



Изв.вузов «ПНД», т.3, № 3, 1995

World Scientific on Nonlinear Science - Series A Vol.8

**DYNAMICAL CHAOS - MODELS AND EXPERIMENTS**

**Appearance Routes and Structure of Chaos  
in Simple Dynamical Systems**

*by V.S. Anishchenko*

ISBN 981 022 1428

*Published by:*

World Scientific Publishing Co.Pte.Ltd.  
PO Box 128, Farrer Road, Singapore 9128

*USA office:* Suite 1B, 1060 Main Street,  
River Edge, NJ 07661

*UK office:* 57 Shelton Street, Covent  
Garden, London WC2H 9HE

© 1995 World Scientific Publishing Co.Pte.Ltd.

*Printed in Singapore*

**Preface**

Dear reader! You are faced with a further book on the problem of dynamical chaos in dissipative systems. This subject has already been covered in a number of notable books. An impression might come to mind that the problem has been exhausted. However, let us not hasten this conclusion. In studies of most of the books dedicated to dynamical chaos in dissipative systems, a group of questions can be separated which, in my opinion, deserve a more detailed analysis. The following issues are the most important:

1. The role of characteristic homoclinic trajectories and Poincaré structures which define general bifurcational mechanisms of the appearance and the main properties of chaotic attractors in the appropriate class of nonlinear systems.

2. Regularities and interplay of the bifurcational mechanisms of transition to chaos, as well as the statistical properties of chaotic attractors which are revealed under the multi-parameter bifurcational study of dynamical systems.

3. The influence of external and internal fluctuations on typical bifurcations, structure transformation, the scenarios of development of chaotic attractors and their statistical properties.

The use of the theory of robust hyperbolic attracting sets (the theory of «truly» strange attractors) is found to be insufficient for investigation of the above problems. The application of the concept of quasiattractors which include regular (periodic and quasiperiodic) subsets of trajectories, along with the «strange» ones appears to be more fruitful. The analysis of dynamical systems from the viewpoint of this quasiattractor conception is more constructive for the examination of experimental results.

In low-dimensional dynamical systems, the occurrence of quasiattractors is mainly due to three types of homoclinic trajectories, and namely: 1) a saddle-focus separatrix loop of the equilibrium state; 2) a homoclinic structure in the form of «Smale's horseshoe» that is realized under intersection between stable and unstable manifolds of a saddle limit cycle; and 3) homoclinic trajectories appearing from breakdown the of a resonant two-dimensional torus.

The study of the typical properties of quasiattractors engendered by the above types of homoclinic trajectories in the basic content of this book.

In Chapters 1 and 2, the elements of the stability and bifurcation theories are briefly outlined, as well as the methods for experimental research of dynamical chaos. The incorporation of this material pursues the goal of assisting young scientists in understanding of the main part of the book and to spare them the necessity of referring to additional literature on the first stage of examination.

Chapter 3 attends to an original system with chaotic dynamics (a modified oscillator with inertial nonlinearity) along with its mathematical model. A simple electronic circuit with one-and-a-half degree of freedom due to a number of its attributes has proved to be extremely convenient for numerical and experimental investigation of a low-dimensional chaos.

The objective of Chapters 4 and 5 is to analyze in more detail, both numerically and experimentally, the dynamics of an autonomous oscillator under variation of the control parameters of the system.

In Chapters 6 and 7, the typical hierarchies of instabilities are discussed which accompany nonlinear phenomena as the regimes of quasiperiodic oscillations with two or three independent frequencies are destroyed. For a basic model the inertial nonlinearity oscillator is used. The successive increase in dimension is achieved for the models under study by introducing an external periodic force or using a system of coupled oscillators.

Chapter 8 is devoted to the problem of synchronization of chaotic oscillations. A more simple class of chaotic attractors is considered for which a pronounced basic frequency in the power spectrum is typical (Shilnikov's attractors). The feasibility of generalization for the concepts of the classical oscillation theory on the external and mutual synchronization of periodic oscillations is shown for the case when these attractors are synchronized.

Recent findings on chaotic oscillations in the well-known Chua's circuit are presented in Chapter 9. Unlike the inertial nonlinearity oscillator, Chua's circuit is described by the equations with distinct symmetry properties and is characterized by three equilibrium states. These properties are responsible for the appearance of attractors with more complicated structure permitting a number of novel effects (e.g. the phenomenon of stochastic resonance) to be observed in this system. Chapter 9 has been written in collaboration with A.B.Neiman and M.A.Safonova.

Chapters 10 and 11 deal with the results of research into the influence of fluctuation on the bifurcations of regular and chaotic attractors. The foremost conclusion is here a justification of a strong sensitivity of particular chaos regimes in quasihyperbolic systems to small external perturbations. These chapters has been written together with A.B.Neiman.

Finally, Chapter 12 is devoted to the problem of reconstruction of dynamical systems with account of homoclinic trajectories and noise. This chapter is written together with M.A. Safonova.

The exposition and discussion of most of the results obtained are performed by starting from a detailed comparison between the data of theory, numerical simulation and

full-scale experiment. This allows many typical regularities to be obviously interpreted from the viewpoint of physics and the role of fluctuations, involved in a physical experiment, to be evaluated.

Finally, a distinctive feature of this book is that it has been written based on the results of original studies performed in the Laboratory of Nonlinear Dynamics of the Saratov State University for the past decade under my supervision.

The book's peculiarities briefly listed above, which make it different from books in print, lead me to hope that you, dear reader, will read this book with particular interest and to your benefit.

I express my profound thanks to V.V.Astakhov, T.E.Vadivasova-Letchford, M.A.Safonova, D.E.Postnov and A.B.Neiman, my students and colleagues, and collaborators in most of the works which have provided the basis for this book.

I am sincerely grateful to Professor L.O. Chua from Berkeley University, Professor J.Kurths from Potsdam University and their colleagues in USA and Germany for an inestimable help and support rendered to our laboratory's team. Their assistance has highly fostered the completion of work on this book.

I would like to thank Prof. Yu.L.Klimontovich, Prof. L.P.Shilnikov, Prof. V.N.Belykh, Prof. Yu.I.Neimark, Prof. M.I.Rabinovich, Prof.V.Ebeling, Prof. F.Moss, Prof. S.P.Kuznetsov, and Prof.L. Schimansky-Geier for numerous discussions and their noteworthy scientific works which undoubtedly have made an impact on the formation of my notions in the field of nonlinear dynamics and statistical physics.

Lastly, the author wishes to express his gratitude to Mrs. S.M.Bormasova and Mrs. N.B.Smirnova-Yanson for great efforts in translation of the text of the book and to Miss O.V.Sosnovtseva for her great assistance in preparation the manuscript for the press.

The work on this book was partly financed by ISF Long-Term Research Grant NRO 000, the State Committee of Russia in Higher Education (Grant 93-8.2-10), the Physical Society of America (individual grant) and Linkage High Technology Grant NATO N HTECH. LG 93 07 49.

*Saratov State University, Russia*

*Vadim S. Anishchenko*

## Contents

<b>Preface</b>	vii
<b>Chapter 1. Stability and Bifurcation of Dynamical Systems</b>	1
1.1 Linear analysis of stability. Variational equations	1
1.2 Spectrum of Lyapunov characteristic exponents of phase trajectories of dynamical systems	2
1.3 Stability of equilibrium states	5
1.4 Stability of periodic solutions. Limit cycle multipliers	6
1.5 Stability of quasiperiodic and chaotic solutions	8
1.6 Discrete-time systems. Poincaré map	10
1.7 Stability of discrete system solutions	13
1.8 Structural stability and bifurcations	15
1.9 Bifurcation of equilibrium states	16
1.9.1 Bifurcation of codimension one - a double equilibrium point	16
1.9.2 Bifurcation of codimension two - a triple equilibrium point	17
1.9.3 Limit cycle birth bifurcation	18
1.9.4 Nonlocal codimension-one bifurcations. The separatrix loop of saddle equilibrium state	20
1.10 Bifurcations of periodic solutions	21
1.10.1 Saddle-node bifurcation of limit cycle	22

1.10.2	Period doubling bifurcation of cycle	23
1.10.3	Two-dimensional torus birth bifurcation	25
1.10.4	Symmetry breaking bifurcation	27
1.10.5	Nonlocal periodic motion bifurcations accompanied by a period becoming infinite	29
1.11	Nonlocal bifurcations in the vicinity of double-asymptotic trajectories	31
<b>Chapter 2. Numerical Methods of Chaos Investigations</b>		33
2.1	Experimental approach to investigations of nonlinear system dynamics	33
2.2	Calculation of the Poincaré map	35
2.3	Numerical analysis of periodic solutions and their bifurcations	42
2.4	Numerical analysis of statistical properties of attractors	50
2.5	Algorithms for calculating the spectrum of Lyapunov characteristic exponents (LCE)	55
2.6	Method of numerical calculating the singular solutions	60
2.7	Dimension calculating algorithms	64
<b>Chapter 3. Inertial Nonlinearity Oscillator. Regular Attractor Bifurcations</b>		69
3.1	General equations of one-and-a-half-freedom degree oscillators	69
3.2	Statement of equations for a modified oscillator with inertial nonlinearity	73
3.3	Periodic oscillation regimes in the oscillator and their bifurcations under variation of the parameters	79
3.3.1	Andronov-Hopf bifurcation	80
3.3.2	Limit cycle bifurcations	83
<b>Chapter 4. Autonomous Oscillation Regimes in Oscillator</b>		90
4.1	Two-parametric analysis of transition to chaos via the cascade of period doubling bifurcations	90
4.2	The Poincaré map	96
4.3	System dynamics in the supercritical range of parameter values. Hysteresis and transition to chaos via intermittency induced by fluctuations	104
4.4	Interaction of chaotic attractors. Intermittency of «chaos-chaos» type	112
4.5	Dissipative nonlinearity influence on attractor bifurcations	116
<b>Chapter 5. Quasiattractor Structure and Properties and Homoclinic Trajectories of Autonomous Oscillator</b>		122
5.1	Oscillator dynamics in the vicinity of a homoclinic trajectory of saddle-focus separatrix loop type	122
5.2	Role of homoclinic saddle cycle trajectories in chaotic attractor bifurcation	131
5.3	Physical interpretation of exciting the nonperiodic oscillations in oscillator with inertial nonlinearity	138
5.4	On the dimension of an attractor	141
<b>Chapter 6. Two-Frequency Oscillation Breakdown</b>		146
6.1	General problem statement	146
6.2	Bifurcation diagram of nonautonomous oscillator in the vicinity of basic resonance. Computer simulation	148
6.3	The bifurcation diagram of system (6.1). Full-scale experiment	151
6.4	Two-dimensional torus doubling bifurcation. Soft transition to chaos	155
6.5	Bifurcation mechanism of torus-chaos birth under two-frequency oscillation breakdown	159
6.6	Universal quantitative regularities of soft transition to chaos via two-dimensional torus breakdown	164

<b>Chapter 7. Breakdown of Two- and Three-frequency Quasiperiodic Oscillations</b>	174
7.1 Transitions to torus-chaos in the system of two coupled oscillators	174
7.2 Qualitative description of bifurcations in the system of coupled oscillators by using a model map	180
7.3 Transitions to chaos via three-frequency quasiperiodic oscillations	188
<b>Chapter 8. Synchronization of Chaos</b>	199
8.1 Introduction and definition of the problem	199
8.2 Experimental system and its mathematical model	200
8.3 Methods of investigation	203
8.4 Forced synchronization of chaos	205
8.5 Bifurcational mechanisms of synchronization in the region of chaos	210
8.6 Mutual synchronization of symmetrically coupled oscillators	211
8.7 Evolution of distribution density of phase spectra difference in the process of synchronization	216
<b>Chapter 9. Nonlinear Phenomena and Chaos in Chua's Circuit</b>	219
9.1 Definition of the problem	219
9.2 Chua's circuit	220
9.3 Chaos-chaos intermittency and 1/f noise in Chua's circuit	231
9.4 Dynamics of non-autonomous Chua's circuit	241
9.5 Stochastic resonance in Chua's circuit	248
9.6 Confirmation of the Afraimovich-Shilnikov torus-breakdown theorem via Chua's torus circuit	255
<b>Chapter 10. Bifurcations of Dynamical System in the Presence of Noise</b>	268
10.1 Some methods of stochastic calculus	268
10.2 Influence of external noise on the bifurcations of equilibrium state	278
10.3 Period doubling bifurcations in the presence of noise	291
<b>Chapter 11. Chaos Structure and Properties in the Presence of Noise</b>	302
11.1 Introduction	302
11.2 Regimes of dynamical chaos under the influence of noise with finite intensity	304
11.3 Hyperbolic and quasihyperbolic attractors under the influence of colored noise	308
11.4 Bifurcations of chaotic attractors in the presence of noise	312
11.5 Transitions in chaotic systems induced by noise	319
11.6 Statistical properties of intermittency in quasihyperbolic systems in the presence of noise	325
<b>Chapter 12. Reconstruction of Dynamical Systems from Experimental data</b>	337
12.1 Introduction	337
12.2 Methods and algorithms	338
12.3 Results of map (12.1) reconstruction	342
12.4 Reconstruction of system (12.2)	345
12.5 Comparison of qualitative characteristics of reconstructed attractors with original ones	350
12.6 Reconstruction of differential system	354
<b>Bibliography</b>	361
<b>Index</b>	381