c\varepsilon_{2n-1} - (c + c^{-1})\varepsilon_{2n} + c^{-1}\varepsilon_{2n+1} + 4\alpha b(\varepsilon_{2n-1} - \varepsilon_{2n+1}), \\ \ddot{\varepsilon}_{2n+1} = c^{-1}\varepsilon_{2n} - (c + c^{-1})\varepsilon_{2n+1} + c\varepsilon_{2n+2} - 4\alpha b(\varepsilon_{2n} - \varepsilon_{2n+2}), \end{cases}$$
$$c = \exp\left(2U(\tau)\right), \quad b = U(\tau), \quad n \in \{1\dots N\}. \tag{8}$$

Note that the right-hand side of the resulting system (8) contains, in addition to the unknowns ε_{2n} and ε_{2n+1}, also terms proportional to ε_{2n-1} and ε_{2n+2}, i.e., it is a system of coupled differential equations. To determine the stability of the π-mode in a chain of N particles, one has to to construct a $2N \times 2N$ monodromy matrix. It is clear that for chains with a large number of particles, the application of this method becomes difficult.

Consider a way to bring the variational system (8) to a form such that the study of the stability of its zero solution is possible for any arbitrarily large number of particles N. Let

$$\boldsymbol{\varepsilon} = [\varepsilon_1, \dots, \varepsilon_N]^T, \quad \mathbf{A} = \begin{bmatrix} A_{11} & \dots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \dots & A_{NN} \end{bmatrix}, \tag{9}$$

where A_{nm} are the coefficients of the linear system defined by (7). In this work, we use unitary transformations $\tilde{\boldsymbol{\varepsilon}} = \mathbf{S}\boldsymbol{\varepsilon}$, that bring the matrix $\tilde{\mathbf{A}} = \mathbf{S}\mathbf{A}\mathbf{S}^{\mathbf{H}}$ of the system under consideration to a block-diagonal form. Then, the transformed variational system

$$\ddot{\tilde{\varepsilon}}_n = \sum_m \tilde{A}_{nm}(\tau)\tilde{\varepsilon}_m \tag{10}$$

consists of independent subsystems, and the stability of the zero solution of each subsystem can be studied independently.

2.3. Decomposition of a variational system using a general group-theoretic method. The first method that we used is based on the theory of irreducible representations of symmetry groups. This method was proposed and described in the work of G. M. Chechin and K. G. Zhukov [31]. To use this method, it is necessary to know the discrete spatial group G of the symmetry of the regime under study, that is, a group of spatial transformations such that for any element $a \in \mathrm{G}$ it is valid

$$au_n^0 = u_n^0, \quad n = 1 \dots N, \qquad \forall a \in \mathrm{G}. \tag{11}$$

In other words, the action of all the elements of G transforms the mode in question into itself.

In [31], it is proved that a variational system corresponding to a certain periodic regime can be decomposed into independent subsystems of finite size using a unitary transformation, the matrix **S** of which is composed of the basis vectors of irreducible representations of the spatial symmetry group G of this mode.

In this work, the symmetry group of the π-mode was taken to be the group $\mathrm{G} = \langle as \rangle$, consisting of all possible powers of the operator as, where the action of the operators a and s is defined as follows:

$$au_n = u_{n+1}, \qquad su_n = -u_n. \tag{12}$$

The works [3,31,33] describe in detail the procedure for finding the basis vectors of irreducible representations and constructing the matrix **S**. In particular, in work [33], this procedure is described for the same symmetry group that is used in this paper. Based on this procedure, the following decomposed variational system can be obtained:

$$\begin{cases} \ddot{\tilde{\varepsilon}}_{2q} = -2\left(c + \dfrac{1}{c}\right)\sin^2 k \cdot \tilde{\varepsilon}_{2n} + i\left(c - \dfrac{1}{c} - 8\alpha b\right)\sin 2k \cdot \tilde{\varepsilon}_{2n+1}, \\ \ddot{\tilde{\varepsilon}}_{2q+1} = -i\left(c - \dfrac{1}{c} - 8\alpha b\right)\sin 2k \cdot \tilde{\varepsilon}_{2n} - 2\left(c + \dfrac{1}{c}\right)\cos^2 k \cdot \tilde{\varepsilon}_{2n+1}, \end{cases} \tag{13}$$

where $k = \frac{2\pi q}{N/2}$, q is an index taking values $q \in \{1 \dots N/2\}$. Note that for each value of the index q, expression (13) represents a closed system of differential equations with respect to the variables $\tilde{\varepsilon}_{2q}$ and $\tilde{\varepsilon}_{2q+1}$. The stability of system (13) for different values of the parameters k, v_0 and α will be investigated in Sections 3 and 4 by analyzing the calculated values of the Floquet multipliers (see 2.1).

2.4. Decomposition of the variational system using discrete Fourier transform. Second method that we used to transform variational system (8) into block-diagonal form is based on the use of discrete Fourier transform. We introduce new variables:

$$\psi_n = \varepsilon_{2n}, \qquad \phi_n = \varepsilon_{2n+1}. \tag{14}$$

Then the system of equations (8) takes the form

$$\begin{cases} \ddot{\psi}_n = c\phi_{n-1} - (c + c^{-1})\psi_n + c^{-1}\phi_n + 4\alpha b(\phi_{n-1} - \phi_n), \\ \ddot{\phi}_n = c\psi_{n+1} - (c + c^{-1})\phi_n + c^{-1}\psi_n - 4\alpha b(\psi_n - \psi_{n+1}). \end{cases} \tag{15}$$

We apply a discrete Fourier transform for the variables ψ_n, ϕ_n independently of each other:

$$\begin{aligned} \hat{\psi}(k) &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N/2} \exp(-ikn)\psi_n, \\ \hat{\phi}(k) &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N/2} \exp(-ikn)\phi_n, \end{aligned} \tag{16}$$

where $k = \frac{2\pi q}{N/2}$, $q \in \{1 \dots N/2\}$. Note that variable transformation (16) is linear and unitary. The variational system after the Fourier transformation will take the form

$$\begin{cases} \ddot{\hat{\psi}} = -(c + c^{-1})\hat{\psi} + \left(4\alpha b\,(1 - \exp(-ik)) + c\exp(-ik) + c^{-1}\right)\hat{\phi}, \\ \ddot{\hat{\phi}} = \left(4\alpha b\,(1 - \exp(ik)) + c\exp(ik) + c^{-1}\right)\hat{\psi} - (c + c^{-1})\hat{\phi}. \end{cases} \tag{17}$$

System (17) is closed with respect to the variables $\hat{\psi}$ and $\hat{\phi}$. As for system (13), its zero solution is not difficult to investigate for stability for any value of q by the standard Floquet method.

2.5. A numerical investigation method. The Floquet multipliers for the variational systems (13) and (17) were found numerically. For specific values of the parameters α, v_0 and k, the calculation procedure was performed in 2 steps.

1. Numerical construction of the monodromy matrix. For the variational systems (13) and (17), the monodromy matrix has a size of 4×4, where each column is composed of the solution of the Cauchy problem x_n at time T (see formula (6)) with initial conditions $x_n(0) = \delta_{nm}$, where x_n is the state variable of the variational system under consideration $n \in \{1 \dots 4\}$, m is the column number of the monodromy matrix, and δ_{nm} is the Kronecker delta. The Cauchy problem was solved using the Dormand–Prince adaptive integration step method [36]. The relative error was less than 10^{-10} (i.e., 10 significant digits of accuracy).
2. The calculation of the Floquet multipliers, which are the eigenvalues of the monodromy matrix obtained in the previous step, was performed numerically using the LAPACK library [37].

The zero solution of the variational system for specific values of α, v_0, and k was considered stable if each of the four calculated Floquet multipliers deviated from the unit circle by no more than 10^{-7}:

$$\big||\mu_i| - 1\big| < 10^{-7}. \tag{18}$$

The studied π-mode was considered to be stable for some specific α and v_0 if the zero solution of the corresponding variational system was stable for any value of $k = \frac{2\pi q}{N/2}$, where $q \in \{1 \dots N/2\}$. The calculations used the value $N = 2000$.

During the stability study, it was found that the decomposed variational systems (13) and (17) lead to the same results for the same values of α and v_0.

3. Stability of the π-mode in the classical Toda lattice

It is known that the unperturbed Toda lattice (i.e., when $\alpha = 0$) is a complete integrable Hamiltonian system (see, for example, [29]). In Poincaré's work [38] (see also [39]) a relationship was established between the number of constants of motion and the values of the Floquet multipliers. The theorem proved in [38] states that if there are M functionally independent constant of motion $F(q_1, p_1 \dots q_N, p_N) = \text{const}$, near the trajectory of a periodic regime, then at least $2M$ Floquet multipliers for that periodic mode will be identically equal to one:

$$\mu_i = 1, \qquad i \in \{1 \dots 2M\}. \tag{19}$$

Note that complete integrability implies the existence of N constant of motion, which means that, according to the theorem, all Floquet multipliers for a certain periodic regime must be equal to one if these constants turn out to be functionally independent in the vicinity of such a periodic regime. This statement is stronger than the necessary condition for the non-asymptotic stability of the periodic regime $|\mu_i| = 1$.

In the present study, it was found that in the unperturbed Toda lattice for the number of particles $N \geqslant 4$, the constants of motion of the system (explicit expressions for which are given in [40]) are functionally dependent near the periodic regime under consideration (5). In particular, for $N = 4$, only 2 of the 4 constants of motion are functionally independent (this can be verified by determining the rank of the Jacobian matrix composed of the integrals of motion), which is confirmed by the calculated values of the Floquet multipliers μ_i:

$$\begin{gathered} \mu_1 = \mu_2 = \mu_3 = \mu_4 \approx 1, \\ \mu_5 = \mu_6 = \mu_7^* = \mu_8^* \approx 0.47 + 0.88i. \end{gathered} \tag{20}$$

Namely, according to the Poincaré theorem, with 2 functionally independent constants of motion, only 4 Floquet multipliers equal to one $\mu_i = 1$ are guaranteed to exist, which corresponds to the results obtained in (20).

We note that as the number of particles N increases, the number of functionally independent constants of motion appears to remain equal to 2. This was verified by a direct calculation of the Floquet multipliers for the monodromy matrix of the full variational system at $N = 10$, 100, 500. At all these values of N, only four Floquet multipliers were found to be equal to one.

Thus, despite the complete integrability of the unperturbed Toda lattice, the stability of the π-mode in it does not follow from the above-mentioned Poincaré theorem. However, this stability can be determined by calculating the Floquet multipliers for the systems (13) and (17), which was done in this work.

The stability of the π-mode in the unperturbed Toda lattice was studied for all values of k in the range $k = \frac{2\pi q}{N/2}$ ($q \in \{1 \dots N/2\}$, $N = 2000$) and values of $v_0 \in [0, 20]$. The step between different values of v_0 was 0.02. The maximum deviation from the unit circle among all Floquet multipliers was 2.6×10^{-9}, which corresponds to the numerical accuracy of integrating the equations (13). Therefore, it can be concluded that the π-mode in the classical Toda lattice (i.e., when $\alpha = 0$) is stable for all values of v_0.

4. Stability of the π-mode in a perturbed Toda lattice

This section presents the results of the study of the stability of the π-mode in the perturbed Toda lattice (i.e., when $\alpha \neq 0$). Generally speaking, for different values of the coefficient α, the Floquet

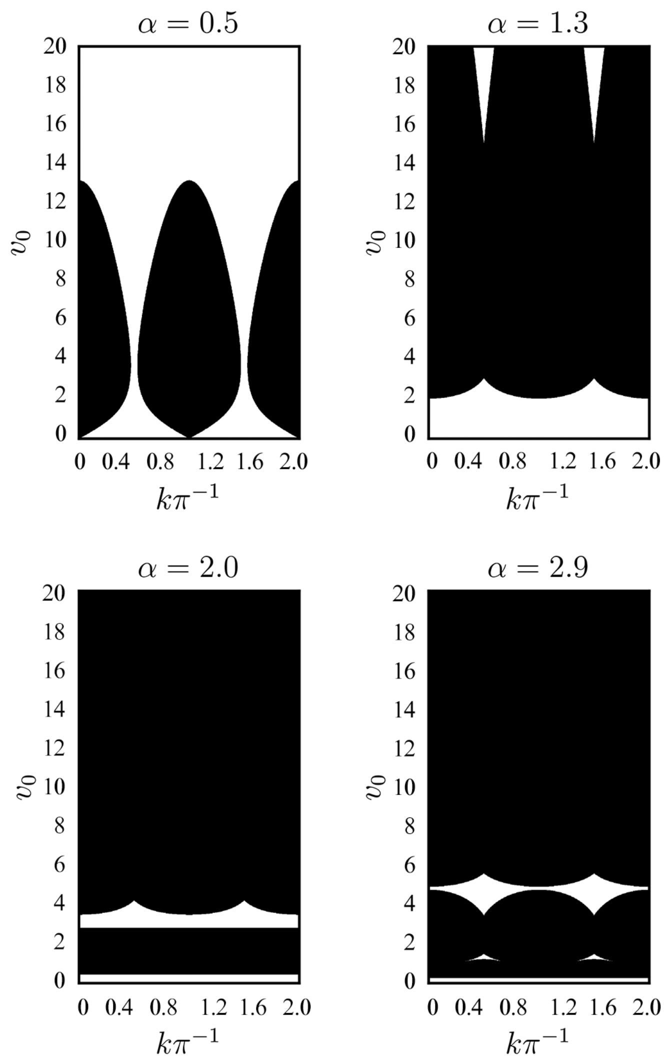

Fig. 2. The stability diagram of the zone boundary mode for various values of k, v_0 and α. White and black colors correspond to the stable and unstable periodic regimes of the investigated mode respectively

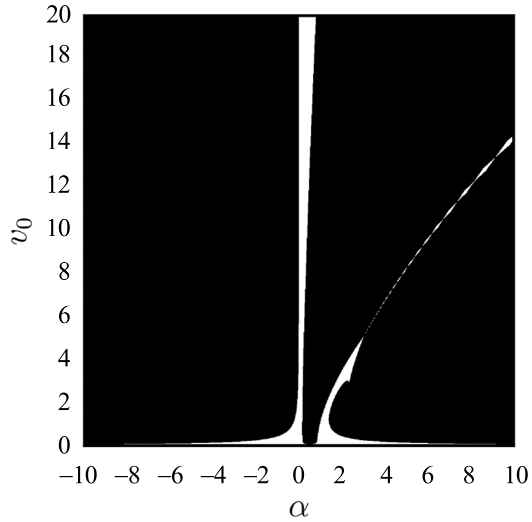

Fig. 3. The stability diagram of the zone boundary mode for various values of α and the initial velocity v_0. White corresponds to the α and v_0 for which the regime is stable for all values of k. Black corresponds to the α and v_0 for which the regime is unstable for at least one value of k

multipliers for each pair of values of v_0 and k are different. For this reason, it is interesting to study the stability for different values of v_0 and k for a fixed value of α, as well as the stability of the system for pairs of values of v_0 and α for an arbitrary value of k. The criteria and methodology for determining stability are described in subsection 2.5.

The results of the study of the stability of the π-mode for different values of k, v_0, and α are presented in Fig. 2. This figure shows the stability diagrams, where white color corresponds to the stability of the zero solution of the corresponding systems (13) and (17), and black color corresponds to instability. Each diagram shows the results of calculations for all values of k in the range $k = \frac{2\pi q}{N/2}$, where $q \in \{1 \ldots N/2\}$ and $N = 2000$, and values of $v_0 \in [0, 20]$, taken in increments of 0.02. It is worth noting that for $\alpha = 2.9$, the stability diagram of the Toda lattice shows a region of values of k and v_0 that is similar to the stability diagram of the π-mode in the FPUT-α chain obtained in [3].

The results of the study of the stability of the π-mode for various values of the parameter α and initial values of v_0 are shown in Fig. 3. As mentioned in subsection 2.5, the regime under study was considered stable for certain values of α and v_0 if the zero solution of the corresponding variational system was stable for any value of k. As in the previous figure, white color corresponds to stability, and black color corresponds to instability. The figure shows that even for large values of the oscillation amplitude and large positive values of α, the chain under study can exhibit stable oscillations corresponding to the π-mode. However, for negative values of α, the nonlinear mode remains stable only for sufficiently small values of the oscillation amplitude.

Conclusion

In this paper, the stability of the π-mode in the Toda lattice and its perturbed version with the addition of a cubic term to the potential is investigated. The study was carried out by decomposing the initial variational system of arbitrary dimension into subsystems of two equations. The decomposition was carried out using both the general group-theoretic method based on the apparatus of irreducible representations of symmetry groups, and the proposed alternative approach using the discrete Fourier transform. The proposed approach can be applied to study the modulation instability of other nonlinear modes.

The π-mode of the completely integrable Toda lattice is shown to be stable for all the considered amplitudes for any number of particles in the system. When a cubic perturbation is introduced into the potential, the stability of the nonlinear mode under study is generally altered: instability intervals appear that depend on the amplitude of the periodic solution and magnitude of the disturbance coefficient. However, even for a perturbed Toda lattice, there are amplitude values at which the π-mode is stable for any number of particles in the chain.

Appendix

A1. Obtaining an exact solution corresponding to the π-mode. This appendix describes a method for solving equation (4). Replacing the variables $V = \dot{U}$ allows one to represent $\ddot{U}$ in the following form:

$$\ddot{U} = \frac{d}{d\tau}V = \frac{dU}{d\tau}\frac{dV}{dU} = V\frac{dV}{dU}. \tag{21}$$

Then equation (4) becomes separable with respect to the variables V and U:

$$\begin{aligned} &V\,dV = -2\sinh\left(2U\right)\,dU,\\ &\int V\,dV = -2\int\sinh\left(2U\right)\,dU,\\ &V^2 = C - 2\cosh\left(2U\right),\\ &V = \pm\sqrt{C - 2\cosh\left(2U\right)}. \end{aligned} \tag{22}$$

Here C is the integration constant. It follows from the type of expression obtained that for the initial conditions $\dot{U}(0) = V(0) = v_0$:

$$C = v_0^2 + 2. \tag{23}$$

By substituting back $\dot{U} = V$, we get the final expression:

$$\begin{aligned} d\tau &= \mathrm{sign}(v_0)\frac{dU}{\sqrt{v_0^2 + 2 - 2\cosh(2U)}},\\ \tau &= D + \mathrm{sign}(v_0)\int\limits_0^U \frac{dU}{\sqrt{v_0^2 - 4\sinh^2(U)}}. \end{aligned} \tag{24}$$

For the initial condition $U(0) = 0$ obviously $D = 0$, then the resulting expression can be reduced to elliptic functions:

$$\begin{aligned} \tau &= \frac{1}{v_0}\int\limits_0^U \frac{dU}{\sqrt{1 + \frac{4}{v_0^2}\sin^2(iU)}},\\ i\tau v_0 &= \int\limits_0^{iU} \frac{d\left(iU\right)}{\sqrt{1 + \frac{4}{v_0^2}\sin^2(iU)}}. \end{aligned} \tag{25}$$

The last expression can be written in terms of the Jacobi amplitude function $\mathrm{am}(u|m)$ [41, p. 16.1.3-4]:

$$iU = \mathrm{am}\left(i\tau v_0\middle| -\frac{4}{v_0^2}\right). \tag{26}$$

A2. Numerical representation of the solution. Expression (26) depends on complex arguments, and the value of the parameter m is negative. Most software packages allow you to calculate elliptic functions only for real arguments and the parameter $m \in [0, 1]$. For this reason, the resulting solution is reduced to the form without complex arguments and $m \in [0, 1]$.

Let us rewrite (5) in the following form:

$$U = -i\arctan\left(\frac{\mathrm{sn}(iw| - k)}{\mathrm{cn}(iw| - k)}\right), \quad w = i\tau v_0, \quad k = \frac{4}{v_0^2}, \tag{27}$$

where sn, cn is elliptic sine and cosine functions defined by the following expressions:

$$\mathrm{sn}(u|m) = \sin\big(\mathrm{am}(u|m)\big), \qquad \mathrm{cn}(u|m) = \cos\big(\mathrm{am}(u|m)\big). \tag{28}$$

First, let us move on from the negative parameter $m = -k$ into $\mu = \frac{k}{1+k}$, $\mu \in [0, 1]$ [41, p. 16.10]:

$$\mathrm{sn}\,(u|-k) = \mu_1^{\frac{1}{2}}\mathrm{sd}\,\left(u\mu_1^{-\frac{1}{2}}\middle|\,\mu\right), \qquad \mathrm{cn}\,(u|-k) = \mathrm{cd}\,\left(u\mu_1^{-\frac{1}{2}}\middle|\,\mu\right), \tag{29}$$

where $\mu_1 = \frac{1}{1+k}$. Also from the definition of the functions sd, cd (see [41, p. 16.3]) it follows that

$$\frac{\mathrm{sd}\,(u|m)}{\mathrm{cd}\,(u|m)} = \frac{\mathrm{sn}\,(u|m)}{\mathrm{cn}\,(u|m)}. \tag{30}$$

Substituting (29), (30) into (27), we get the following expression:

$$U = -i\arctan\left(\mu_1^{\frac{1}{2}}\frac{\mathrm{sn}\,\left(iw\mu_1^{-\frac{1}{2}}\middle|\,\mu\right)}{\mathrm{cn}\,\left(iw\mu_1^{-\frac{1}{2}}\middle|\,\mu\right)}\right). \tag{31}$$

The transition from the imaginary argument is carried out by the following transformation [41, p. 16.20]:

$$\mathrm{sn}\,(iw|\mu) = i\frac{\mathrm{sn}\,(w|m_1)}{\mathrm{cn}\,(w|\mu_1)}, \qquad \mathrm{cn}\,(iw|\mu) = \frac{1}{\mathrm{cn}\,(w|\mu_1)}, \tag{32}$$

where $\mu_1 = \sqrt{1-\mu^2}$. Total:

$$U = \mathrm{arctanh}\left(\mu_1^{\frac{1}{2}}\mathrm{sn}\,\left(w\mu_1^{-\frac{1}{2}}\middle|\,\mu\right)\right). \tag{33}$$

Using (27) to express w and k via the original variables τ, v_0, we get

$$U = \mathrm{arctanh}\left(\frac{\mathrm{sn}\,\left(\tau\sqrt{v_0^2+4}\middle|\,\frac{1}{1+4/v_0^2}\right)}{\sqrt{1+4/v_0^2}}\right). \tag{34}$$

It was expression (34) that was used in the work to calculate the values of U.

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