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Frequency repulsion in ensembles of general limit-cycle oscillators synchronized by common noise in the presence of global desynchronizing coupling

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Topic. We study the interaction of two fundamentally different synchronization mechanisms: by means of coupling and by means of the driving by a random signal, which is identical for all oscillators – common noise. Special attention is focused on the effect of frequency divergence arising from the competition between these mechanisms. **Aim.** The aim of the paper is to construct a universal theory describing such an interaction for a general class of smooth limit-cycle oscillators with a global coupling. The effect of intrinsic noise, which is individual for each oscillator, is also to be taken into account. We also plan to assess how the results obtained earlier for the systems of the Ott–Antonsen type reflect the situation in general case. **Method.** For a general class of oscillators, the phase description is introduced. For the Fokker–Planck equation, which corresponds to the stochastic equations of phase dynamics, a rigorous averaging procedure is performed in the limit of high oscillation frequency (the conventional multiple scale method is used). With the derived equations one can obtain the conditions for synchronization of ensembles of identical oscillators, and for weakly nonidentical oscillators, one can find the average oscillation frequencies in quadratures. Analytical results are verified by direct numerical simulations for large but finite ensembles of van der Pol, Rayleigh and van der Pol–Duffing oscillators, as well as for FitzHugh–Nagumo neuron-like oscillators. **Results.** For the case of identical oscillators without intrinsic noise, a sufficiently strong common noise synchronizes an ensemble with a repulsive global coupling, and the dynamics of localization of the oscillator distribution is investigated. The latter clearly indicates that during the transition to the state of perfect synchrony, the distribution of oscillators possesses «heavy» power-law tails, even with an arbitrary strong attracting global coupling – without common noise, such tails do not apper. For the case of oscillators with intrinsic noise, the equilibrium distribution of phase differences always possesses «heavy» power-law tails, and the parameters of these tails are determined. The asymptotic behavior for the average frequency of an oscillator as a function of the natural frequency is derived analytically; in particular, the effect of divergence of the average frequencies is reported to accompany the synchronization by common noise in the presence of a repulsive global coupling. Examples of the application of the constructed theory for van der Pol, Rayleigh, van der Pol–Duffing and FitzHugh–Nagumo oscillators are presented. The results of direct numerical simulation for large finite ensembles of these oscillators are consistent with the theory. **Discussion.** Arbitrary weak general noise, on the one hand, increases the stability of the synchronous state, and on the other hand, it always creates «heavy» power-law tails for the distribution of phase differences. This indicates a significantly intermittent character of synchronization by common noise – epoches of synchronous behavior are interrupted by the phase difference slips – and is consistent with the fact that in the presence of common noise, a perfect frequency locking becomes impossible. For a repulsive coupling, a nontrivial effect occurs: sufficiently strong common noise synchronizes the states of oscillators, but their average frequencies are mutually repelled. The effect of individual intrinsic noise on the average frequencies is effectively equivalent to the effect of synchrony imperfection.

Key words: Synchronization, stochastic processes, frequency repulsion, synchronization by common noise.

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Introduction

In recent decades, the importance of the synchronization effect has been widely elucidated in the physical, geophysical, biological, and social sciences (see, for example, [1]). Systems in which the internal dynamics of the elements are quite simple, and the complexity of behavior arises exclusively as a collective effect are of particular interest. From a mathematical point of view, such a system is an ensemble of oscillators having a stable limit cycle and interacting with each other through weak mutual coupling or the influence of a common force. It is known that the states of oscillators with a limit cycle experiencing a weak external action can be completely characterized by their phases. It allows us to describe the dynamics of such a system in the framework of the phase approximation [2–4]. Recently, a number of works have appeared that demonstrate the practical application of the phase approximation for self-oscillations in spatially distributed systems [5,6], as well as for collective oscillations of networks of coupled dynamic elements [7].

For ensembles of identical oscillators with a limit cycle (that is, phase oscillators), three fundamentally different synchronization mechanisms can be distinguished: (1) global coupling, (2) general periodic action, and (3) common noise [8–13]. The difference between these mechanisms is most pronounced when considering the dynamics of a pair of oscillators with close, but not identical natural frequencies (a discussion of the interaction of coupling synchronization mechanisms and the common periodic action from a different point of view can be found, for example, in [14]). An important distinction of the synchronization mechanism by common noise is that in this case there is no frequency locking [15], which is not only typical of two other mechanisms, but can also be used as a criterion for determining the synchronization state. The states of two slightly different oscillators in the presence of common noise are close to each other most of the time, but from time to time in such a system jumps in phase difference occur. Moreover, the average frequencies of the oscillators are not attracted to each other.

The question of whether different mechanisms can compensate each other, exerting the opposite effect on the system, is of particular interest. For example, can synchronizing common noise compete with desynchronizing global coupling? The interaction of coupling and common noise was first investigated in [16, 17]. The development of the Ott–Antonsen theory [18] made it possible to look at the effect of synchronization in ensembles of oscillators from a new perspective and expand the understanding of the mechanisms of the emergence of synchronous states [19–21]. In particular, an unexpected effect was discovered and investigated: in the presence of a repulsive coupling, a sufficiently strong common noise synchronizes the elements of the ensemble, while the average frequencies of the oscillators in the synchronized state are more dispersed than the natural frequencies of the oscillators. This effect is especially noteworthy due to the fact that in earlier works, synchronization was understood as the «phenomenon of the pulling together of frequencies» [22]. In this case, on the contrary, synchronization is observed, leading to repulsion of frequencies.

It is worth highlighting the system of phase oscillators of the following form:

$$\dot{\varphi}_j = \omega(t) + \mathrm{Im}(H(t)e^{-i\varphi_j})\,, \tag{1}$$

where $\omega(t)$ and $H(t)$ can depend on time. It was for such systems that the theories of Watanabe–

Strogatz and Ott–Antonsen were constructed [18, 23–25]. The Ott–Antonsen theory allows us to write an equation for a complex order parameter $Z = \langle e^{i\varphi_j} \rangle$

$$\dot{Z} = i\omega(t)\, Z + \frac{H(t)}{2} - \frac{H^*(t)}{2} Z^2 \,. \quad (2)$$

Based on this equation, a mathematically rigorous description of the dynamics of the order parameter for an arbitrary degree of synchronism in the system can be obtained, whereas in the case of general oscillators, an analytical study of collective behavior is possible only for states with a high degree of synchronism (this will be shown in Section 1). Therefore, ensembles of the form (1) are of particular interest: they provide an opportunity to study in detail the process of transition from a state of full synchronism to asynchronous states.

(1) Kuramoto oscillator ensemble with multiplicative common noise [19,20]

$$\dot{\varphi}_j = \Omega_j + \frac{\mu}{N} \sum_{k=1}^{N} \sin(\varphi_k - \varphi_j - \beta) + \varepsilon\xi(t) \sin\varphi_j \,,$$

where $N \to \infty$, μ is the coupling coefficient, β is the phase shift of the coupling, $\xi(t)$ is the normalized common noise, the natural frequencies of the oscillators Ω_j are equal or have a Lorentzian distribution. In this case, $\omega_j(t) = \Omega_j$, $H(t) = \mu e^{-i\beta} Z - \varepsilon\xi(t)$ (see equation (1)).

(2) Ensemble of active rotators with global coupling and additive common noise [21]

$$\dot{\varphi}_j = \Omega_j - B\sin\varphi_j + \frac{\mu}{N} \sum_{k=1}^{N} \sin(\varphi_k - \varphi_j) + \varepsilon\xi(t) \,.$$

The nonuniformity of rotation of the «phase» described by the term $B\sin\varphi_j$ is significant, because otherwise the common noise does not have a synchronizing effect [11,12]. This case corresponds to $\omega_j(t) = \Omega_j + \varepsilon\xi(t)$ и $H(t) = B + \mu Z$.

In [19–21], the basic principles of the interaction of two mechanisms of synchronization are described; in particular, the effect of frequency divergence in the synchronous state is shown. Nevertheless, the existing theoretical description is limited to a rather narrow class of Ott–Antonsen-type systems, and in such systems the dynamic formation and redistribution of clusters is impossible (the distribution of elements between clusters is frozen). Also, the presence of intrinsic noise in oscillators significantly violates the properties of such systems; only recently, a new approach was proposed in [26, 27], which allows one to solve this problem and construct a perturbation theory for Ott–Antonsen-type systems. In addition, a significant limitation of the Ott–Antonsen theory is that it does not allow consideration of terms containing harmonics of higher orders (for example, proportional to $\sin 2\varphi_j$).

In this regard, ensembles of oscillators with a limit cycle are the third important case, allowing a complete comprehensive analysis of collective dynamics. This case is especially interesting in connection with the fact that it is applicable for experimental studies of systems not described by the Ott–Antonsen equations [28].

In this paper, we do not consider the effects of desynchronization caused by the influence of common noise [8,9,15,29,30], since they appear at least for a moderate noise intensity, and the phase approximation is applicable only at a low noise level. In this work, we consider the synchronization of oscillators with a limit cycle caused by the action of synchronizing common noise and global coupling. The influence of intrinsic noise in the elements on the dynamics of the system is considered. The effect of the divergence of the observed frequencies of oscillators with different natural frequencies under the action of a desynchronizing (repulsive) coupling is also described.

The article is organized as follows. Section 1 describes the procedure for constructing the phase approximation for a system of general-type oscillators with a limit cycle in the presence of common and intrinsic noise and global coupling. The correctness of the phase description for systems in which amplitude degrees of freedom are important is shown. From the phase equations of the individual oscillators, the equations of averaged dynamics in the high-frequency limit are obtained. Section 2 describes the conditions for the appearance of a synchronous state in an ensemble, as well as the characteristics of violation of ideal synchronism under the influence of intrinsic noise, and the distribution of the deviations of the oscillator phases from the average value is obtained. In Section 3, we construct an analytical theory for the case of an ensemble of non-identical oscillators, obtain the laws of asymptotic behavior for the average frequency of the oscillators depending on the detuning of the natural frequency, and study the effect of frequency divergence for repulsive coupling. The analytically obtained results are illustrated by the results of numerical simulations for ensembles of van der Pol, Rayleigh, van der Pol–Duffing and FitzHugh–Nagumo oscillators. The results are discussed in the Conclusion.

1. The basic model of a limit-cycle oscillators ensemble of the general form

Consider an ensemble of N identical general oscillators under the influence of common noise and global coupling

$$\dot{\mathbf{x}}_j = \mathbf{F}(\mathbf{x}_j) + \frac{\mu}{N}\sum_{k=1}^{N}\mathbf{H}(\mathbf{x}_j,\mathbf{x}_k) + \varepsilon\mathbf{B}(\mathbf{x}_j)\circ\xi(t) + \sigma\mathbf{C}(\mathbf{x}_j)\circ\zeta_j(t)\,, \tag{3}$$

where $\mathbf{x}_j$ describes the state of the jth oscillator, $j = 1, 2, ..., N$; μ is the coupling coefficient; ε and σ are the amplitudes of the common and intrinsic noise; $\xi(t)$ and $\zeta_j(t)$ are independent signals with normalized δ-correlated Gaussian noise:$\langle\xi\rangle = \langle\zeta_j\rangle = 0$, $\langle\xi(t)\,\xi(t')\rangle = 2\delta(t-t')$, $\langle\xi(t)\,\zeta_j(t')\rangle = 0$, $\langle\zeta_j(t)\,\zeta_k(t')\rangle = 2\delta_{jk}\delta(t-t')$; the symbol «$\circ$» indicates that the equations are considered in the sense of Stratonovich. A coupling in which all oscillators interact in pairs in the same way, described by the term $\mathbf{H}(\mathbf{x}_j,\mathbf{x}_k)$ is meant by «global coupling». Without loss of generality, we can assume that the coupling between the matching elements disappears $\mathbf{H}(\mathbf{x},\mathbf{x}) = 0$. If this is not so, then one can always get rid of the term $\mathbf{H}(\mathbf{x},\mathbf{x})$ by redefining $\mathbf{F}(\mathbf{x})$ as follows: $\mathbf{F}(\mathbf{x}) \to \mathbf{F}(\mathbf{x}) - \mu\mathbf{H}(\mathbf{x},\mathbf{x})$. If there is neither noise nor global coupling in the system, the equations of state of the oscillators admit a stable periodic solution $\mathbf{x}^{(0)}(t) = \mathbf{x}^{(0)}(t + 2\pi/\Omega)$, where Ω is the natural frequency of the oscillators. This solution can be parameterized by the phase φ uniformly growing with time: $\mathbf{x}^{(0)}(\varphi) = \mathbf{x}^{(0)}(\varphi + 2\pi)$. A phase can be introduced in a finite neighborhood of the limit cycle – the basin of its attraction: $\varphi = \varphi(\mathbf{x})$.

The dynamics of system (3) in the presence of weak noise and weak coupling in the leading order can be described in the framework of the phase approximation [2–4]

$$\dot{\varphi}_j = \Omega + \frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}(\varphi_j, \varphi_k - \varphi_j) + \varepsilon\mathcal{B}(\varphi_j)\circ\xi(t) + \sigma\mathcal{C}(\varphi_j)\circ\zeta_j(t), \tag{4}$$

where

$$\mathcal{B}(\varphi) \equiv \left(\frac{\partial\varphi}{\partial\mathbf{x}}\cdot\mathbf{B}\right)_{\mathbf{x}=\mathbf{x}^{(0)}(\varphi)}, \qquad \mathcal{C}(\varphi) \equiv \left(\frac{\partial\varphi}{\partial\mathbf{x}}\cdot\mathbf{C}\right)_{\mathbf{x}=\mathbf{x}^{(0)}(\varphi)}$$

are 2π-periodic functions characterizing the sensitivity of the phase to noise. $(\mu/N)\mathcal{H}(\varphi,\psi)$ is the increase in the growth rate of the phase of the oscillator in the state $\mathbf{x}^{(0)}(\varphi)$, caused by the coupling with another oscillator in the state $\mathbf{x}^{(0)}(\varphi+\psi)$

$$\mathcal{H}(\varphi,\psi)\equiv\left(\frac{\partial\varphi}{\partial\mathbf{x}}\right)_{\mathbf{x}=\mathbf{x}^{(0)}(\varphi)}\cdot\mathbf{H}\Big(\mathbf{x}^{(0)}(\varphi),\mathbf{x}^{(0)}(\varphi+\psi)\Big)\,.$$

Since $\mathbf{H}(\mathbf{x},\mathbf{x})=0$, we find $\mathcal{H}(\varphi,0)=0$.

Note that for δ-correlated noise, the derivation of equation (4) is not rigorous.

A rigorous consideration [31, 32] is carried out in the next section 1.1 and gives the same results, but with a frequency Ω shifted by some correction of the order of $(\varepsilon^2+\sigma^2)$, which is associated with the presence of amplitude degrees of freedom. This correction is important when considering the average frequency of oscillations, since it has the same order of smallness as the correction due to the noise term in equation (4). Nevertheless, equation (4) can be considered as mathematically rigorous, adjusted for the fact that Ω is not the natural frequency but the shifted frequency of the oscillators.

1.1. Phase approximation taking into account the amplitude degrees of freedom. Let us derive the phase approximation for a system in which the noise autocorrelation time is small compared to the relaxation time of perturbations along the amplitude degrees of freedom. In such a system, taking into account the amplitude degrees of freedom in constructing the phase description is mandatory [32]. For simplicity, we will consider two-dimensional oscillators (that is, systems with one amplitude degree of freedom). The results obtained can be easily generalized to the case of multidimensional systems.

In the general case, equations (3) for oscillators with a limit cycle can be written in variable amplitudes and phases

$$\dot{\varphi}_j=\Omega+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}(\varphi_j,\varphi_k-\varphi_j,r_j,r_k)+\varepsilon\mathcal{B}(\varphi_j,r_j)\circ\xi(t)+\sigma\mathcal{C}(\varphi_j,r_j)\circ\zeta_j(t)\,, \qquad (5)$$

$$\dot{r}_j=-\lambda r_j+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{P}(\varphi_j,\varphi_k-\varphi_j,r_j,r_k)+\varepsilon\mathcal{S}(\varphi_j,r_j)\circ\xi(t)+\sigma\mathcal{R}(\varphi_j,r_j)\circ\zeta_j(t)\,, \qquad (6)$$

where λ is the Lyapunov exponent (the possibility of reducing the equations to this form is explained in detail in supplementary materials to [32]). Expanding in a series of r functions $\mathcal{H}(\varphi,\psi,r_j,r_k)=\mathcal{H}_{00}(\varphi,\psi)+\mathcal{H}_{10}(\varphi,\psi)r_j+\mathcal{H}_{01}(\varphi,\psi)r_k+\mathcal{O}(r_jr_k,r_j^2,r_k^2)$, $\mathcal{B}(\varphi,r)=\mathcal{B}_0(\varphi)+$ $+\mathcal{B}_1(\varphi)r+\mathcal{O}(r^2)$ etc., the following can be obtained

$$\begin{aligned}\dot{\varphi}_j=\Omega+\frac{\mu}{N}\sum_{k=1}^{N}\big[\mathcal{H}_{00}(\varphi_j,\varphi_k-\varphi_j)+r_j\mathcal{H}_{10}(\varphi_j,\varphi_k-\varphi_j)+r_k\mathcal{H}_{01}(\varphi_j,\varphi_k-\varphi_j)+\ldots\big]+&\\+\varepsilon\big[\mathcal{B}_0(\varphi_j)+r_j\mathcal{B}_1(\varphi_j)+\ldots\big]\circ\xi(t)+\sigma\big[\mathcal{C}_0(\varphi_j)+r_j\mathcal{C}_1(\varphi_j)+\ldots\big]\circ\zeta_j(t)\,,& \qquad (7)\end{aligned}$$

$$\dot{r}_j=-\lambda r_j+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{P}_{00}(\varphi_j,\varphi_k-\varphi_j)+\varepsilon\mathcal{S}_0(\varphi_j)\circ\xi(t)+\sigma\mathcal{R}_0(\varphi_j)\circ\zeta_j(t)+\ldots\,. \qquad (8)$$

Using the standard phase approximation procedure for δ-correlated noise [31, 32], we find first-order corrections

$$\begin{aligned}\dot{\varphi}_j\approx\Omega+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}_{00}(\varphi_j,\varphi_k-\varphi_j)+\varepsilon^2\mathcal{S}_0(\varphi_j)\,\mathcal{B}_1(\varphi_j)+\sigma^2\mathcal{R}_0(\varphi_j)\,\mathcal{C}_1(\varphi_j)+&\\+\varepsilon\mathcal{B}_0(\varphi_j)\circ\xi(t)+\sigma\mathcal{C}_0(\varphi_j)\circ\zeta_j(t)\,.& \qquad (9)\end{aligned}$$

The growth rate of the phase φ_j now depends not only on the influence of the coupling and the two noise terms, but also has an additional deterministic drift $\varepsilon^2\mathcal{S}_0(\varphi_j)\,\mathcal{B}_1(\varphi_j)+$ $+\sigma^2\mathcal{R}_0(\varphi_j)\,\mathcal{C}_1(\varphi_j)$. We introduce the «true» phase φ, which should grow uniformly with time,

$$\dot{\phi}=\Omega_*\equiv\Omega+\varepsilon^2\langle\mathcal{S}_0(\varphi)\,\mathcal{B}_1(\varphi)\rangle_\varphi+\sigma^2\langle\mathcal{R}_0(\varphi)\,\mathcal{C}_1(\varphi)\rangle_\varphi\,,\quad\text{where}\quad\langle...\rangle_\varphi=(2\pi)^{-1}\int_0^{2\pi}...\mathrm{d}\varphi.$$

The relation between φ and ϕ has the following form:

$$\mathrm{d}\phi=\frac{\Omega_*\mathrm{d}\varphi}{\Omega+\varepsilon^2\mathcal{S}_0(\varphi)\,\mathcal{B}_1(\varphi)+\sigma^2\mathcal{R}_0(\varphi)\,\mathcal{C}_1(\varphi)}\,.$$

Up to leading corrections μ, ε^2 and σ^2, equation (9) can be transformed to the form

$$\dot{\phi}_j\approx\Omega_*+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}_{00}(\phi_j,\phi_k-\phi_j)+\varepsilon\mathcal{B}_0(\phi_j)\circ\xi(t)+\sigma\mathcal{C}_0(\phi_j)\circ\zeta_j(t)\,. \tag{10}$$

Equations (4) and (10) are identical up to the substitution $(\Omega,\varphi_j)\leftrightarrow(\Omega_*,\phi_j)$, i.e., equation (4) in the general case is true taking into account the corresponding correction of the natural frequency.

1.2. Averaging the dynamics of the ensemble over the period of natural oscillators revolutions. In the case of imperfectly identical oscillator frequencies, the system of equations (4) in the leading order takes the form

$$\dot{\varphi}_j=\Omega_j+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}(\varphi_j,\varphi_k-\varphi_j)+\varepsilon\mathcal{B}(\varphi_j)\circ\xi(t)+\sigma\mathcal{C}(\varphi_j)\circ\zeta_j(t), \tag{11}$$

where Ω_j is the natural frequency of the jth oscillator.

To describe the dynamics of the ensemble in states close to the synchronous one, it is convenient to introduce reference phase φ_0 defined by the equation

$$\dot{\varphi}_0=\Omega_0+\varepsilon\mathcal{B}(\varphi_0)\circ\xi(t), \tag{12}$$

where Ω_0 is the average natural frequency of the elements of the ensemble. Then the phase mismatch $\theta_j=\varphi_j-\varphi_0$ obeys the equation

$$\dot{\theta}_j=\omega_j+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}(\varphi_0+\theta_j,\theta_k-\theta_j)+\varepsilon[\mathcal{B}(\varphi_0+\theta_j)-\mathcal{B}(\varphi_0)]\circ\xi(t)+\sigma\mathcal{C}(\varphi_0+\theta_j)\circ\zeta_j(t), \tag{13}$$

where $\omega_j=\Omega_j-\Omega_0$ is the detuning of the natural frequency.

From the system of equations (12)–(13), the Fokker–Planck equation can be obtained for probability densities $w(\varphi_0,\theta_1,...,\theta_N,t)$

$$\frac{\partial w}{\partial t}+\frac{\partial}{\partial\varphi_0}\Big(\Omega_0 w\Big)+\sum_{j=1}^{N}\frac{\partial}{\partial\theta_j}\Big(\big(\omega_j+\frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}(\varphi_0+\theta_j,\theta_k-\theta_j)\big)w\Big)-$$
$$-\varepsilon^2\hat{Q}_\Sigma^2 w-\sigma^2\sum_{j=1}^{N}\frac{\partial}{\partial\theta_j}\Big(\mathcal{C}(\varphi_0+\theta_j)\frac{\partial}{\partial\theta_j}\big(\mathcal{C}(\varphi_0+\theta_j)w\big)\Big)=0\,,$$

where

$$\hat{Q}_\Sigma(\cdot) \equiv \frac{\partial}{\partial\varphi_0}\Big(\mathcal{B}(\varphi_0)(\cdot)\Big) + \sum_{j=1}^{N}\frac{\partial}{\partial\theta_j}\Big([\mathcal{B}(\varphi_0+\theta_j) - \mathcal{B}(\varphi_0)](\cdot)\Big)\,.$$

Integrating the Fokker–Planck equation over all θ_j, except for $j = l$, we can obtain for

$$w_l(\varphi_0, \theta_l, t) = \textstyle\int \mathrm{d}\theta_1 ... \mathrm{d}\theta_{l-1}\mathrm{d}\theta_{l+1} ... \mathrm{d}\theta_N w(\varphi_0, \theta_1, ..., \theta_N, t)$$

the following equation:

$$\frac{\partial w_l}{\partial t} + \frac{\partial}{\partial\varphi_0}\Big(\Omega_0 w_l\Big) + \frac{\partial}{\partial\theta_l}\Big(\omega_l w_l + \int \mathrm{d}\theta_1 ... \mathrm{d}\theta_{l-1}\mathrm{d}\theta_{l+1} ... \mathrm{d}\theta_N \frac{\mu}{N}\sum_{k=1}^{N}\mathcal{H}(\varphi_0+\theta_l, \theta_k - \theta_l) w\Big) -$$
$$- \varepsilon^2\hat{Q}_l^2 w_l - \sigma^2\frac{\partial}{\partial\theta_l}\Big(\mathcal{C}(\varphi_0+\theta_l)\frac{\partial}{\partial\theta_l}\big(\mathcal{C}(\varphi_0+\theta_l) w_l\big)\Big) = 0\,,$$

where

$$\hat{Q}_l(\cdot) \equiv \frac{\partial}{\partial\varphi_0}\Big(\mathcal{B}(\varphi_0)(\cdot)\Big) + \frac{\partial}{\partial\theta_l}\Big([\mathcal{B}(\varphi_0+\theta_l) - \mathcal{B}(\varphi_0)](\cdot)\Big).$$

In the thermodynamic limit $N \to \infty$, each oscillator is indicated by the frequency ω instead of index l and we go from summation to integration. In this case, the Fokker–Planck equation for $w_\omega(\varphi_0, \theta_\omega, t)$ takes the form

$$\frac{\partial w_\omega}{\partial t} + \frac{\partial}{\partial\varphi_0}\Big(\Omega_0 w_\omega\Big) + \frac{\partial}{\partial\theta_\omega}\bigg(\Big(\omega + \mu\int\limits_{-\infty}^{+\infty}\mathrm{d}\omega_1 g(\omega_1)\int\limits_0^{2\pi}\mathrm{d}\theta\, w_{\omega_1}(\varphi_0,\theta)\,\mathcal{H}(\varphi_0+\theta_\omega, \theta-\theta_\omega)\Big) w_\omega\bigg) -$$
$$- \varepsilon^2\hat{Q}_\omega^2 w_\omega - \sigma^2\frac{\partial}{\partial\theta_\omega}\Big(\mathcal{C}(\varphi_0+\theta_\omega)\frac{\partial}{\partial\theta_\omega}\big(\mathcal{C}(\varphi_0+\theta_\omega) w_\omega\big)\Big) = 0\,, \quad (14)$$

where $g(\omega)$ is the distribution of the natural frequencies of the elements of the ensemble.

In the high-frequency limit, Ω_0 is large compared to μ, ε^2, σ^2, and ω, we can use the rigorous averaging procedure over the rapidly rotating phase φ_0 using the multi-scale method [33] (detailed examples of the application of such a procedure are given in [19, 20]). In this case, we have $w_\omega(\varphi_0, \theta, t) = W_\omega(\theta, t) + \mathcal{O}(\mu/\Omega_0, \varepsilon^2/\Omega_0, \sigma^2/\Omega_0, \omega/\Omega_0)$, and the evolution equation for $W_\omega(\theta, t)$ takes the form

$$\frac{\partial W_\omega(\theta,t)}{\partial t} + \frac{\partial}{\partial\theta}\bigg(\Big(\omega + \mu\int\limits_{-\infty}^{+\infty}\mathrm{d}\omega_1 g(\omega_1)\int\limits_0^{2\pi}\mathrm{d}\theta_1 W_{\omega_1}(\theta_1,t)\,h(\theta_1-\theta)\Big) W_\omega(\theta,t)\bigg) -$$
$$- \frac{\partial^2}{\partial\theta^2}\Big(\big(2\varepsilon^2[f(0) - f(\theta)] + \sigma^2\big) W_\omega(\theta,t)\Big) = 0, \qquad (15)$$

where

$$f(\theta) \equiv \langle\mathcal{B}(\varphi+\theta)\,\mathcal{B}(\varphi)\rangle_\varphi, \quad h(\psi) \equiv \langle\mathcal{H}(\varphi,\psi)\rangle_\varphi.$$

The normalization condition $\langle[\mathcal{C}(\varphi)]^2\rangle_\varphi = 1$ is introduced here. The obtained equation can be written in the form

$$\frac{\partial W_\omega}{\partial t} + \frac{\partial}{\partial\theta}\Big(\big(\omega + \mu\hbar(-\theta)\big) W_\omega\Big) - \frac{\partial^2}{\partial\theta^2}\Big(\big(2\varepsilon^2[f(0) - f(\theta)] + \sigma^2\big) W_\omega\Big) = 0\,, \qquad (16)$$

$$\hbar(-\theta) = \int\limits_{-\infty}^{+\infty}\mathrm{d}\omega\, g(\omega)\int\limits_0^{2\pi}\mathrm{d}\theta_1 W_\omega(\theta_1,t)\,h(\theta_1-\theta). \qquad (17)$$

Equation (16) is the basic equation with which we will work further. It should be noted that $h(0) = 0$, and the function

$$f(\theta) = \langle \mathcal{B}(\varphi)\, \mathcal{B}(\varphi+\theta)\rangle_\varphi = \Big\langle \mathcal{B}(\varphi) \sum_{k=0}^{\infty} \frac{\mathrm{d}^k \mathcal{B}(\varphi)}{\mathrm{d}\varphi^k} \frac{\theta^k}{k!} \Big\rangle_\varphi = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Big\langle \Big(\frac{\mathrm{d}^n \mathcal{B}(\varphi)}{\mathrm{d}\varphi^n}\Big)^2 \Big\rangle_\varphi \theta^{2n}$$

when expanded into a Taylor series, it contains only even terms.

Equations (16)–(17) are a complete closed mathematical description of the system, which makes it possible for given functions $\hbar(\theta)$ and $f(\theta)$ to find $W_\omega(\theta)$, which, in turn, allows us to find $\hbar(\theta)$ from $h(\theta)$ and $g(\omega)$. Unfortunately, this problem can be solved analytically only for systems of a certain type (for example, for a phase ensemble of the Ott–Antonsen type [17–21]). However, with the introduction of some simplifying assumptions that allow one to calculate $\hbar(\theta)$ from $h(\theta)$, the problem can be solved for a wider range of systems. For ensembles with a high degree of synchronism, it is natural to assume $\hbar(\theta) \approx h(\theta)$.

The effective Langevin equation for (16) has the following form:

$$\dot{\theta} = \omega + \mu\hbar(-\theta) + \varepsilon^2 f'(\theta) + \varepsilon\sqrt{2[f(0) - f(\theta)]} \circ \xi(t) + \sigma\zeta(t)\,. \tag{18}$$

Equations (16) and (18) can be compared with similar equations for the Kuramoto and Kuramoto–Sakaguchi ensembles with a sinusoidal noise term (see [19, 20]), for which $h(\theta) = \sin(\theta+\beta) - \sin\beta$, $\mathcal{B}(\varphi) = \sin\varphi$ and $f(\theta) = 0.5\cos\theta$. Substituting these functions in equations (16) and (18) for $\sigma = 0$ makes them completely equivalent to the corresponding equations in [19, 20] in the limit of a high degree of synchronism

$$\frac{\partial W_\omega}{\partial t} + \frac{\partial}{\partial\theta}\Big(\big(\omega - \mu\big[\sin(\theta+\beta) - \sin\beta\big]\big) W_\omega\Big) - \varepsilon^2 \frac{\partial^2}{\partial\theta^2}\Big(\big(1 - \cos\theta\big) W_\omega\Big) = 0$$

and

$$\dot{\theta} = \omega - \mu\big[\sin(\theta+\beta) - \sin\beta\big] - \frac{\varepsilon^2}{2}\sin\theta + \varepsilon\frac{\sin\theta}{\sqrt{2}} \circ \xi_1(t) + \varepsilon\frac{\cos\theta - 1}{\sqrt{2}} \circ \xi_2(t)\,,$$

where $\xi_1(t)$ and $\xi_2(t)$ are independent normalized functions, δ-correlated Gaussian noises.

The demonstration of the validity of this statement for equation (16) is purely technical task. For equation (18), we must additionally bear in mind that two independent Gaussian noise $\sqrt{0.5}\sin\theta \circ \xi_1(t) + \sqrt{0.5}(\cos\theta - 1) \circ \xi_2(t)$, appearing in [19, 20], act as a single noise, the intensity of which is the sum of the intensities of the independent noise terms, that is, the term $[0.5\sin^2\theta + 0.5(\cos\theta - 1)^2]^{1/2} \circ \xi(t) = \sqrt{1 - \cos\theta} \circ \xi(t)$ corresponds to them $\circ\xi(t)$. When considering states of imperfect synchronism, one can notice differences in the equations: this is because the reference phase used in this work is equivalent to the phase of the order parameter only when the ensemble is perfectly synchronized.

We define the normalization conditions for $h(\theta)$ and $f(\theta)$ (and, therefore, $\mathcal{B}(\varphi)$). These conditions can be specified by choosing the scales for μ and ε, respectively. It can be seen from equation (18) that the global coupling for small deviations θ (near complete synchronization) is substantially determined by the function $h(\theta)$. In this regard, it is natural to choose the following normalization condition:

$$\lim_{\theta\to 0} \frac{h(\theta)}{\theta} = 1.$$

It will be shown later in the text that the integral parameter $\langle[\mathcal{B}'(\varphi)]^2\rangle_\varphi$ is an important characteristic of the phase susceptibility to noise intensity $\mathcal{B}(\varphi)$. Since traditionally in the works on this problem we take $\mathcal{B}(\varphi) = \sin\varphi$, it is advisable to use the following normalization condition for convenience of comparison with earlier works [19, 20]:

$$\langle[\mathcal{B}'(\varphi)]^2\rangle_\varphi = \frac{1}{2} \quad \text{or} \quad f(0) - f(\theta) = \frac{\theta^2}{4} + \mathcal{O}(\theta^4)\,.$$

2. Ensembles of identical oscillators

2.1. A case without intrinsic noise. For oscillators with the same natural frequencies ($\omega = 0$) without intrinsic noise, a state of full synchronization is possible, and the study of the stability properties of such a state is of primary interest. In contrast to the Ott–Antonsen systems, for the dynamics of the order parameter in which a finite-dimensional system of equations can be obtained, and, accordingly, the transition to complete synchronization can be described in terms of the order parameter, for general-type oscillators, the choice of a quantitative characteristic of the degree of synchronism is not so straightforward. In this paper, we will consider two stability characteristics: (1) the Lyapunov exponent for an oscillator deviating from a synchronous cluster, and (2) the dynamics of the probability density function $W(\theta, t)$ near the state of complete synchronization.

For an oscillator that deviates infinitesimally from the synchronous cluster, we have $\hbar(\theta) = h(\theta)$, $|\theta| \ll 1$. Then Langevin equation (18) for $\omega = \sigma = 0$ gives

$$\dot{\theta} = -\mu\theta - \varepsilon^2\langle[\mathcal{B}'(\varphi)]^2\rangle_\varphi\theta + \varepsilon\big[\langle[\mathcal{B}'(\varphi)]^2\rangle_\varphi\big]^{1/2}\theta \circ \xi(t)\,, \tag{19}$$

from which

$$\lambda \equiv \langle\frac{\mathrm{d}}{\mathrm{d}t}\ln\theta\rangle = -\mu - \varepsilon^2\langle[\mathcal{B}'(\varphi)]^2\rangle_\varphi = -\mu - \frac{\varepsilon^2}{2}\,. \tag{20}$$

The state of full synchronization is attractive at $\lambda < 0$, i.e., under the influence of both attractive and small repulsive coupling.

Consider the dynamics of $W(\theta)$ near the state of complete synchronization. From equations (16) and (17) it follows

$$\frac{\partial}{\partial t}W - \mu\frac{\partial}{\partial\theta}(\theta W) - \frac{\varepsilon^2}{2}\frac{\partial^2}{\partial\theta^2}\left(\theta^2 W\right) = 0\,. \tag{21}$$

If the distribution $W(\theta)$ is even, we can multiply equation (21) by θ^n and integrate from 0 to $+\infty$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle|\theta|^n\rangle + n\mu\langle|\theta|^n\rangle - \frac{n(n-1)}{2}\varepsilon^2\langle|\theta|^n\rangle = 0\,,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\langle|\theta|^n\rangle = n\left(-\mu - (1-n)\frac{\varepsilon^2}{2}\right) = n\left(\lambda + n\frac{\varepsilon^2}{2}\right). \tag{22}$$

Equation (22) describes the process of localization (or delocalization) of the distribution over time. It is important to note that the convergence of the integral $\langle|\theta|^n\rangle$ requires that $W(\theta)$ decrease for large θ no more slowly than $1/\theta^{1+n+\epsilon}$, where $\epsilon > 0$. For $n \to +0$, this integral converges for any distribution that can be normalized (that is, it is not a δ-function). In this case, the condition for the «collapse» of the distribution is $\lambda < 0$. The different rates and damping conditions $\langle|\theta|^n\rangle$ for $n > 0$ reflect the localization properties of the distribution of $W(\theta)$ over θ. The resulting distribution always has «heavy» power tails, which is reflected in the convergence properties of the integral $\int W(\theta)|\theta|^n\mathrm{d}\theta$ and the conditions for its decay with time.

2.2. The case of internal noise. For $\sigma \neq 0$, equations (16) and (17) yield

$$\frac{\partial}{\partial t}W - \mu\frac{\partial}{\partial\theta}(\theta W) - \frac{\partial^2}{\partial\theta^2}\left(\left(\frac{\varepsilon^2\theta^2}{2} + \sigma^2\right)W\right) = 0\,. \tag{23}$$

We restrict our consideration to the situation $\sigma \ll \varepsilon$, since otherwise the ensemble state will always be far from synchronization. The stationary solution of equation (23) has the form

$$W_{\omega=0}(\theta) = \frac{\Gamma(1+m)}{\sqrt{2\pi}\,\Gamma\left(\frac{1}{2}+m\right)}\frac{\varepsilon}{\sigma}\left(1+\frac{\varepsilon^2\theta^2}{2\sigma^2}\right)^{-(1+m)}, \tag{24}$$

where

$$m \equiv \frac{\mu}{\varepsilon^2}\,.$$

Distribution (24) is localized for an arbitrary small amplitude of internal noise σ; it can be normalized only if $m > -1/2$, which corresponds to $\lambda < 0$.

For $m = 0$, equation (24) transforms into the Lorentz distribution, which was obtained for the case of a system without coupling in [15].

3. Low frequency detuning oscillator ensemble

3.1. Analytic theory. Find the probability density flux for stationary distribution. To do this, we integrate equation (16)

$$q = \big(\omega + \mu\hbar(-\theta)\big)W_\omega - \frac{\partial}{\partial\theta}\Big(\big(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2\big)W_\omega\Big)\,, \tag{25}$$

where the integration constant q is a probability density flux. Then the mismatch of the average frequency is $\langle\dot{\theta}\rangle = 2\pi q$. The formal solution of equation (25) has the form

$$W_\omega(\theta) = \frac{q}{2\varepsilon^2[f(0)-f(\theta)]+\sigma^2}\,\frac{\displaystyle\int\limits_{\theta}^{\theta+2\pi} d\psi \exp\left(-\int\limits_{\theta}^{\psi} d\vartheta \frac{\omega+\mu\hbar(-\vartheta)}{2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2}\right)}{\exp\left(\displaystyle\int\limits_{0}^{2\pi} d\vartheta \frac{\omega+\mu\hbar(-\vartheta)}{2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2}\right)-1}\,. \tag{26}$$

The probability flow q can be found from the normalization condition $\int_0^{2\pi} W_\omega(\theta)\,d\theta = 1$. Equation (26) can be presented in a more informative form if we notice that for $\sigma \ll \varepsilon$ the main contribution to the integral over ϑ is made by the interval of small ϑ at which the denominator is small. One can use a substitution for this interval

$$\mu\hbar(-\vartheta) = m(2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2)' + \mu\hbar_{\mathrm{res}}(-\vartheta)\,,$$

where

$$m = \lim_{\vartheta\to 0}\frac{\mu\hbar(-\vartheta)}{(2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2)'} = \frac{\mu}{\varepsilon^2}\,,$$

the expansion of the function $\hbar_{\mathrm{res}}(-\vartheta)$ in a Taylor series begins with a term proportional to ϑ^2 (that is, it becomes not small only when the integrand is already substantially «suppressed» by the denominator), and, moreover, for systems in which $f(\vartheta)\sim\cos\vartheta$ and $h(\vartheta)\sim-\sin\vartheta$ (see, for example, [19,20]), the contribution $\hbar_{\mathrm{res}}(\vartheta))$ disappears. Then equation (26) can be written as

$$W_\omega(\theta)=\frac{q\int\limits_{\theta}^{\theta+2\pi}\mathrm{d}\psi\frac{(2\varepsilon^2[f(0)-f(\psi)]+\sigma^2)^{-m}}{(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2)^{1-m}}\exp\left(-\int\limits_{\theta}^{\psi}\mathrm{d}\vartheta\frac{\omega+\mu\hbar_{\mathrm{res}}(-\vartheta)}{2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2}\right)}{\exp\left(\int\limits_{0}^{2\pi}\mathrm{d}\vartheta\frac{\omega+\mu\hbar_{\mathrm{res}}(-\vartheta)}{2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2}\right)-1}\,,\qquad(27)$$

and

$$\frac{\langle\dot\theta\rangle}{2\pi}=\frac{\exp\left(\int\limits_{0}^{2\pi}\mathrm{d}\vartheta\frac{\omega+\mu\hbar_{\mathrm{res}}(-\vartheta)}{2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2}\right)-1}{\int\limits_{0}^{2\pi}\mathrm{d}\theta\int\limits_{\theta}^{\theta+2\pi}\mathrm{d}\psi\frac{(2\varepsilon^2[f(0)-f(\psi)]+\sigma^2)^{-m}}{(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2)^{1-m}}\exp\left(-\int\limits_{\theta}^{\psi}\mathrm{d}\vartheta\frac{\omega+\mu\hbar_{\mathrm{res}}(-\vartheta)}{2\varepsilon^2[f(0)-f(\vartheta)]+\sigma^2}\right)}\,.\qquad(28)$$

Later in this subsection we will derive asymptotic laws for the dependence of $\langle\dot\theta\rangle$ on ω described by equation (28).

For equation (28) with $\sigma\ll\varepsilon$, two characteristic regions of the dependence of $\langle\dot\theta\rangle$ on ω can be distinguished. First, we consider the region where ω is not small compared to $\mu\hbar_{\mathrm{res}}$, and the neighborhood $\theta=0$ makes the main contribution to the integral ϑ, since in this case the denominator of the integrand is small. Neglecting σ and using the same approximations as in Appendix B of [21], we can obtain

$$\langle\dot\theta\rangle\approx\frac{\sqrt{\pi}\Gamma\left(|m+\frac{1}{2}|+\frac{1}{2}\right)\varepsilon^2}{\Gamma\left(|m+\frac{1}{2}|\right)\Gamma(|2m+1|)}\left(\frac{\omega}{\varepsilon^2}\right)^{|2m+1|}.\qquad(29)$$

Here $\Gamma(\cdot)$ is the Gamma function.

We now consider the neighborhood $\omega=0$. In this neighborhood, the contribution of the term $\mu\hbar_{\mathrm{res}}$ to the integral over ϑ cannot be neglected. For $\omega\ll\sigma^2$, a small correction $W_\omega=W_0+W_1+\ldots$, $q=q_1+\ldots$ can be introduced into equation (25) in the form $W_n,q_n\sim\omega^n$;

$$0=\mu\hbar(-\theta)W_0-\frac{\partial}{\partial\theta}\Big(\big(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2\big)W_0\Big)\,,\qquad(30)$$

$$q_1=\omega W_0+\mu\hbar(-\theta)W_1-\frac{\partial}{\partial\theta}\Big(\big(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2\big)W_1\Big)\,.\qquad(31)$$

The unnormalized solution of equation (30) has the form

$$W_0(\theta)=\frac{\exp\left(\int_0^\theta\frac{\mu\hbar_{\mathrm{res}}(-\vartheta)\,\mathrm{d}\vartheta}{2\varepsilon^2[f(0)-f(\theta)]+\sigma^2}\right)}{\left(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2\right)^{1-m}}\,,$$

where the integral in the exponent is finite, since the integrand is finite over entire integration interval. The Hermitian-conjugate problem for (30) and its solution have the form

$$0 = \mu\hbar(-\theta)W_0^+ + \left(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2\right)\frac{\partial}{\partial\theta}W_0^+\,,$$

$$W_0^+(\theta) = \frac{\exp\left(-\displaystyle\int_0^\theta \frac{\mu\hbar_{\mathrm{res}}(-\vartheta)\,\mathrm{d}\vartheta}{2\varepsilon^2[f(0)-f(\theta)]+\sigma^2}\right)}{\left(2\varepsilon^2[f(0)-f(\theta)]+\sigma^2\right)^m}\,.$$

Multiplying equation (31) by W_0^+ and integrating over θ, we obtain

$$\langle\dot\theta\rangle \approx 2\pi q_1 = \frac{2\pi\langle W_0^+W_0\rangle_\theta}{\langle W_0^+\rangle_\theta\langle W_0\rangle_\theta}\omega\,. \tag{32}$$

Thus, for $\omega\to 0$, the power law of dependence (29) is replaced by a linear dependence.

For $\sigma \ll \varepsilon$, the expression for the proportionality coefficient of this dependence can be simplified, which allows one to see some of its properties in explicit form. So, for $\sigma\ll\varepsilon$

$$\langle W_0^+W_0\rangle_\theta \approx \frac{\pi\sqrt{2}}{\sigma\varepsilon}$$

$$\langle W_0^+\rangle_\theta \approx \begin{cases} \dfrac{1}{(2\varepsilon^2)^m}\left\langle \dfrac{e^{\frac{m}{2}\int_0^\theta \frac{-\hbar_{\mathrm{res}}(-\vartheta)\,\mathrm{d}\vartheta}{f(0)-f(\vartheta)}}}{[f(0)-f(\theta)]^m}\right\rangle_\theta, & m<\dfrac{1}{2}\,,\\[2ex] \sqrt{\dfrac{\pi}{2}}\dfrac{1+e^{mI_{\mathrm{res}}}}{\varepsilon\sigma^{2m-1}}\dfrac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}\,, & m>\dfrac{1}{2}\,,\end{cases}$$

$$\langle W_0\rangle_\theta \approx \begin{cases} \sqrt{\dfrac{\pi}{2}}\dfrac{1+e^{-mI_{\mathrm{res}}}}{\varepsilon\sigma^{1-2m}}\dfrac{\Gamma(\frac{1}{2}-m)}{\Gamma(1-m)}\,, & m<\dfrac{1}{2}\,,\\[2ex] \dfrac{1}{(2\varepsilon^2)^{1-m}}\left\langle \dfrac{e^{-\frac{m}{2}\int_0^\theta \frac{-\hbar_{\mathrm{res}}(-\vartheta)\,\mathrm{d}\vartheta}{f(0)-f(\vartheta)}}}{[f(0)-f(\theta)]^{1-m}}\right\rangle_\theta, & m>\dfrac{1}{2}\,,\end{cases}$$

where $I_{\mathrm{res}} \equiv \dfrac{1}{2}\displaystyle\int_0^{2\pi}\frac{-\hbar_{\mathrm{res}}(-\vartheta)\,\mathrm{d}\vartheta}{f(0)-f(\vartheta)}$. Thus,

$$\langle\dot\theta\rangle \approx \begin{cases} \dfrac{\dfrac{2^{m+2}\pi^{3/2}}{1+e^{-mI_{\mathrm{res}}}}\left(\dfrac{\varepsilon}{\sigma}\right)^{2m}\dfrac{\Gamma(1-m)}{\Gamma(\frac{1}{2}-m)}}{\left\langle \dfrac{e^{\frac{m}{2}\int_0^\theta \frac{-\hbar_{\mathrm{res}}(-\vartheta)\,\mathrm{d}\vartheta}{f(0)-f(\vartheta)}}}{[f(0)-f(\theta)]^m}\right\rangle_\theta}\omega\,, & m<\dfrac{1}{2}\,,\\[4ex] \dfrac{\dfrac{2^{3-m}\pi^{3/2}}{1+e^{mI_{\mathrm{res}}}}\left(\dfrac{\sigma}{\varepsilon}\right)^{2m-2}\dfrac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}}{\left\langle \dfrac{e^{-\frac{m}{2}\int_0^\theta \frac{-\hbar_{\mathrm{res}}(-\vartheta)\,\mathrm{d}\vartheta}{f(0)-f(\vartheta)}}}{[f(0)-f(\theta)]^{1-m}}\right\rangle_\theta}\omega\,, & m>\dfrac{1}{2}\,.\end{cases} \tag{33}$$

Equation (33) can be represented as

$$\langle\dot{\theta}\rangle \approx C(m)\left(\frac{\sigma}{\varepsilon}\right)^{|2m-1|-1}\omega\,,$$

where it is clearly indicated that the coefficient C depends only on the ratio $m=\mu/\varepsilon^2$. The exact form of this dependence is determined by the properties of a particular system. Note that for systems with small $\hbar_{\mathrm{res}}$ (that is, for oscillators close to harmonic), the integral $I_{\mathrm{res}}\to 0$ and expressions (33) can be simplified

$$\langle\dot{\theta}\rangle \approx \begin{cases} \dfrac{2^{m+1}\pi^{3/2}}{\langle[f(0)-f(\theta)]^{-m}\rangle_\theta}\left(\dfrac{\varepsilon}{\sigma}\right)^{2m}\dfrac{\Gamma(1-m)}{\Gamma(\frac{1}{2}-m)}\omega\,, & m<\dfrac{1}{2}\,,\\[2ex] \dfrac{2^{2-m}\pi^{3/2}}{\langle[f(0)-f(\theta)]^{m-1}\rangle_\theta}\left(\dfrac{\sigma}{\varepsilon}\right)^{2m-2}\dfrac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}\omega\,, & m>\dfrac{1}{2}\,. \end{cases}$$

In the next section, the obtained analytical expressions (28) and the asymptotic laws (29) and (32) are compared with the results of direct numerical simulation for several typical systems.

3.2. Comparison of theory and numerical simulation results. Fig. 1 illustrates the effect of global coupling on the ensemble of a finite number of nonlinear oscillators in the presence of common noise. One may notice that in this case, the frequency locking becomes impossible with an attractive coupling of finite strength (Fig. 1, *b*), whereas in the absence of noise with a sufficiently strong coupling, the frequencies become the same (Fig. 1, *a*). The common noise has a synchronizing effect on the system; therefore, the order parameter $R=\left\langle|N^{-1}\sum_j e^{i\varphi_j}|\right\rangle_t$ can be quite large even in the presence of a slight negative coupling. It should be noted that the synchronization of oscillator states (which can also occur with moderate repulsive coupling) does not necessarily entail the approximation of the average frequencies.

On the contrary, in the case of negative coupling, the frequencies are more scattered than in the absence of coupling (see Fig. 1, *b*). However, in the case where there is no common noise, the influence of the negative coupling on the average frequencies almost disappears, since in the term describing the global coupling, the mean field becomes almost zero (some small nonzero values of R appear due to the finit size of the system – in the thermodynamic limit $N\to\infty$, the average field is $R\to 0$).

As an example, consider a few typical oscillators.

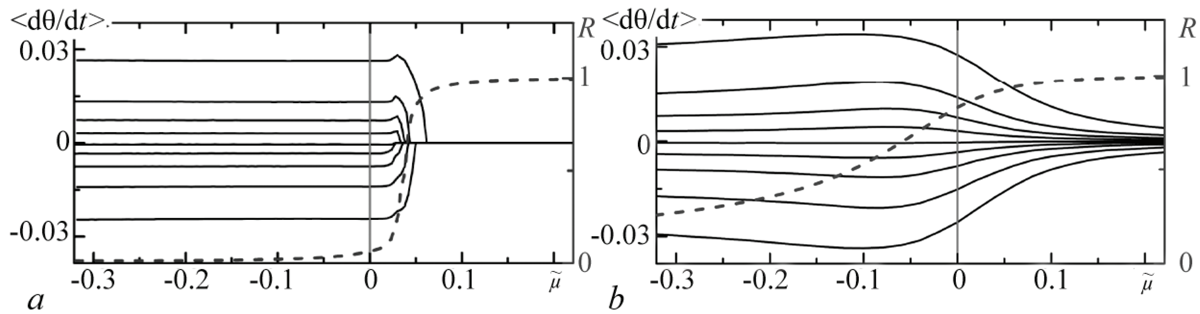

Fig. 1. Ensemble of 201 van der Pol oscillators with global coupling (34): *a* – without common noise, *b* – with common noise of strength $\widetilde{\varepsilon}=0.2$. Parameters: $a=0.5$, intrinsic noise strength $\widetilde{\sigma}=10^{-5}$. Linear oscillation frequencies $\widetilde{\omega}_j$ obey Gaussian distribution with mean value $\widetilde{\omega}_0=1$. The deviation of the average frequency of an individual oscillator from the ensemble-mean frequency is plotted for 9 arbitrary chosen oscillators. The order parameter $R=\langle|N^{-1}\sum_j e^{i\varphi_j}|\rangle_t$ is plotted as a dashed line

(a) Van der Pol Oscillator Ensemble

$$\dot{x}_j = y_j, \quad \dot{y}_j = a(1 - 4x_j^2)y_j - \widetilde{\omega}_j^2 x_j + \frac{\widetilde{\mu}}{N}\sum_{k=1}^{N}(y_k - y_j) + \widetilde{\varepsilon}\xi(t) + \widetilde{\sigma}\zeta_j(t), \tag{34}$$

where a is the bifurcation parameter (the greater the a, the more pronounced the anharmonicity of the oscillators). Hereinafter, the tilde sign over the coefficients means that they are not normalized, like the coefficients in the phase approximation equations; the parameter ω may also differ from the cyclic frequency of nonlinear oscillations.

(b) Rayleigh Oscillator Ensemble

$$\dot{x}_j = y_j, \quad \dot{y}_j = a\Big(1 - \frac{4}{3}y_j^2\Big)y_j - \widetilde{\omega}_j^2 x_j + \frac{\widetilde{\mu}}{N}\sum_{k=1}^{N}(y_k - y_j) + \widetilde{\varepsilon}\xi(t) + \widetilde{\sigma}\zeta_j(t). \tag{35}$$

The Rayleigh oscillator, when differentiating the equations with respect to time, passes into the van der Pol oscillator; therefore, for these systems, the susceptibility of the phase to global coupling should be identical. A comparison of the results of calculating this susceptibility for the van der Pol and Rayleigh oscillators can serve as a test of the accuracy of the algorithms used to numerically find the phase approximation characteristics. At the same time, the effect of intrinsic and common noise on the phase dynamics of systems (34) and (35) is different, since when integrating system (34) in time, it passes into system (35) up to a change of variables $(x_j, y_j) \to (\dot{x}_j, \dot{y}_j)$. However, the noises do not become δ-correlated, but become time integrals of those, that is, Wiener processes.

(c) Van der Pol–Duffing Oscillator Ensemble

$$\dot{x}_j = y_j, \quad \dot{y}_j = a(1 - 4x_j^2)y_j - \widetilde{\omega}_j^2 x_j - b\,x_j^3 + \frac{\widetilde{\mu}}{N}\sum_{k=1}^{N}(y_k - y_j) + \widetilde{\varepsilon}\xi(t) + \widetilde{\sigma}\zeta_j(t), \tag{36}$$

where the parameter b determines the nonisochronism of the oscillators: the larger is b, the more anisochronous are oscillations with different amplitudes.

(d) The FitzHugh–Nagumo System Ensemble [34.35]

$$\dot{v}_j = v_j - v_j^3/3 - w_j + I_{\mathrm{ext},j} + \frac{\widetilde{\mu}}{N}\sum_{k=1}^{N}(v_k - v_j) + \widetilde{\varepsilon}\xi(t) + \widetilde{\sigma}\zeta_j(t), \quad \dot{w}_j = 0.12(v_j + 0.7 - 0.8w_j), \tag{37}$$

where v_j is the voltage on the neuron membrane, w_j is the linear recovery variable, $I_{\mathrm{ext},j}$ is external stimulus.

Fig. 2 shows the phenomenon of attraction (or repulsion) of frequencies in such systems in the presence of an attractive or repulsive global coupling.

Fig. 3–6 illustrate the properties of the phase model calculated for systems (34)–(37). The calculations were performed using a modified version of the program for the Maple 6 package of analytical calculations, which was provided in the supplementary materials to work [36]. For the van der Pol and Rayleigh oscillators, the calculated susceptibility of the phase dynamics to global coupling coincides with an accuracy of about 10^{-4}: in Fig. 3, *d, e* and Fig. 4, *d, e* curves are visually indistinguishable.

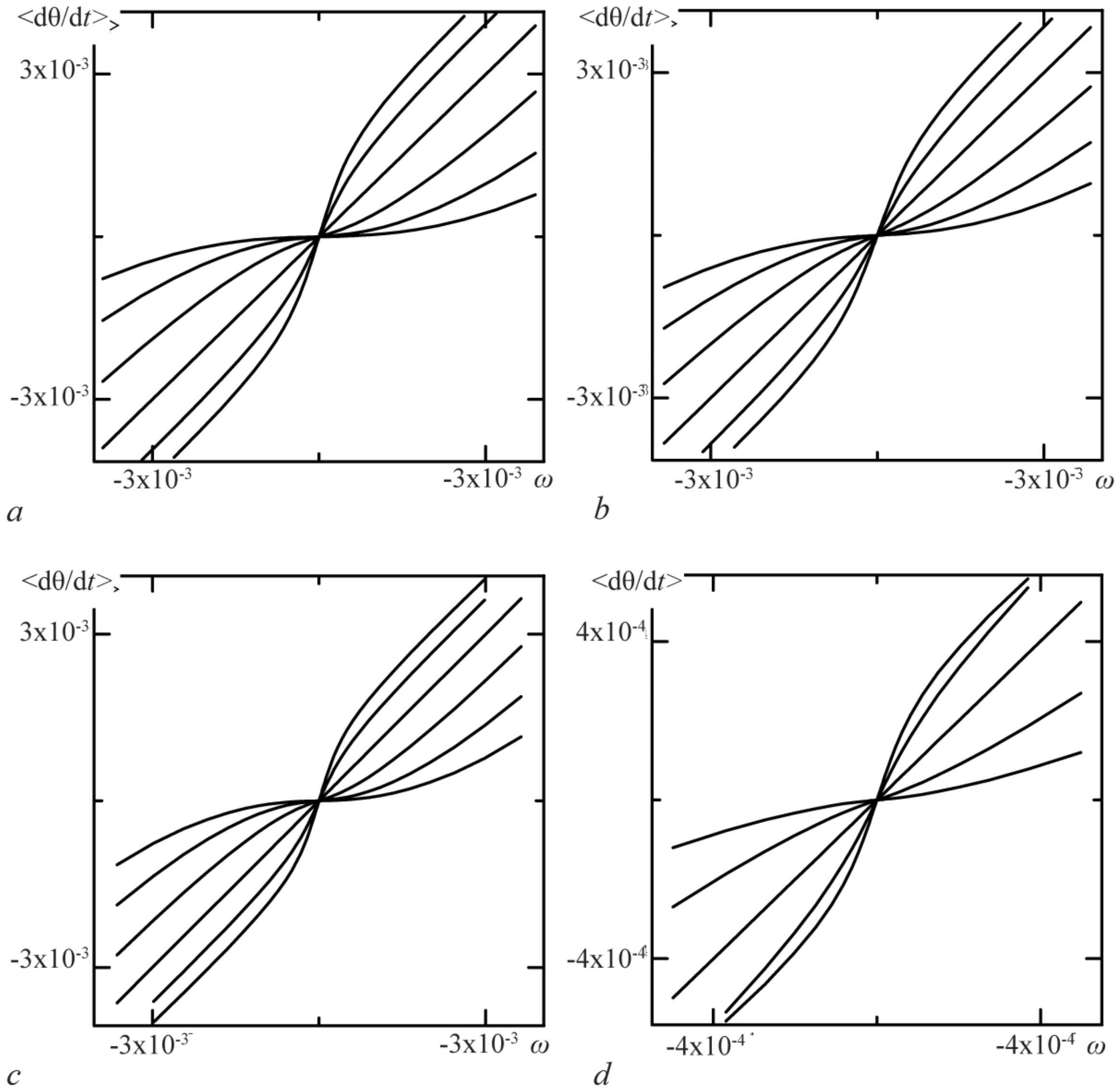

Fig. 2. The average frequency shift $\langle\dot{\theta}\rangle$ as a function of the natural frequency mismatch ω for a large ensemble of oscillators subject to common and intrinsic noises (results of direct numerical simulations): *a* – van der Pol oscillators (34) for $a = 1$, $\widetilde{\mu} = 0.021, 0.014, 0.007, 0, -0.007, -0.014$ (see from bottom to top on the graphic right hand side), the others parameters are as in Fig. 8, *a*; *b* – Rayleigh oscillators (35) for the same parameters $a = 1$, $\widetilde{\mu}$, the others parameters are as in Fig. 8, *b*; *c* – van der Pol–Duffing oscillators (36) for $a = b = 1$, $\widetilde{\mu} = 0.0135, 0.009, 0.0045, 0, -0.0045, -0.009$, the others parameters are as in Fig. 9; *d* – FitzHugh–Nagumo systems (37) for $\widetilde{\mu} = 0.006, 0.003, 0, -0.003, -0.006$; the others parameters are as in Fig. 10

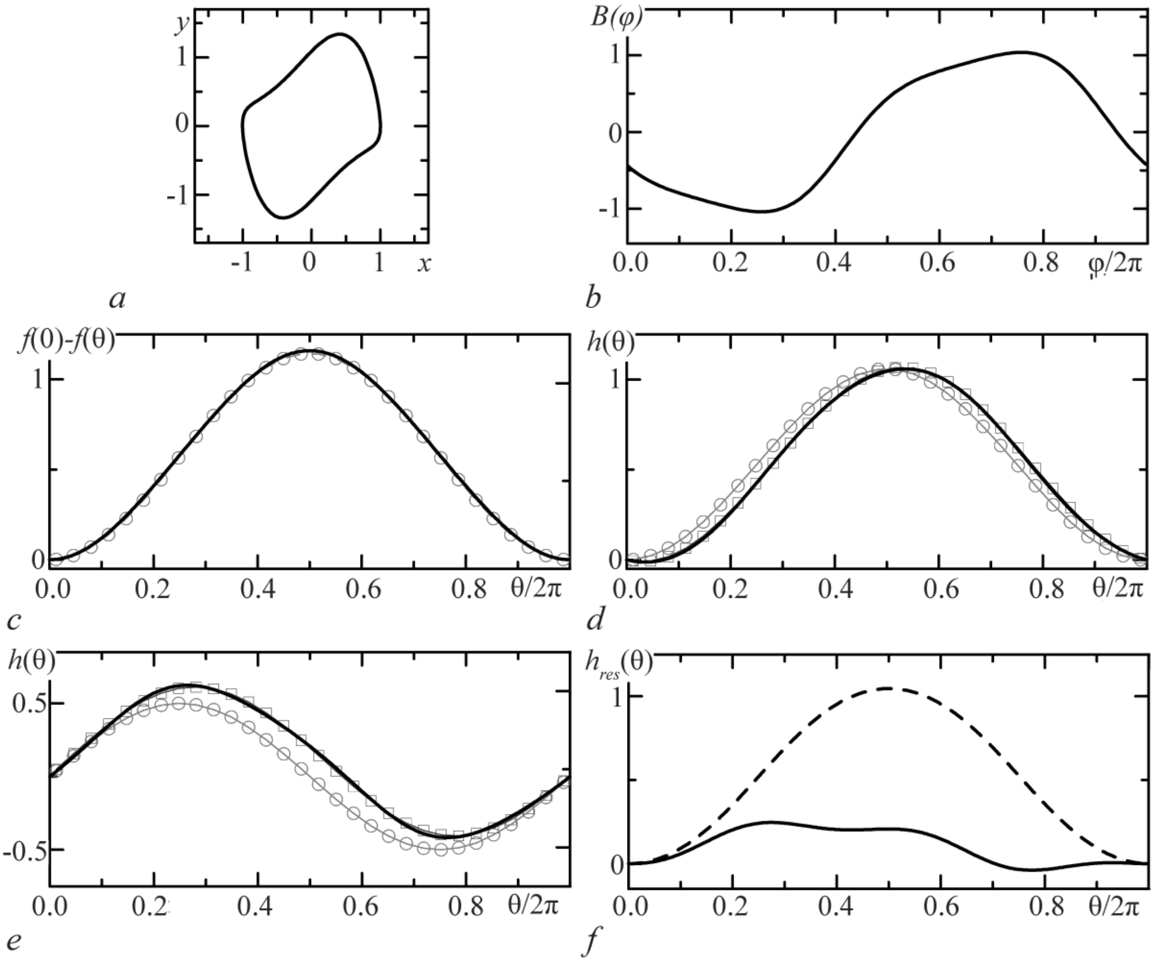

Fig. 3. Phase reduction properties of van der Pol oscillator (34) for $a = 1$: *a* – limit cycle orbit; *b* – phase resetting curve $\mathcal{B}(\varphi)$ (susceptibility of the phase to noise); *c* – function $f(\theta)$ determining the synchronizing action of noise (see eqs. (18), (25) or (28)) (the harmonic approximation of $f(\theta)$ is plotted with the circles); *d,e* – the susceptibility $h(\theta)$ of the phase to the coupling term for the cases of $(\widetilde{\mu}/N)\sum_k(x_k - x_j)$ and $(\widetilde{\mu}/N)\sum_k(y_k - y_j)$ terms in $\dot{y}$, respectively (the circles and the squares represent the harmonic and Ott–Antonsen approximations of $h(\theta)$, respectively); *f* – the residual part of $h(\theta)$ for the cases of the y- and x-coupling are plotted with solid and dashed lines, respectively

Fig. 7 presents the results of direct numerical simulations for an ensemble of oscillators (34). Direct numerical simulation in this and subsequent cases was carried out for an ensemble of 81 oscillators. It is known that the finiteness of the ensemble significantly affects the dynamics of not only individual elements, but also order parameters (see, for example, [37, 38]). Since the purpose of this direct numerical simulation is to verify the correctness of the analytical theory that is being built in the thermodynamic limit of large ensembles, the question is relevant: is the ensemble of 81 oscillators sufficiently large? Two points are important here. Firstly, the finiteness of an ensemble with a random set of frequencies with a given distribution leads to significant fluctuations in the realized distribution. For different ensembles with a similar realization of the frequency distribution, the macroscopic dynamics turns out to be substantially similar [37]. In this connection, by generating a regular set of frequencies with a given distribution, that is, by eliminating the random component of fluctuations in the frequency distribution, one can significantly reduce the effect of the discreteness of

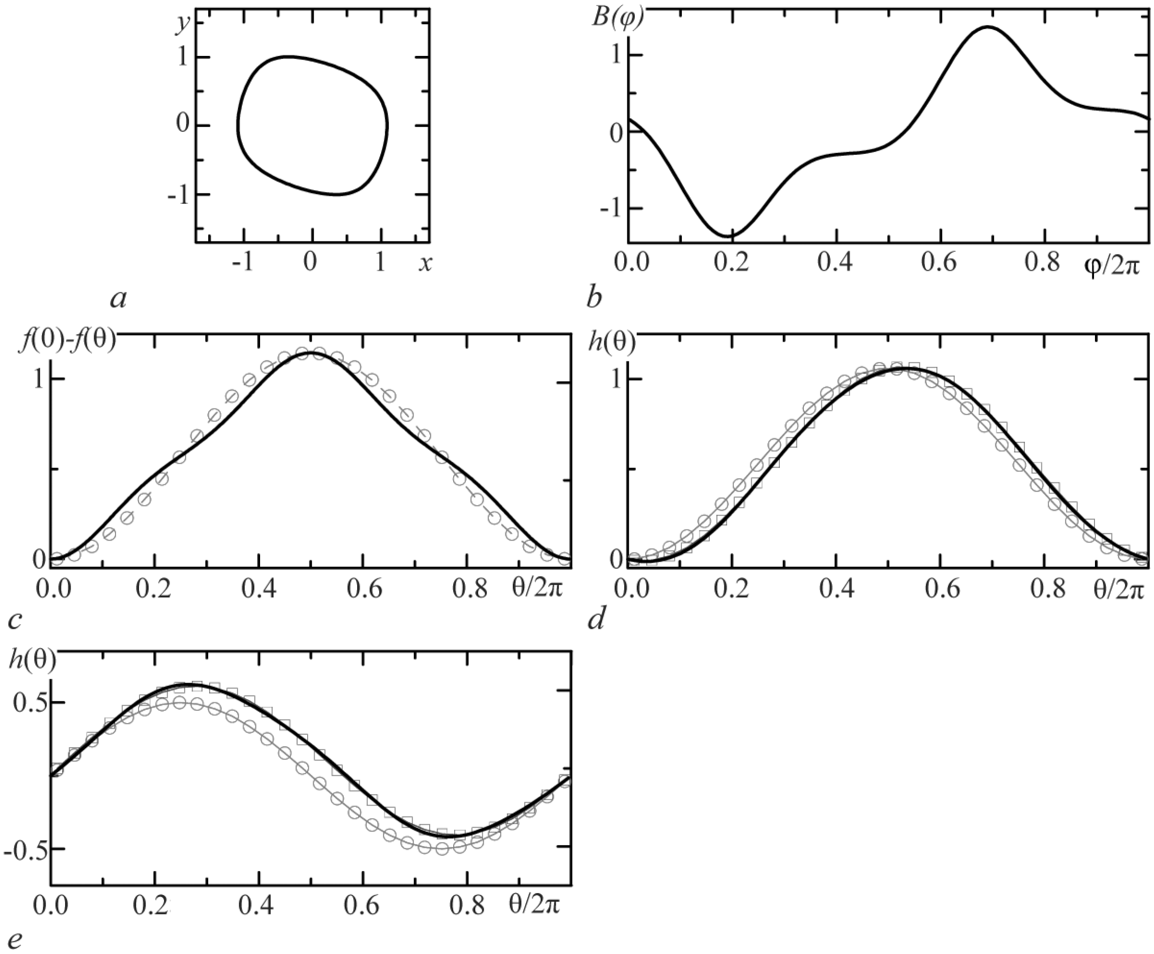

Fig. 4. Phase reduction properties of Rayleigh oscillator (36) with $a = 1$. For description see caption to Fig. 3

the set. Secondly, the theory is valid for situations with a high degree of synchronicity of the ensemble, when the imperfectness of synchrony in the numerical account manifests itself as a small correction to the dynamics of the system. Correction of such an amendment due to the finiteness of the ensemble is insignificant for the ensemble of 81 oscillators. A detailed study of the influence of the ensemble finiteness is beyond the scope of this work.

It can be seen that the results of direct numerical simulation are in good agreement with the constructed analytical theory (dashed lines). In addition, one can observe the agreement of the results with the power law (29) for large ω and the linear law (32) for small ω.

In the graphs presented, one can notice a regular shift between the analytical and numerical results as $\omega \to 0$. In order to understand the nature of this shift, we turn to the results obtained for the Kuramoto and Kuramoto–Sakaguchi ensembles [19, 20], for which it is possible to take into account the non-ideal synchronization rigorously. For cases of perfect synchronization as $\omega \to 0$, only power law (29) is observed (this law was analytically obtained in Appendix B of [21]). A transition to a linear dependence is observed at very small ω solely as a result of imperfectness synchronization. In this paper, effects associated with imperfect synchronization cannot be introduced into consideration in a simple way. On the other hand, the influence of intrinsic noise can be taken into account in this work. Such a consideration is correct from the point of view of the hierarchy of small parameters which can

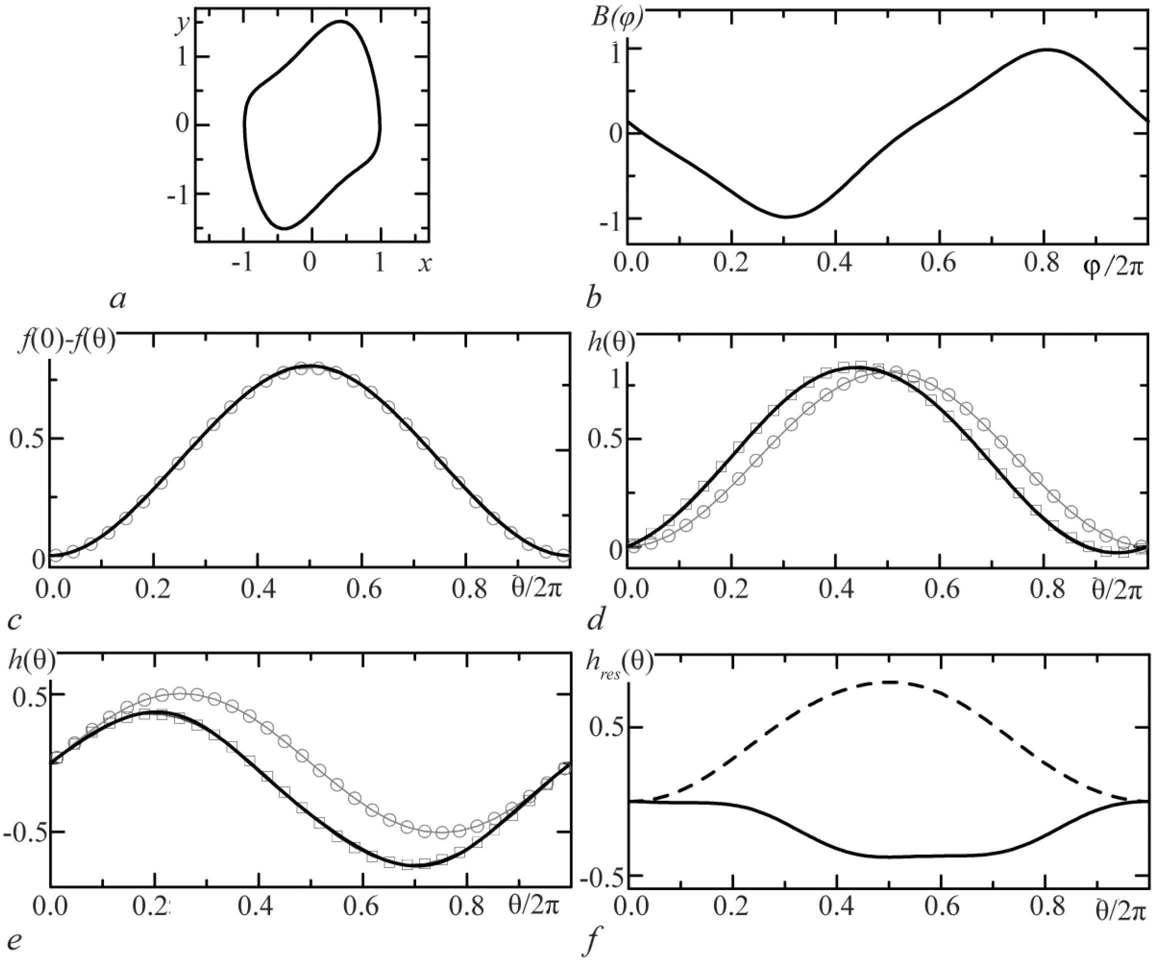

Fig. 5. Phase reduction properties of van der Pol–Duffing oscillator (36) with $a = b = 1$. For description see caption to Fig. 3

be seen from the procedure for deriving the basic equations, starting from equation (11). However, it cannot be extended beyond the leading order of the perturbation theory, since in the coupling term the deviation of the order parameter from 1 is considered negligibly small. Mathematically, this assumption of a high degree of synchronism of the ensemble is expressed by the approximation $\hbar(\theta) = h(\theta)$. In considering this work, the influence of intrinsic noise (in general form) ultimately manifests itself in the term with σ^2 in equation (25), while in [19, 20] the term of exactly the same form appears due to the imperfectness of synchronization. Thus, some fixed level of additive intrinsic noise can mimic the effect of imperfectness of synchronization. Indeed, with the noise intensity $\Delta\widetilde{\sigma}^2 = 0.28 \cdot 10^{-4}$, the analytical results (the thick line in Fig. 7) almost completely correspond to the results of direct numerical simulation.

Fig. 7–10 show that the results of the constructed analytical theory with the functions $f(\theta)$ and $h(\theta)$ presented in Fig. 3–6 are in good agreement with the results of direct numerical simulations for strongly nonlinear oscillators of van der Pol, Rayleigh and van der Pol–Duffing and FitzHugh–Nagumo systems (equations (34), (35), (36) and (37), respectively).

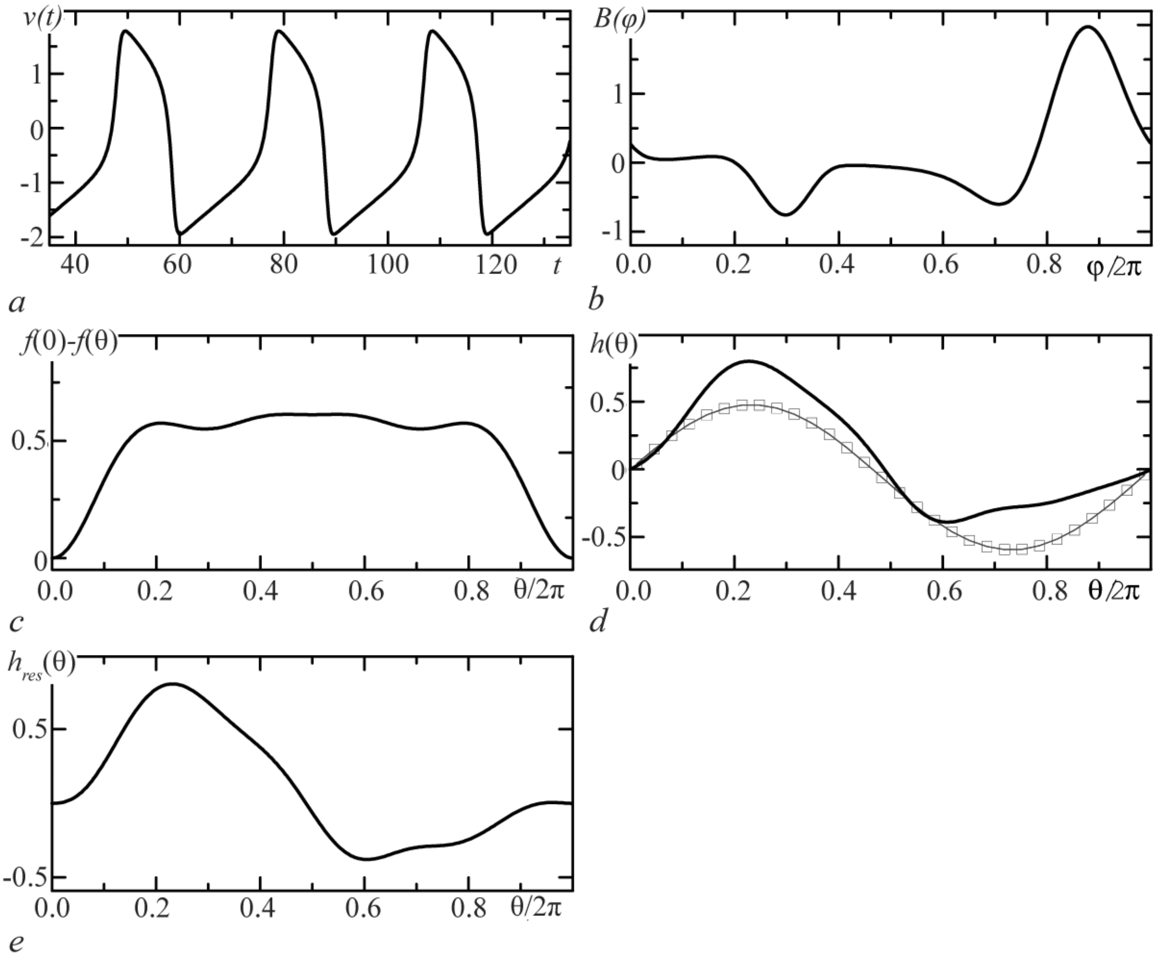

Fig. 6. Phase reduction properties of FitzHugh–Nagumo model (37): *a* – the regime of stable periodic oscillations; *b, c* – see caption to Fig. 3; *d* – susceptibility $h(\theta)$ of the phase to the coupling term (the squares show its Ott–Antonsen approximation); *e* – the residual part of $h(\theta)$

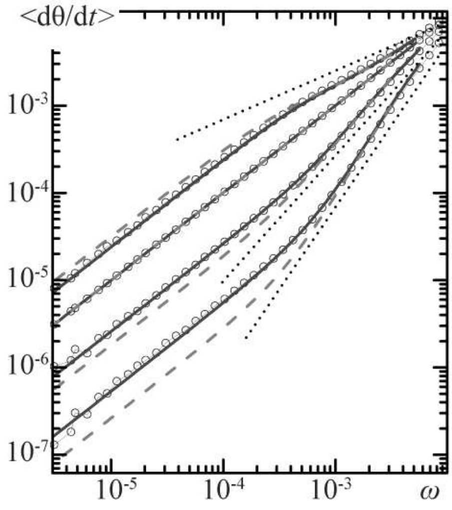

Fig. 7. Frequency entrainment (anti-entrainment) for a large ensemble of the globally coupled van der Pol oscillators (34) subject to common white Gaussian noise of strength $\widetilde{\varepsilon} = 0.1$ and intrinsic noise of strength $\widetilde{\sigma} = 0.005$. The nonlinearity parameter $a = 0.5$; the natural frequencies of linear oscillations $\widetilde{\omega}_j$ are distributed according to the Gaussian with the mean value $\widetilde{\omega}_0 = 1$ and standard deviation 10^{-4}; from bottom to top the coupling coefficient $\widetilde{\mu} = 0.014$, 0.007, 0, -0.007. Circles – the results of numerical simulation; dashed lines – the results of the analytical theory (28) with the phase reduction characteristics similar to those shown in Figs. 3–6; solid lines – the results of the theory (28) corrected by an additional effective intrinsic noise of intensity $\Delta\widetilde{\sigma}^2 = 2.8 \cdot 10^{-5}$ in order to resemble the effect of imperfect synchrony of the population; the dotted lines – the power law $\langle\dot{\theta}\rangle \propto \omega^{2m+1}$ for $m = 0.44$, 0.22, -0.22 (see from bottom to top), which is suggested by Eq. (29). The others phase reduction characteristics similar to those presented in Figs. 3–6: $\mu/\widetilde{\mu} = 0.4853...$, $\varepsilon^2/\widetilde{\varepsilon}^2 = 1.082...$, $\sigma^2/\widetilde{\sigma}^2 = 0.5254...$

Particularly, graphs *c–e* in Fig. 3–5 should be discussed. For oscillators close to harmonic, we can write

$$\ddot{x}_j + x_j + \mathcal{N}(x_j, \dot{x}_j) = \frac{\widetilde{\mu}}{N}\sum_{k=1}^{N}(\dot{x}_k - \dot{x}_j) + \widetilde{\varepsilon}\xi(t) + \widetilde{\sigma}\zeta_j(t),$$

where $\mathcal{N}(x_j, \dot{x}_j)$ means a small nonlinear term. This equation corresponds to Kuramoto-type equations with $h = \sin\theta$ (or $h = 1 - \cos\theta$ for the case of reactive coupling) and $f = 0.5\cos\theta$. For a wider class of oscillators that allow for the use of the Ott–Antonsen approach, we have $h = \sin(\theta + \beta) - \sin\beta$ and $f = 0.5\cos\theta$. In Fig. 3–5 (see fragments *c* and *d*) it is seen that f and h for essentially nonlinear oscillators are close to their harmonic approximations (marked with circles) and are practically indistinguishable from the Ott–Antonsen approximation (marked with squares), even if the form of $\mathcal{B}(\varphi)$ is far from sinusoidal. Since the effect of attraction (repulsion) of frequencies is completely determined by the form of the functions $f(\theta)$ and $h(\theta)$, it can be assumed that the results for the van der Pol, Rayleigh, and van der Pol–Duffing oscillators should be similar to the results obtained for the systems to which the Ott–Antonsen approach is applicable [19–21]. Thus, the results obtained earlier using the Ott–Antonsen approach turn out to be even more significant than could be expected initially, since, on the one hand, they allow one to strictly describe the role of imperfectness of synchronism, and on the other hand, are equivalent to the results for systems for which this approach is not applicable.

For strongly nonlinear FitzHugh–Nagumo systems, the functions $f(\theta)$ and $h(\theta)$ turn out to be quite far from the sinusoidal approximation (Fig. 6, *c*, *d*), therefore there is no reason to expect that

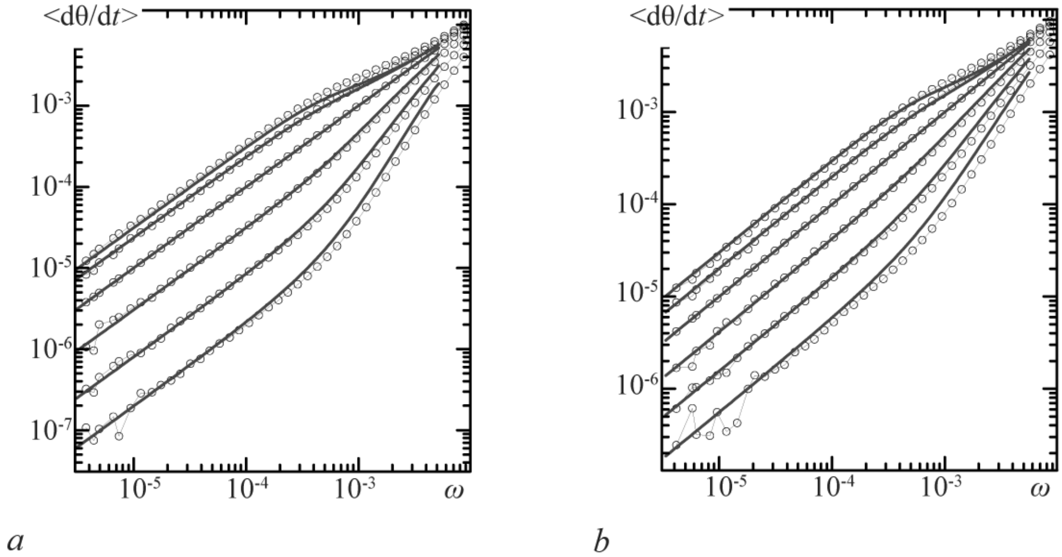

Fig. 8. Frequency entrainment (anti-entrainment) as in Fig. 7 for: *a* – van der Pol oscillators (35); *b* – Rayleigh oscillators (35); $a = 1$, $\widetilde{\mu} = 0.021$, 0.014, 0.007, 0, -0.007, -0.014. Compared to Fig. 7, a weaker additional effective intrinsic noise is needed to resemble the effect of the imperfectness of the population synchrony: $\Delta\widetilde{\sigma}^2 = 1.5 \cdot 10^{-5}$ (*a*) and 10^{-5} (*b*). The others phase reduction characteristics presented in Fig. 3 and 4: *a* – $\mu/\widetilde{\mu} = 0.4466...$, $\varepsilon^2/\widetilde{\varepsilon}^2 = 2.183...$, $\sigma^2/\widetilde{\sigma}^2 = 0.5560...$; *b* – $\mu/\widetilde{\mu} = 0.4466...$, $\varepsilon^2/\widetilde{\varepsilon}^2 = 2.183...$, $\sigma^2/\widetilde{\sigma}^2 = 0.5560...$

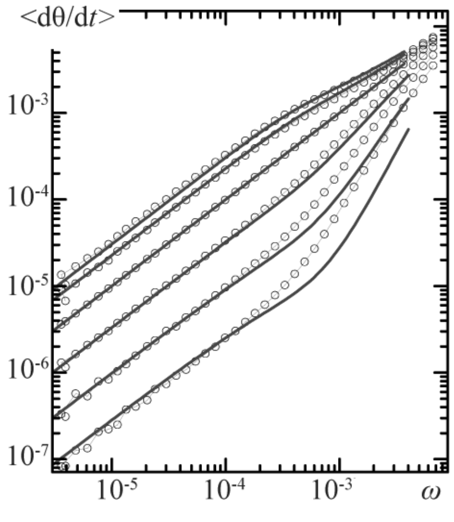

Fig. 9. Frequency entrainment (anti-entrainment) as in Fig. 7 for van der Pol–Duffing oscillators (36) for $a = b = 1$ and $\widetilde{\mu} = 0.0135$, 0.009, 0.0045, 0, -0.0045, -0.009 (see from bottom to top on the graphic left hand side). Here $\Delta\widetilde{\sigma}^2 = 0.14 \cdot 10^{-4}$, all other parameters are the same as in Fig. 7. From the phase reduction characteristics presented in Fig. 5, one finds $\mu/\widetilde{\mu} = 0.4658...$, $\varepsilon^2/\widetilde{\varepsilon}^2 = 0.8806...$, $\sigma^2/\widetilde{\sigma}^2 = 0.4050...$

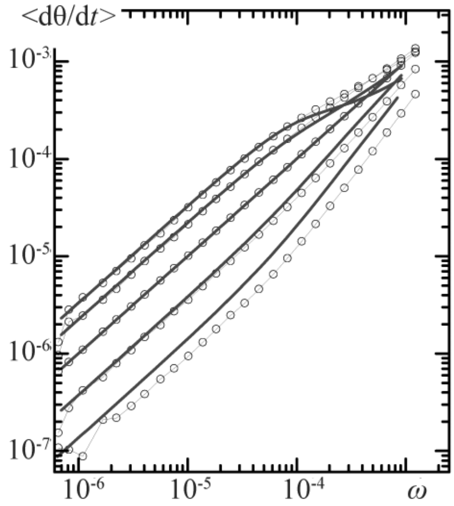

Fig. 10. Frequency entrainment/anti-entrainment as in Fig. 7 for FitzHugh–Nagumo systems (37) in the regime of periodic spiking. Here $\widetilde{\varepsilon} = 0.05$, $\widetilde{\sigma} = 0.001$, $I_{\mathrm{ext},j}$ is distributed according to the Gaussian distribution with the mean value $I_{\mathrm{ext},0} = 0.5$ and standard deviation $0.5 \cdot 10^{-6}$, $\widetilde{\mu} = 0.006$, 0.003, 0, -0.003, -0.006 (see from bottom to top on the graphic left hand side), and $\Delta\widetilde{\sigma}^2 = 10^{-6}$. From the phase reduction characteristics presented in Fig. 6, one finds $\mu/\widetilde{\mu} = 0.3101...$, $\varepsilon^2/\widetilde{\varepsilon}^2 = 4.730...$, $\sigma^2/\widetilde{\sigma}^2 = 0.4881...$

the results will be close to the results obtained for systems of the Ott–Antonsen type. Nevertheless, an analytic theory based on the phase approximation gives fairly accurate results (Fig. 10), and these results are qualitatively similar to the results for other nonlinear oscillators.

Conclusion

In this paper, we have constructed an analytical theory of the combined influence of common noise and global coupling on synchronization processes in ensembles of oscillators with a smooth limit cycle of the general form. The theory is mathematically rigorous for highly synchronous states; it takes into account the possible non-identity of the oscillators, as well as the influence of intrinsic noise. The theory is developed in the framework of the phase approximation constructed taking into account the amplitude degrees of freedom.

The synchronization condition for an ensemble of identical oscillators is obtained for the negative coupling μ. It is shown that the probability density of the phase deviations of the oscillators θ caused by the presence of intrinsic noise always has «heavy» tails $\propto 1/|\theta|^{2(1+m)}$, where $m = \mu/\varepsilon^2$ and ε is the amplitude of the common noise. If the amplitude of the common noise ε tends to zero, the exponent m tends to infinity, but the law is still power-law. Nevertheless, for $\varepsilon = 0$ phase mismatch θ with a low noise have a Gaussian distribution. Thus, the presence of total noise qualitatively changes the statistical properties of phase deviations in states of a high degree of synchronism.

For ensembles of weakly non-identical oscillators, the laws of asymptotic behavior were obtained for the average frequencies $\langle\dot{\theta}\rangle$ as functions of the natural frequencies ω. Two areas were identified for which asymptotic laws can be obtained: (1) for moderately small ω, we have $\langle\dot{\theta}\rangle \propto \omega^{|2m+1|}$ and (2) as $\omega \to 0$, we have $\langle\dot{\theta}\rangle \approx c(m)\,\omega$. The obtained law for the first region with an attractive coupling ($\mu > 0$) means a strong mutual attraction of the average frequencies of the oscillators: the slope of the dependence $\langle\dot{\theta}\rangle(\omega)$ tends to 0 for small ω. In the linear dependence region, the coefficient $c(m) < 1$, i.e., for different oscillators in the ensemble, the average frequencies differ weaker than their natural frequencies. For the repulsive coupling ($\mu < 0$) for moderately small ω, the obtained dependence corresponds to a strong discrepancy in the average frequencies: the slope of the dependence $\langle\dot{\theta}\rangle(\omega)$ tends to the vertical for small ω. In the linear dependence region, the coefficient $c(m) > 1$, i.e., the average frequencies of the oscillators are more dispersed than their natural ones.

Comparing equation (25) with a similar equation in [19, 20], we can see that the intrinsic noise intensity σ^2 affects the average frequencies $\langle\dot{\theta}\rangle$ in exactly the same way as the non-ideal synchronization caused by the frequency spread in Ott–Antonsen-type systems.

The constructed theory uses the phase response curve $\mathcal{B}(\varphi)$ and the phase susceptibility to the global coupling $h(\theta)$, which can be calculated for any oscillators with a stable limit cycle. As an example, the results of calculations for nonlinear oscillators van der Pol, Rayleigh, van der Pol–Duffing, as well as for FitzHugh–Nagumo systems in the regime of periodic spiking (see Fig. 3–6) are presented. The results of the analytical theory are in good agreement with the results of direct numerical modeling (see Fig. 7–10).

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