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Synchronization of oscillators with hyperbolic chaotic phases

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Abstract. *Topic and aim*. Synchronization in populations of coupled oscillators can be characterized with order parameters that describe collective order in ensembles. A dependence of the order parameter on the coupling constants is well-known for coupled periodic oscillators. The goal of the study is to extend this analysis to ensembles of oscillators with chaotic phases, moreover with phases possessing hyperbolic chaos. *Models and methods*. Two models are studied in the paper. One is an abstract discrete-time map, composed with a hyperbolic Bernoulli transformation and with Kuramoto dynamics. Another model is a system of coupled continuous-time chaotic oscillators, where each individual oscillator has a hyperbolic attractor of Smale–Williams type. *Results*. The discrete-time model is studied with the Ott–Antonsen ansatz, which is shown to be invariant under the application of the Bernoulli map. The analysis of the resulting map for the order parameter shows, that the asynchronouis state is always stable, but the synchronous one becomes stable above a certain coupling strength. Numerical analysis of the continuous-time model reveals a complex sequence of transitions from an asynchronous state to a completely synchronous hyperbolic chaos, with intermediate stages that include regimes with periodic in time mean field, as well as with weakly and strongly irregular mean field variations. *Discussion*. Results demonstrate that synchronization of systems with hyperbolic chaos of phases is possible, although a rather strong coupling is required. The approach can be applied to other systems of interacting units with hyperbolic chaotic dynamics.

Keywords: hyperbolic attractor, synchronization, collective dynamics.

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Синхронизация осцилляторов с гиперболическими хаотическими фазами

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Аннотация. *Тема и цель.* Синхронизация в популяциях связанных осцилляторов может быть охарактеризована параметрами порядка, описывающими коллективный порядок в ансамблях. Зависимость параметра порядка от коэффициентов связи хорошо известна для связанных периодических осцилляторов. Целью данного исследования является обобщение этого анализа на ансамбли осцилляторов с хаотическими фазами, а именно, с фазами, распределёнными на гиперболическом аттракторе. *Модели и методы*. В работе исследуются две модели. Первая – абстрактное отображение в дискретном времени, составленное из гиперболического преобразования Бернулли и динамики Курамото. Вторая – это система связанных хаотических осцилляторов в непрерывном времени, где каждый отдельный осциллятор имеет гиперболический аттрактор типа Смейла–Вильямса. *Результаты*. Модель в дискретном времени изучается с помощью подхода Отта–Антонсена, который, как показано, инвариантен при применении отображения Бернулли. Анализ полученного отображения по параметрам порядка показывает, что асинхронное состояние всегда устойчиво, а синхронное состояние становится устойчивым выше определенной силы связи. Численный анализ модели в непрерывном времени показывает сложную последовательность переходов из асинхронного состояния в полностью синхронный гиперболический хаос с промежуточными стадиями, которые включают режимы с периодическим во времени средним полем, а также со слабо и сильно нерегулярными вариациями среднего поля. *Обсуждение*. Результаты показывают, что синхронизация систем с гиперболическим фазовым хаосом возможна, хотя требуется довольно сильная связь. Данный подход может быть применен и к другим системам взаимодействующих звеньев с гиперболической хаотической динамикой.

Ключевые слова: гиперболический аттрактор, синхронизация, коллективная динамика.

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Introduction

Synchronization of chaotic oscillators has many aspects [1], one generally distinguishes complete, generalized, and phase synchronization. The latter property is related to chaotic oscillators with well-defined phases. Many chaotic oscillators, like the Rössler system, possess chaotic amplitudes, while the phase in such systems is not chaotic and corresponds to a zero Lyapunov exponent. The dynamics of the phase is of diffusion type, and correspondingly the phase synchronization phenomena for such oscillators are close to those for periodic oscillators with a certain level of noise in the phase dynamics.

In a seminal paper [2] S.P. Kuznetsov constructed a physical model of an oscillator with a *chaotic phase*. In this construction, the process has amplitude modulation, and at each period of modulation the phase experience a doubling map. The overall attractor is hyperbolic and belongs to a Smale–Williams solenoid class. In a series of subsequent publications, summarized in the book [3], S.P. Kuznetsov and co-authors provided many examples of systems with hyperbolic phase chaos.

In this paper we study synchronization properties of the oscillators with chaotic phases. First, we construct a rather abstract model, where phase chaos and synchronizing interactions are separated in time (Section 1). Namely, the process consists of two epochs: in one epoch phase oscillators interact according to the Kuramoto global coupling scheme, and in another epoch the phases undergo a chaotic Bernoulli map. This model demonstrates, for certain values of parameters, a bistability between a desynchronized and a synchronized states. In Section 2 we consider coupled autonomous oscillators with chaotic phases, constructed by S.P. Kuznetsov and the author in [4]. This system demonstrates a rather reach behavior with asynchronous, completely synchronous, and complex partially synchronous states.

1. Kuramoto–Bernoulli model

In this section we construct a model of interacting phase oscillators, which combines features of the Kuramoto model [5] (global attractive coupling of the phases) with the hyperbolic chaotic dynamics of the phases described by a Bernoulli map.

1.1. Kuramoto ensemble and Ott–Antonsen ansatz evolution. Consider N phase oscillators φ_k interacting via Kuramoto mean-field coupling

$$\dot{\varphi}_k = \mu R \sin(\Theta - \varphi_k), \quad Z = Re^{i\Theta} = \frac{1}{N}\sum_k e^{i\varphi_k} \,. \tag{1}$$

Here Z is the complex mean field, and μ is the coupling constant. Quantity R is called Kuramoto order, it characterizes asynchronous ($R < 1$) and synchronous ($R = 1$) regimes. We assume that all the oscillators have the same frequency, and write equations in the reference frame where this frequency vanishes, so it does not enter in (1). For $\mu > 0$ the coupling is attractive, and in this situation all the oscillators eventually synchronize: $R \to 1$, and a state where $\varphi_1 = \varphi_2 = \ldots \varphi_N$ establishes.

Synchronization transition is monotonous (in fact, there exists a Lyapunov function that governs it), but it can be generally hardly expressed analytically. An analytic solution is, however, possible, if the Ott–Antonsen (OA) ansatz [6], which applies to the thermodynamic limit $N \to \infty$, is peformed. In the OA ansatz it is assumed that the distribution of the phases is a wrapped Cauchy distribution, and the complex circular moments

$$Z_k = \langle e^{ik\varphi} \rangle \tag{2}$$

can all be expressed via the complex mean field $Z_k = Z^k$. Then the equation for the order parameter reads [6]

$$\dot{R} = \frac{\mu}{2} R(1 - R^2) \,.$$

Evolution of the complex mean field during a time interval T is

$$R(T) = \frac{R(0)}{\sqrt{R^2(0) + (1 - R^2(0))\exp[-\mu T]}} \,. \tag{3}$$

One can see that the only parameter in this transformation is $\gamma = \exp(-\mu T)$. Evidently, $R \to 1$ as $T \to \infty$, and the rate of this convergence is larger for smaller γ.

1.2. Bernoulli map of phases. Consider a Bernoulli map acting on the phases

$$\varphi(n+1) = K\varphi(n)\,, \tag{4}$$

with an integer parameter K. For an ensemble of Bernoulli maps (4), it is easy to express the evolution of the probability density of phases through the complex circular moments (2):

$$Z_m(n+1) = Z_{Km}(n)\,.$$

One can see that the OA ansatz is invariant under Bernoilli maps. Indeed, if $Z_m(n) = Z^m$, then $Z_m(n+1) = Z^{Km} = (Z^K)^m$. Thus, the evolution of the complex mean field under the Bernoulli map is

$$Z(n+1) = Z^K(n)\,. \tag{5}$$

1.3. Kuramoto ensemble and Bernoilli map. We construct a Kuramoto-Bernoulli (KB) model as a sequence of applications of the Kuramoto dynamics (3) and of the Bernoulli dynamics (5). Application of the expressions (3), (5) leads to the following map for the order parameter

$$R(n+1) = \frac{R^K(n)}{(R^2(n) + (1 - R^2(n))\gamma)^{K/2}}.$$

This map has always a stable asynchronous fixed point $R_{\text{as}} = 0$, and synchronous fixed point $R_{\text{s}} = 1$. The fixed point $R = 1$ is stable for

$$\gamma < \frac{1}{K}\,, \tag{6}$$

in this case also an unstable partially synchronous fixed point with $0 < R_{\text{ps}} < 1$ exists, so there is a bistability asynchrony–synchrony.

The threshold for synchrony stability (6) is valid not only in the OA approximation, but generally. Indeed, close to the synchronous state the deviations of the phases satisfy, in the Kuramoto stage, the linear equation

$$\frac{d}{dt}\delta\varphi = -\mu\delta\varphi$$

so that combined map for the linear deviations is

$$\delta\varphi(n+1) = K\gamma\delta\varphi(n)$$

from which (6) follows.

We illustrate the dynamics of the KB model in Fig. 1. There we show the evolution of the oder parameter R for different values of parameter γ and different initial states. The fully synchronous state

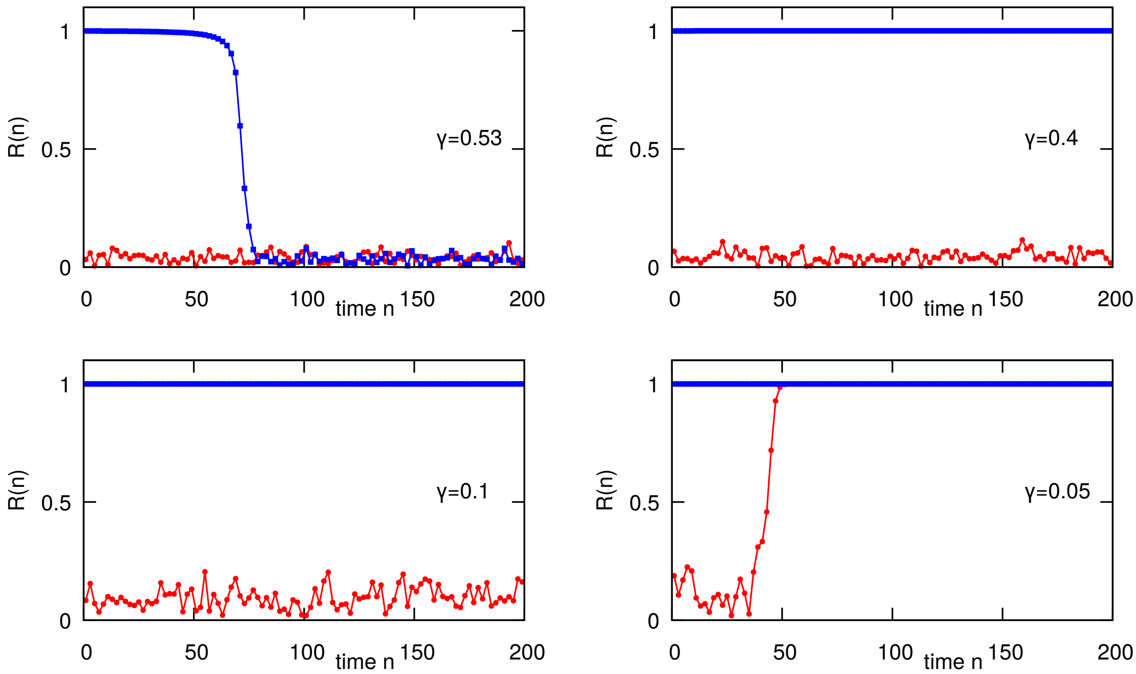

Fig. 1. Illustration of the dynamics in a Kuramoto–Bernoulli model with $K = 2$ and different γ, ensemble size $N = 1000$. Red curves: disordered initial state; blue curves: ordered initial state (distribution of phases in an interval $(0, 0.1)$)

is absorbing (exactly the same phases remain the same) for all system sizes N, while there are finite-size fluctuations around the disordered state. For small γ, one observes a finite-size induced transition to the synchronous state.

2. Globally coupled Kuznetsov–Pikovsky oscillators

Here we study globally coupled chaotic phase oscillators introduced by S.P. Kuznetsov and the author in Ref. [4].

2.1. One Kuznetsov–Pikovsky oscillator. An individual Kuznetsov–Pikovsky (KP) oscillator consists of three modes described by their complex amplitudes u, v, w. The equation of one unit are

$$\begin{aligned} \dot{u} &= -iu + (1 - |u|^2 - \frac{1}{2}|v|^2 - 2|w|^2)u + \varepsilon \mathrm{Im}(v^2) \ , \\ \dot{v} &= -iv + (1 - |v|^2 - \frac{1}{2}|w|^2 - 2|u|^2)v + \varepsilon \mathrm{Im}(w^2) \ , \\ \dot{w} &= -iw + (1 - |w|^2 - \frac{1}{2}|u|^2 - 2|v|^2)u + \varepsilon \mathrm{Im}(u^2) \ . \end{aligned} \tag{7}$$

Below we fix the internal coupling parameter $\varepsilon = 0.075$. For $\varepsilon = 0$, system (7) has a stable homoclinic cycle, where the modes are excited consequentially $w \to v \to u \to w \to \ldots$, with increasing periods of the cycle. The effect of coupling $\varepsilon > 0$ is twofold: first, the cycle period is limited from above (see Fig. 2, *b*), and second, at each stage where a mode amplitude passes close to zero, its phase attains the doubled value of the exciting mode. The latter property is described in Ref. [4] in details; here we illustrate it with figure 3. Thus, the KP oscillator (7) has a chaotic phase obeying a Bernoulli map.

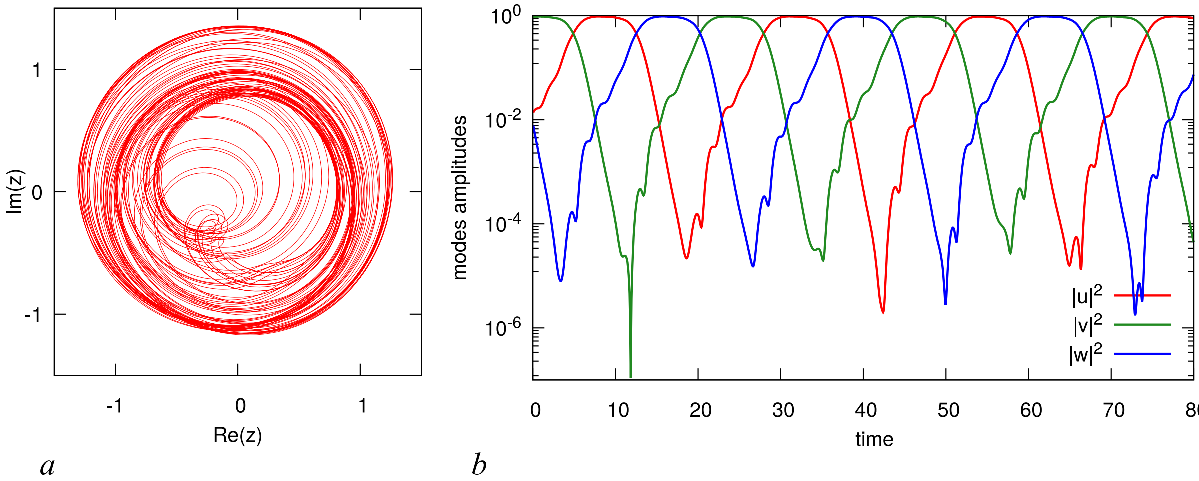

Fig. 2. Illustration of the dynamics in a KP model. Panel *a*: phase portrait of the observable $z = u + v + w$ (complex amplitude of oscillations). Panel *b*: amplitudes of the modes

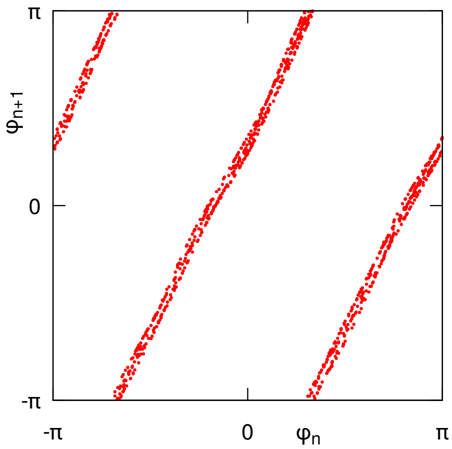

Fig. 3. Illustration of the phase transformation in model (7). The consecutive phases $\arg(u) \to \arg(v) \to \arg(w)$ are depicted. The transformation $\arg(u) \to \arg(u)$ would be $\varphi \to 2^3\varphi$

2.2. Globally coupled KP oscillators. Here we introduce a global coupling of N oscillators (numbered by index k), such that complete synchrony is possible:

$$\begin{aligned}
\dot{u}_k &= -iu_k + (1 - |u_k|^2 - \frac{1}{2}|v_k|^2 - 2|w_k|^2)u + \varepsilon \mathrm{Im}(v_k^2) + \mu |u_k|^2 (U - u_k),\\
\dot{v}_k &= -iv_k + (1 - |v_k|^2 - \frac{1}{2}|w_k|^2 - 2|u_k|^2)v + \varepsilon \mathrm{Im}(w_k^2) + \mu |v_k|^2 (V - v_k),\\
\dot{w}_k &= -iw_k + (1 - |w_k|^2 - \frac{1}{2}|u_k|^2 - 2|v_k|^2)u + \varepsilon \mathrm{Im}(u_k^2) + \mu |w_k|^2 (W - w_k),\\
U &= \frac{1}{N}\sum_k u_k, \quad V = \frac{1}{N}\sum_k v_k, \quad W = \frac{1}{N}\sum_k w_k\,.
\end{aligned} \tag{8}$$

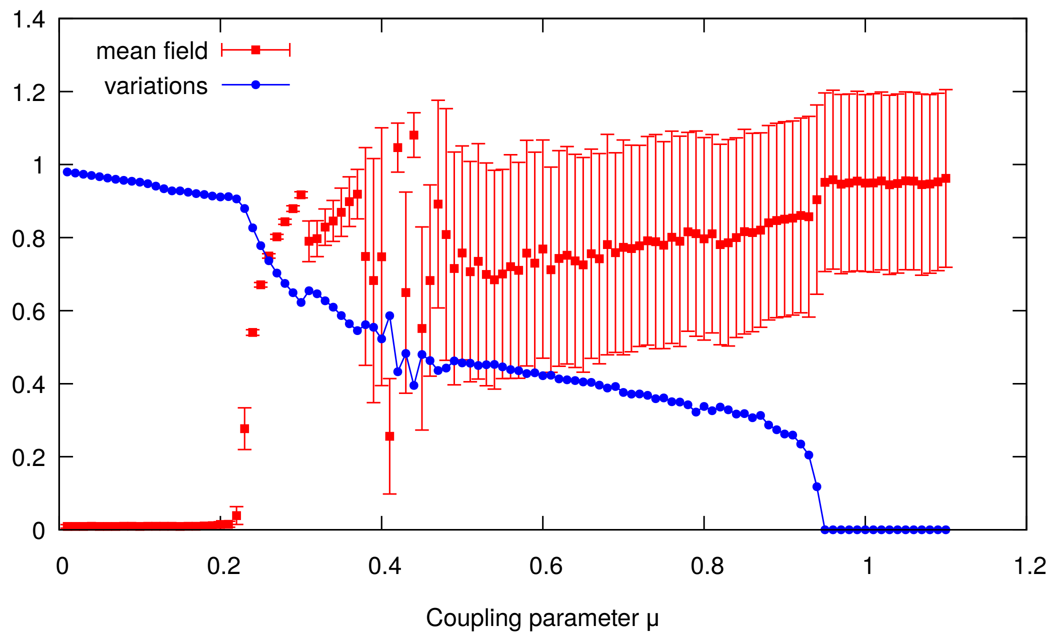

Fig. 4. Bifurcation diagram of model (8) for $N = 10^4$

The coupling term is proportional to the parameter μ, it contains three complex mean fields U, V, W, corresponding to three modes of each oscillator.

Figure 4 intends to characterize the dynamical regimes in the system, in dependence on the coupling parameter μ. Here two quantities have been calculated. First, I present the dynamics of the global complex mean field

$$Z(t) = U(t) + V(t) + W(t)\ .$$

I calculated the time average $\langle|Z|\rangle_t$ and its fluctuations $\langle(|Z| - \langle|Z|\rangle_t)^2\rangle_t$, these quantities are shown in Fig. 4 with red (fluctuations as error bars). Additionally, for each moment of time, I calculated the spread in the ensemble

$$D(t) = \frac{1}{N}\sum_k \left[|u_k(t) - U(t)|^2 + |v_k(t) - V(t)|^2 + |w_k(t) - W(t)|^2\right]$$

and then average this quantity over time. This quantity is shown with blue.

Below I describe different states on the bifurcation diagram.

1. *Complete synchronization*. This regime is observed for $\mu > 0.95$. Here $u_k = u_j$, $v_k = v_j$, $w_k = w_j$ for all k, j. In this state $D = 0$.
2. *Asynchronous state*. This regime is observed for $\mu \lesssim 0.22$. Here the mean field vanishes, and one has effectively a set of non-interacting oscillators.
3. *Periodic mean field*. This regime is observed in the range $0.22 \lesssim \mu \lesssim 0.31$. Here the complex mean field $Z(t)$ is nearly periodic. We illustrate this in Fig. 5, *a*, *b*. There are visible fluctuations for $\mu = 0.23$, but for $\mu = 0.3$ periodicity is nearly perfect. The transition to a periodic mean field at $\mu \approx 0.22$ is very much similar to one described in Ref. [7].
4. *Weakly irregular mean field*. This state is illustrated in Fig. 5, *c*. At $\mu = 0.35$ the mean field is close to periodic one, but has a seemingly nearly quasi-periodic modulation.

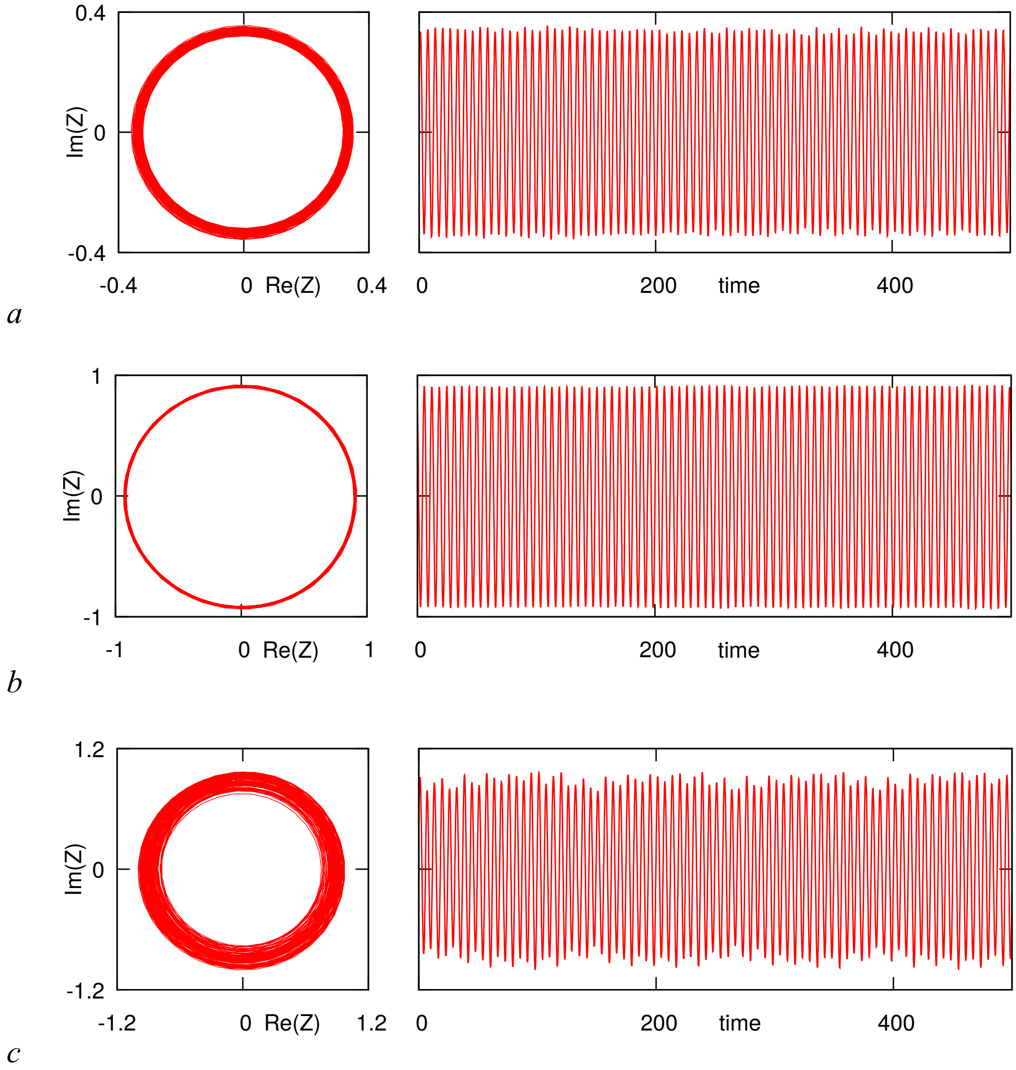

Fig. 5. The dynamics of the global complex order parameter for $\mu = 0.23$ (panel *a*), $\mu = 0.3$ (panel *b*), and $\mu = 0.35$ (panel *c*)

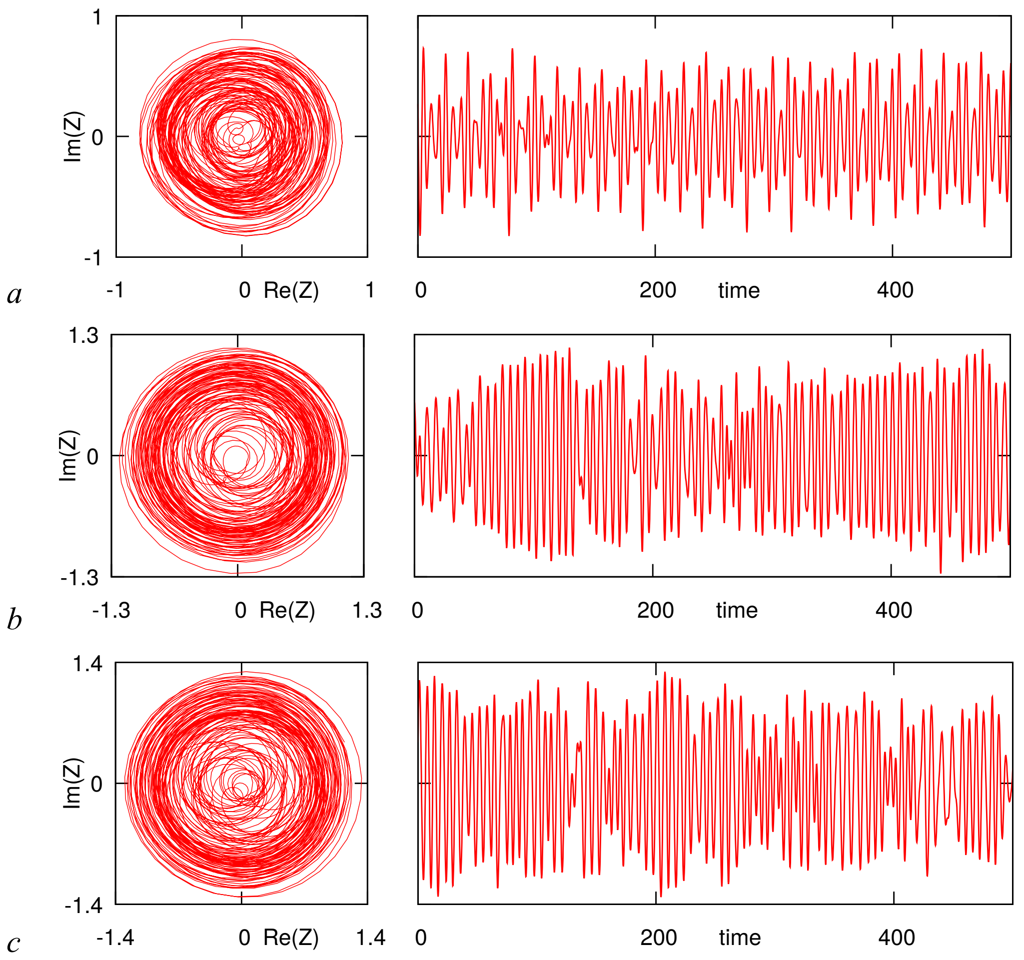

Fig. 6. The dynamics of the global complex order parameter for $\mu = 0.45$ (panel *a*), $\mu = 0.6$ (panel *b*), and $\mu = 0.8$ (panel *c*)

5. *Irregular mean field*. This state is observed for $0.4 \lesssim \mu \lesssim 0.95$, we illustrate it in Fig. 6, *a–c*. Fluctuations of the mean field are essential, eventually for large μ they become close to the fluctuations of the field $z(t)$ in one chaotic oscillator.

Discussion

In this paper I studied effects of coupling on oscillators with hyperbolic chaotic dynamics of the phases. In the simplest, rather artificial Kuramoto–Bernoulli model, an exact mapping for the order parameter has been derived in the Ott–Antonsen approximation in the thermodynamic limit. The dynamics here, beyond a certain level of coupling, is bistable: synchronous and asynchronous states coexist. In relatively small ensembles, for strong enough coupling, only synchronous states survives as it is a truly absorbing one. A more realistic model of coupled autonomous continuous-time oscillators with hyperbolic dynamics of the phases demonstrated much more rich dynamics. Together with a fully asynchronous state at small coupling strengths, and a completely synchronous at strong coupling strengths, it demonstrates different states with partial synchrony. Close to the asynchronous state, the mean field is nearly periodic; and with increase of coupling strength it becomes irregular through presumably a quasi-periodic state. Detailed exploration of partially synchronous states in this model will be a subject of a separate study.

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