Review of actual problems of nonlinear dynamics

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Mathematical theory of dynamical chaos and its applications: Review
Part 1. Pseudohyperbolic attractors

A. S. Gonchenko[1], *S. V. Gonchenko*[1], *A. O. Kazakov*[2,1,3], *A. D. Kozlov*[1]

[1] N.I. Lobachevsky State University of Nizhni Novgorod – National Research University
23 Prospekt Gagarina (Gagarin Avenue) BLDG 2, 2nd floor, 603950 Nizhny Novgorod, Russia
[2] National Research University Higher School of Economics
25/12 Bolshaya Pecherskaya Ulitsa, 603155 Nizhny Novgorod, Russia
[3]Udmurt State University, Universitetskaya 1, 426034 Izhevsk, Russia
E-mail: agonchenko@mail.ru, sergey.gonchenko@mail.ru,
kazakovdz@yandex.ru, kozzzloff@list.ru
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We consider important problems of modern theory of dynamical chaos and its applications. At present, it is customary to assume that in the finite-dimensional smooth dynamical systems three fundamentally different forms of chaos can be observed. They are: the dissipative chaos, whose mathematical image is a strange attractor; the conservative chaos, for which the whole phase space is a large «chaotic sea» with elliptic islands randomly disposed within it; and the mixed dynamics which is characterized by the principle inseparability, in the phase space, of attractors, repellers and orbits with conservative behavior.

In the first part of this series of our works, we present some elements of the theory of pseudohyperbolic attractors of multidimensional maps. Such attractors, the same as hyperbolic ones, are genuine strange attractors, however, they allow homoclinic tangencies. We also give a description of phenological scenarios of the appearance of pseudohyperbolic attractors of various types for one parameter families of three-dimensional diffeomorphisms, and, moreover, consider some examples of such attractors in three-dimensional orientable and nonorientable Henon maps.

In the second part, we will give a review of the theory of spiral attractors. Such type of strange attractors are very important and are often observed type in dynamical systems. The third part will be dedicated to mixed dynamics – a new type of chaos which is typical, in particulary, for (time) reversible systems i.e. systems which are invariant with respect to some changes of coordinates and time reversal. It is well known that such systems occur in many problems of mechanics, electrodynamics, and other areas of natural sciences.

Keywords: Strange attractor, pseudohyperbolicity, homoclinic tangency, discrete Lorenz attractor, three-dimensional generalized Hénon map.

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Introduction

At present, three relatively independent various forms of dynamical chaos of smooth finite-dimensional systems can be distinguished: "dissipative chaos", "conservative chaos" and "mixed dynamics". The first of them is characterized by existence of a *strange attractor* – nontrivial attractive closed invariant set, lying in the phase space of the system inside an absorbing domain, which attract all the orbits crossing the boundary of this domain. Unlike dissipative chaos, conservative chaos is "spread out" throughout the phase space – in this case, all points are nonwandering (according to the Poincaré's recurrence theorem). Speaking in terms of attractors $\mathcal{A}$ and repellers $\mathcal{R}$ (attractors at time reversal), which, by the well-known Conley theorem [1], exist in any system with a compact phase space, then $\mathcal{A} \cap \mathcal{R} = \emptyset$ in the case of dissipative chaos and $\mathcal{A} = \mathcal{R}$ for conservative chaos. For comparison, "mixed dynamics" is the new type of chaos that is characterized by the fact that stable dynamics elements (e.g. stable periodic orbits) coexist with completely unstable, and they are inseparable apart from each other.

The very phenomenon of mixed dynamics was discovered in the work by Gonchenko–Shilnikov–Turaev [2]. In this work, in particular, it was proved that in the case of two-dimensional diffeomorphisms, there are Newhouse domains in which diffeomorphisms are dense with a countable set of stable, completely unstable and saddle periodic orbits; moreover, the closures of the sets of orbits of different types have a nonempty intersection. Naturally, for any definition of an attractor, it must be, in any case, a closed invariant set that is stable and which must contain stable periodic orbits, if they exist. The same thing (with respect to completely unstable periodic orbits) should be performed for repellers. Thus, in [2] it was shown that the attractor can intersect with the repeller, which formally agrees with Conley's theorem [1].

A mathematical justification for this phenomenon is given, for example, in [3,4]. Then for "mixed dynamics" the following conditions are formally satisfied: $\mathcal{A} \cap \mathcal{R} \neq \emptyset$ and $\mathcal{A} \neq \mathcal{R}$. These conditions are additional to ones for "conservative chaos" ($\mathcal{A} = \mathcal{R}$) and "dissipative chaos" ($\mathcal{A} \cap \mathcal{R} = \emptyset$).

As for the strange attractors to which this article is devoted, their generally accepted definition, which would be suitable for all occasions, still does not exist. The exception is the so-called *genuine strange attractors*, the definition of which includes two main points: 1) the existence of an absorbing region in phase space (which includes all orbits crossing the boundary of this region) and 2) the instability of orbits on the attractor, which is postulated by the fact that each orbit on the attractor has a positive maximal Lyapunov exponent. It is also assumed that the properties "1" and "2" are satisfied for all close systems.

On the other hand, (see the discussion in [5–7]) we can, with proper reason, reckon to strange attractors the so-called *quasiattractors* – nontrivial attracting invariant sets that either themselves contain stable periodic orbits of very large periods (and with very narrow regions of attraction), or such orbits appear for arbitrarily small smooth perturbations. The last circumstance is connected with the fact that quasiattractors allow the existence of saddle periodic orbits, in which stable and unstable invariant manifolds intersect non-transversally. In turn, bifurcations of such homoclinic tangencies under certain conditions-criteria [8–12] lead to the birth of asymptotically stable periodic orbits, attracting invariant tori, small strange Hénon-like attractors [13–17] and even Lorenz-like attractors [18–22] etc.

We also note that in dissipative systems with special structures strange attractors of other types may occur that do not formally fit to this scheme. For example, non-smooth or discontinuous systems can have attractors with singularly hyperbolic behavior of orbits, in the sense that for some orbits their Lyapunov exponents are not defined (due to the non-smoothness of the system itself), although there is an absorbing region, and necessary runaway of the orbits on the attractor. Examples of such attractors are the Lozi attractor [23] and the Belykh attractor [24].

The completely different type of complex non-periodic behaviour of the orbits is demonstrated by the so-called strange non-chaotic attractors that arise in special models, having the structure of a direct product of a non-specific dynamical system and a quasi periodic system. They are characterized by the fact that for any orbit one of the Lyapunov exponents is equal to zero, and the rest are less than zero; the existence of a small number (measures zero) orbits with a positive exponent is also allowed. For more details on such attractors, see, for example, in [25].

We can say that most of the known strange attractors of smooth dynamical systems, including those found in applications, are, essentially, quasi attractors. Examples of such attractors are: numerous "torus-chaos" attractors appearing from destruction of a two-dimensional torus, [26]; attractors in Chua circuits [27]; the Hénon attractor [28,29]; attractors in periodically perturbed two-dimensional systems with homoclinic eights of saddles [30] and many others. A special class of quasi attractors are the so-called spiral attractors that contain saddle-foci. They often arise in applications, and examples of such attractors are well known. There are, for example, spiral three-dimensional flow attractors such as the Rössler attractor [31,32], attractors in Arneodo–Calle–Tresse models [33–35] etc. Note that in the work by L.P. Shilnikov [36] there was proposed a rather simple and universal phenomenological scenario of the appearance of spiral attractors in one-parameter families of three-dimensional flows, moreover, these were such attractors that contain a saddle-focus with a two-dimensional unstable manifold. With some modifications, this scenario can easily be carried over to case of three-dimensional maps [37]. Therefore, for such spiral attractors, we proposed in [38] a generalizing term "Shilnikov attractor" for flows (these include, in particular, the Rössler attractors and ACT attractors, mentioned above), or "discrete Shilnikov attractor" for maps. Various examples of the latter were found in three-dimensional Henon maps [37–41], here we have obtained quite interesting results that we intend to present in the second part of our work.

Until recent time we could attribute only hyperbolic attractors and Lorenz attractors as genuine strange attractors of smooth dynamical systems. However, the situation changed after the work by Turaev and Shilnikov [42], in which there was introduced a new class of genuine strange attractors, the so-called *wild hyperbolic attractors*. These attractors, unlike hyperbolic ones, admit the existence of homoclinic tangencies, but do not contain stable periodic orbits and any other proper stable invariant subsets that also do not arise with small smooth perturbations. Systems with wild hyperbolic attractors belong to Newhouse domains, i.e. open (C^2-topology) domains from the space of dynamical systems, in which systems with homoclinic tangencies are dense.[1] However, these tangencies, unlike homoclinic tangencies in systems with quasiattractors, are such that their bifurcations do not lead to the birth of stable periodic orbits [10–12] (see also section 1 below).

In [42] these was also constructed an example of a four-dimensional flow with a wild spiral attractor containing a saddle-focus equilibrium state. One of the main features of the Turaev — Shilnikov spiral attractor is that it possesses a *pseudohyperbolic structure*. In two words, this feature is manifested in the fact that in a neighbourhood of the attractor, in some of its absorbing regions D, there is a weak version of hyperbolicity: there is a partition of the neighbourhood into invariant with respect to the differential and transversal subspaces, so that on one of them there is exponential compression along all directions, and on the other – exponential expansion of the volume. It is also required that this partition depends continuously on a point of D; the corresponding compression and stretching coefficients, as well as the angles between the tangent vectors of subspaces are uniformly bounded; if there are any contractions in the expanding subspace, then they are all uniformly weaker than any compression in the strongly compressive space.

Note that the conditions of pseudohyperbolicity are checked for points of the absorbing region D. If they are fulfilled, then, as shown in [42]: the attractor exists and it is the only one; each of its orbits has a positive maximum Lyapunov exponent (this follows from properties of volumes expanding), and the attractor itself is a Ruelle attractor [44], i.e. closed, invariant, asymptotically stable, and chain transitive set (see more details below in section 2).

[1] The term "wild" itself goes back to the Newhouse's paper [43], in which there was introduces the concept of "wild hyperbolic set", i.e. such hyperbolic invariant set, which among its stable and unstable invariant manifolds always has nontransversally intersecting ones, and this property is preserved for all small C^2-smooth perturbations.

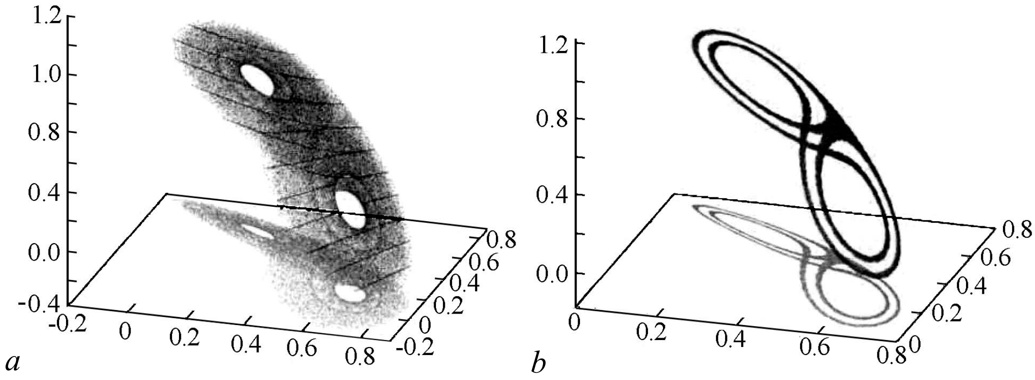


Fig. 1. Phase portraits of discrete Lorenz attractors in the case of maps (1) for $M_1 = 0$, $B = 0.7$ and different M_2: a – $M_2 = 0.85$, b – $M_2 = 0.825$. In both cases, the order of 10^5 iterations of one starting point on the attractor is shown. There are also shown the projections of the attractor to the plane (x, y) and some sections of the attractor by the plane $z = \mathrm{const}$ (although these sections look like lines, they actually have a complex Cantor structure). This figure is taken from the work [45]

In fact, in [42] it was laid the foundation for a very promising theory of pseudohyperbolic strange attractors. New examples of such attractors were also soon found. Thus, in [45] it was shown that in three-dimensional Hénon maps of the form

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = M_1 + Bx + M_2 y - z^2, \tag{1}$$

where M_1, M_2, B are the parameters (B is the Jacobian of map), discrete Lorenz attractors exist for parameter values from some area adjacent to the point $A^* = (M_1 = 1/4,\ M_2 = 1,\ B = 1)$.[2] In Fig. 1 one can see some examples of discrete Lorenz attractors in the case of map (1). Let's note that their phase portraits are very similar to the flow Lorenz attractors. Neverhteless, the values of the parameters, at which these attractors observed, are not at all close to A^*: here $M_1 = 0$, $M_2 = 0.85$, $B = 0.7$ (Fig. 1, a) and $M_1 = 0$, $M_2 = 0.825$, $B = 0.7$ (Fig. 1, b). Therefore, the conditions of pseudohyperbolicity of such attractors must be checked additionally.

For the authors of this paper, this task looks to be very difficult. In fact, we can only check some necessary conditions. For example, for a pseudohyperbolic attractor of a three-dimensional map, its Lyapunov exponents $\Lambda_1 > \Lambda_2 > \Lambda_3$ must satisfy the following conditions:

$$\Lambda_1 > 0, \quad \Lambda_1 + \Lambda_2 > 0, \quad \Lambda_1 + \Lambda_2 + \Lambda_3 < 0. \tag{2}$$

The first and third conditions indicate that the observed attractor is strange, and the second – that two-dimensional areas are expanded.

However, Lyapunov exponents are averaged characteristics of the orbits on the attractor, therefore, in principle, there is not excluded a situation when the attractor has very small "holes" (which size may be less than any reasonable accuracy), where the conditions (2) for corresponding orbits are violated. At our request, the conditions of pseudohyperbolicity of the attractors (see Fig. 1) were checked with the methods of interval arithmetic by the mathematicians G. Figueros and V. Tucker from Uppsala University (Sweden) who got very interesting and very thin results. So, in the case of attractor from

[2]The pseudohyperbolicity of such attractors was tested analytically in [45] based on the fact that for parameter values close to A^*, the square of the map in some neighbourhood of a saddle fixed point can be presented as a Poincare map of periodically perturbed Shimizu–Morioka system, which has the Lorenz attractor [46, 47]. If the perturbation is sufficiently small (which is determined by the closeness of the parameters to A^*), then the desired pseudohyperbolicity should naturally be inherited from the Lorenz attractor (which itself is of this kind) [48, 49]). In particular, in the work [50] it was shown that the property of pseudohyperbolicity of flows is preserved also for their Poincare maps for small periodic perturbations.

Fig. 1, *a*, inside it, there was found a stable periodic orbit with an attraction area having size with the order of 10^{-40}, whereas the discrete Lorenz attractor from Fig. 1, *b* turned out to be genuine pseudohyperbolic attractor. Similar results were obtained independently by other methods by Saratov mathematicians S.P. Kuznetsov and P.V. Kuptsov. Wherein some others attractors from our work [40] were also checked for pseudohyperbolicity. We hope these interesting results will be published soon.

On the other hand, the very fact of the existence of homoclinic tangencies in the attractor can already tell whether the attractor is genuine (pseudohyperbolic) attractor or quasi attractor. So, for example, attractors from Fig. 1 contain a saddle fixed point with multipliers λ_1, λ_2, λ_3 such, that $\lambda_1 < -1$, $0 < \lambda_2 < 1$, $-1 < \lambda_3 < 0$, $|\lambda_2| > |\lambda_3|$ and $\lambda_1\lambda_2\lambda_3 = B = 0.7 < 1$, and besides, the saddle value $\sigma = |\lambda_1\lambda_2|$ is greater than 1. Then inevitably arising homoclinic tangencies in the general case will be as in Fig. 3 (see below) and bifurcations of such tangencies do not lead to the birth of stable periodic orbits [10–12][3].

Remark 1. On the other hand, stable periodic orbits are necessarily born if $\sigma < 1$, or if a fixed (periodic) point on the attractor is a saddle focus (it doesn't matter, with one-dimensional or two-dimensional unstable manifold). In particular, spiral attractors of three-dimensional smooth maps or flows always are quasiattractors. In this regard, the following problem seems quite interesting: *let a three-dimensional diffeomorphism in R^3 have a strange attractor containing a saddle fixed point with the two-dimensional unstable invariant manifold, then this attractor is a quasiattractor.*[4].

For this reason, in this paper we consider only such strange attractors of three-dimensional maps that contain fixed points of saddle type with one-dimensional unstable varieties and with a saddle value σ greater than 1. Moreover, we focus on the so-called homoclinic attractors that contain *exactly one* fixed point. As it is shown in the works [37, 38, 40, 41, 45], this direction is very promising.

Content of the work. Section 1 considers the main properties of pseudohyperbolic maps (diffeomorphisms) and the types of homoclinic tangencies that support or destroy pseudohyperbolicity. Section 2 discusses phenomenological scenarios of the emergence of strange homoclinic attractors in one-parameter families of three-dimensional diffeomorphisms, both orientable and nonorientable. Section 3 gives examples of such attractors in the case of three-dimensional generalized Henon maps. In the Appendix we give definition of pseudohyperbolic diffeomorphism.

1. Pseudohyperbolicity and homoclinic tangency

In this section, we consider the main concepts of the theory of pseudohyperbolic strange attractors. In the case of flows, the definition of pseudohyperbolicity was given in the work of Turaev and Shilnikov [42] (see also [50, 54]), and in the case of maps, the definition is given in [55] (see also Appendix to this article). Briefly speaking, pseudohyperbolicity of the diffeomorphism f on some region $\mathcal{D}$ means that at every point of this domain there are two transversal linear subspaces N_1 and N_2, continuously depending on the point and invariant with respect to the differential Df of the map, such that Df is exponentially strongly contracting on N_1 and stretching (exponentially) volumes on

[3] In the general case, such (quadratic) homoclinic tangencies are called "simple" [12], and in the case $\sigma > 1$ they do not destroy pseudohyperbolicity if the fixed point itself is pseudohyperbolic, although they lead to "wild hyperbolicity" (for more details, see [10, 12, 51]).

[4] This problem seems to be very difficult, and its solution is connected, for example, with proof of the existence of the so-called non-simple homoclinic tangencies [21, 52, 53], examples of which are shown in Fig. 4, only the direction of the arrows must be reversed, so that the unstable manifold of the point O become two-dimensional. Let's note that bifurcations of such touches lead to the birth of stable periodic orbits [52]. In turn, the appearance of non-simple touches in this case should be expected due to the fact that the two-dimensional unstable manifold itself should infinitely many times "fold" in different directions, so that it belongs to the attractor (it is like if to try to "pack" a two-dimensional plane into a three-dimensional cube, avoiding sharp corners)

N_2 (here the word "strongly" means that any possible contraction in N_2 is uniformly weaker than any contraction in N_1). Thus, unlike hyperbolicity, it is not required that stretching on N_2 exists in all directions. However, pseudohyperbolicity, as well as hyperbolicity, persists [42]. Therefore if diffeomorphism f has an attractor in $\mathcal{D}$, then this attractor is strange, because volume expansion on N_2 guarantees the existence of a positive maximal Lyapunov exponent for any orbit. In other words, pseudohyperbolic attractors are genuine attractors.

However, unlike hyperbolic attractors and the Lorenz attractors, pseudohyperbolic attractors may have *homoclinic tangencies*. Moreover, if it isn't known in advance that a strange attractor is hyperbolic, then, in addition to transverse homoclinic orbits (at which points stable and unstable invariant manifolds of saddle periodic orbits intersect transversally), an attractor must also have nontransversal ones. The occurrence of a certain homoclinic tangency is not exceptional – this is the bifurcation moment of codimension one in general case (when the tangency is quadratic). However, as Newhouse [43] showed, this bifurcation entails an extremely complex structure of bifurcation set. In particular, there arise infinitely many secondary homoclinic tangencies. These tangencies may be degenerate [56], in turn means the possibility of the appearance of many degenerate periodic orbits, etc. All this leads to the fact that bifurcation of homoclinic tangencies cannot be fully studied, for example, by means of finite-parameter families, which is a traditional apparatus of the classical bifurcation theory. Here, if necessary, there arise problems of different kind related to the study of basic bifurcations and the main characteristic properties of such systems. Moreover (which is very important and interesting), the question of which bifurcations and what characteristic properties are the main ones, the researcher must decide himself.

In the theory of strange attractors of smooth dynamical systems, one of the most important questions relates to determining whether a given attractor is a quasiattractor or a genuine attractor (in particular, pseudohyperbolic). Sometimes, this question is solved simply and in parallel with the main computer calculating. In the case of strange attractors of two-dimensional diffeomorphisms (if they are not hyperbolic), bifurcations of the inevitable in them homoclinic tangencies lead to the emergence of stable periodic orbits of very large periods [8], and correspondingly any attractor of this type should be considered a quasi attractor.[5]

In the case of strange attractors of three-dimensional diffeomorphisms, which are the main subject of this article, the question of definition of their types (quasiattractor or real attractor) is more difficult. However, in this case, the homoclinic tangencies found in attractors, are specific indicators, too. So if the attractor allows homoclinic tangencies to a fixed or periodic point, such as in the Fig. 2, then it is actually a quasiattractor. In the first case (Fig. 2, *a*) the fixed point is a saddle with the saddle value σ less than one, in the second case (Fig. 2, *b*) it is a saddle-focus. Here it is only required that the Jacobian J of the fixed point be less than one, and in the case of a saddle its unstable manifold be one-dimensional (in the case of a saddle-focus, it can be either one-dimensional or two-dimensional).

On the other hand, it is very important that there are homoclinic tangencies that do not destroy pseudohyperbolicity. In the case of three-dimensional diffeomorphisms such are *simple homoclinic tangencies* [10,51] under condition $\sigma > 1$. Suppose, for example, that a diffeomorphism f has a saddle fixed (periodic) point O with real multipliers λ_1, λ_2, λ_3, such that $|\lambda_1| > 1 > |\lambda_2| > |\lambda_3| > 0$ and $\sigma = |\lambda_1||\lambda_2| > 1$. For such tangencies, the point itself is pseudohyperbolic: for it $N_1(O)$ is a straight line, passing through O in the direction of the eigenvector of the linearization matrix A, corresponding to its strong stable eigenvalue (multiplier) λ_3, and $N_2(O)$ is a plane stretched over the eigenvectors of the matrix A and corresponding to multipliers λ_1 and λ_2. Obviously at any point p from a small neighbourhood $U(O)$ of the saddle O, there should be the same invariant decompositions into subspaces

[5]This is true, for example, for Hénon attractors, in which stable periodic orbits may be missing for parameter values forming nowhere dense set of positive measure according to the Benedix–Carleson theory [29]. However, they immediately arise with arbitrarily small perturbations.

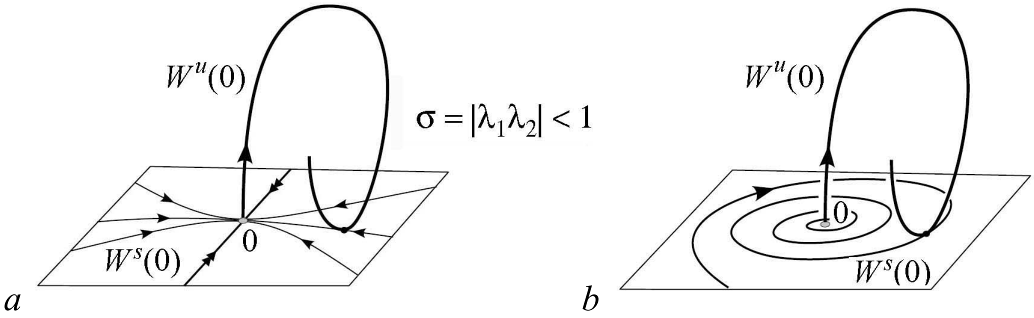


Fig. 2. Homoclinic tangencies whose bifurcations lead to the birth of stable periodic orbits

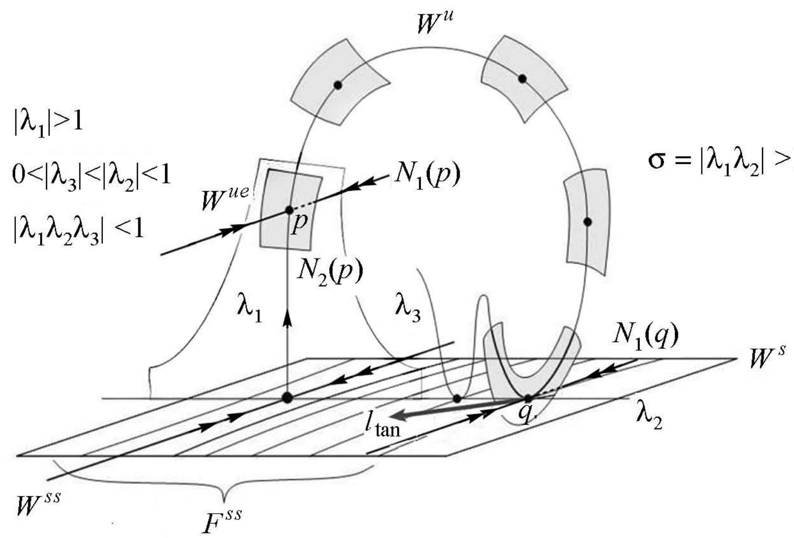


Fig. 3. To the definition of simple homoclinic tangency

$N_1(p)$ and $N_2(p)$. The similar invariant decompositions near the entire homoclinic orbit can also be obtained if the homoclinic tangency is simple.

The latter means the following. Let's take two arbitrary homoclinic points p and q in $U(O)$, such that $p \in W^u_{\text{loc}}(O)$ and $q \in W^s_{\text{loc}}(O)$, by which we define the so-called global map T_1, constructed along the orbits of the considered diffeomorphism and acting from a small neighbourhood $V(p)$ of p into a small neighbourhood of q so that $T_1(p) = q$ (note that if $f^s(p) = q$ for some positive integer s, then $T_1 = f^s|_{V(p)}$).[6] Then it is required that the plane $DT_1(N_2(p))$ intersect transversally with $N_1(q)$ and with $W^s_{\text{loc}}(O)$. Note that the curve $T_1(W^u_{\text{loc}}(O))$ is tangent to the two-dimensional plane $W^s_{\text{loc}}(O)$ along the vector ℓ_{tan}, which, in turn, forms a nonzero angle with the line $N_1(q)$ (see Fig. 3).

If an attractor of three-dimensional smooth map is pseudohyperbolic, then it can contain only simple homoclinic tangency.[7] For any small smooth perturbations, pseudohyperbolicity is preserved, but if these perturbations are not too small; it can be destroyed. Wherein the destruction itself can be caused by the appearance of such homoclinic tangencies as in Fig. 2 (for example, the fixed point, initially with $\sigma > 1$, during evolution can become a point with $\sigma < 1$, or even a saddle-focus). A more

[6] Note that the local invariant manifolds $W^u_{\text{loc}}(O)$ and $W^s_{\text{loc}}(O)$ can always be stightened by introducing into $U(O)$ such C^r-smooth coordinates (x, y, z) in which $W^u_{\text{loc}}(O) = \{x = 0, y = 0\}$ and $W^s_{\text{loc}}(O) = \{z = 0\}$, [12].

[7] In addition to quadratic tangencies, there may exist homoclinic tangencies of arbitrarily large orders [56], however they are all simple in the sense that at any homoclinic point p, the subspaces $N_2(p)$ and $N_1(p)$ intersect (see details in [57]).

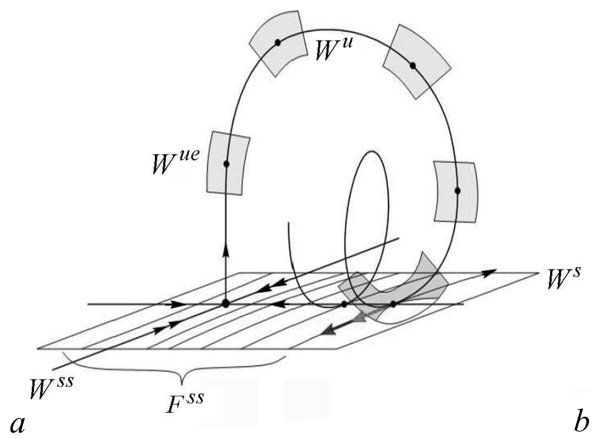


Fig. 4. Two types of non-simple homoclinic tangencies: *a* – the surface $T_1(N_2(p))$ intersects transversally with $W^s_{\rm loc}(O)$, but vector $\ell_{\rm tan}$ lies in $N_1(q)$; *b* – the surface $T_1(N_2(p))$ touches $W^s_{\rm loc}(O)$

subtle mechanism of destruction of pseudohyperbolicity is associated with the emergence of so-called non-simple homoclinic tangencies, examples of which are shown in Fig. 4. Moreover, how it was established in [52, 53, 58], with bifurcations of such homoclinic tangencies, there can appear stable periodic orbits, closed invariant curves, and even nontrivial attracting invariant sets, for example, small Lorenz-like attractors [58].

An important conclusion can be made from this for the theory of strange attractors of 3D smooth maps: if such an attractor is genuine, then it must be either hyperbolic or pseudohyperbolic. As for hyperbolic attractors, their mathematical theory is quite well developed (see, for example, [59]). Moreover, after excellent works of S.P. Kuznetsov [60–63] it is known that such attractors can also be found in applications. Note that, for proving hyperbolicity of an attractor in a partial model, successful qualitative and computer methods are developed. As we know, similar methods are now being created for the detection of pseudohyperbolic attractors in scientific schools from Nizhny Novgorod, Saratov and Uppsala (Sweden). In particular, in our recent works we presented new qualitative research methods of such attractors, including the building of phenomenological scenarios of their occurrences in one-parameter families [37, 38, 40, 41], the search methods based on effective use of so-called “saddle maps” [40] and Lyapunov diagrams, etc. In the next sections, we will give some overview of these methods and show examples of applying them to obtain some attractors, which at the first consideration seem to be genuine (pseudohyperbolic) attractors.

2. Of phenomenological scenarios of the emergence of strange attractors of three-dimensional maps

In this section, we consider the issues of a qualitative study of strange attractors of three-dimensional maps. In this case, we will focus on those attractors that can be pseudohyperbolic. Here we mean that for the attractors under consideration we only check necessary condition (2) (see also *Remark 1*). In addition, we restrict ourselves to the study of the so-called *homoclinic attractors*, i.e. attractors contain that only one fixed point O and its unstable manifold. Moreover, by the attractor of a map f, following Ruelle [44], we mean *closed, invariant, stable, and chain-transitive set* $\mathcal{A}$. Under stability, here we consider the asymptotic stability meaning that the attractor lies inside some absorbing domain $\mathcal{D}$, all points of which under positive iterations of the map f tend to $\mathcal{A}$. We recall that chain transitivity (see, for example, [42]) means that any two points on the attractor can be connected by a ε-orbit for any $\varepsilon > 0$. This means that for any points $a, b \in \mathcal{A}$ and any $\varepsilon > 0$ in $\mathcal{A}$ there exists points $a = x_0, x_1, ..., x_{N-1}, x_N = b$ (here $N = N(\varepsilon)$), such that $x_i \in \mathcal{A}$ and dist $(x_{i+1}, f(x_i)) < \varepsilon$, $i = 0, ..., N-1$. The sequence of points $\{x_i\}$ is called ε-orbit of a point x_0 of length $N+1$, and the point b is called ε-attainable from point a. Then the homoclinic attractor with point O is a closed, invariant set consisting of points ε-attainable from the points O or, one can say, $\mathcal{A}$ is a prolongation of the point O.

In this case, an attractor $\mathcal{A}$, as a set in R^3, can be geometrically considered as a closure (extension) of an unstable manifold of its fixed point O. From this generally obvious notice, we can conclude that the geometric, as well as dynamical properties of homoclinic attractor is largely determined by its homoclinic structure, i.e. by the character of intersections of stable and unstable invariant manifolds of the point O. In this regard, in [37] we proposed fairly simple phenomenological scenarios of birth of some types of homoclinic attractors in one-parameter families of maps, beginning from simple attractor – stable fixed point. Two such scenarios are shown schematically in Fig . 5.

We mark in these scenarios the two main features (from our point of view). The first is that a stable fixed point O loses stability when the parameter changes as a result of period doubling bifurcation. Right after this bifurcation the point O becomes a saddle with a one-dimensional unstable manifold, and in its neighbourhood a stable cycle (p_1, p_2) of period 2 is born (i.e. $f(p_1) = p_2$ and $f(p_2) = p_1$), which becomes an attractor. In this case, the point O must have eigenvalues λ_1, λ_2, λ_3, such that $\lambda_1 < -1$, $|\lambda_{2,3}| < 1$ and $\lambda_2\lambda_3 < 0$. Let us suppose that with further changing of the parameter, the point O no longer undergoes bifurcations, and the cycle (p_1, p_2) loses its stability. The way it happens does not matter so far, but it is important that (and this is the second of main peculiarities of the named scenarios) there takes place a global bifurcation associated with appearance of homoclinic intersections of the one-dimensional unstable W^u and two-dimensional stable W^s invariant manifolds of the point O. Moreover, the configuration of these manifolds will be similar to that we see in Fig. 5, *c* and *d*.

To explain how such different configurations are formed, we assume for definiteness that $\lambda_2 > 0$

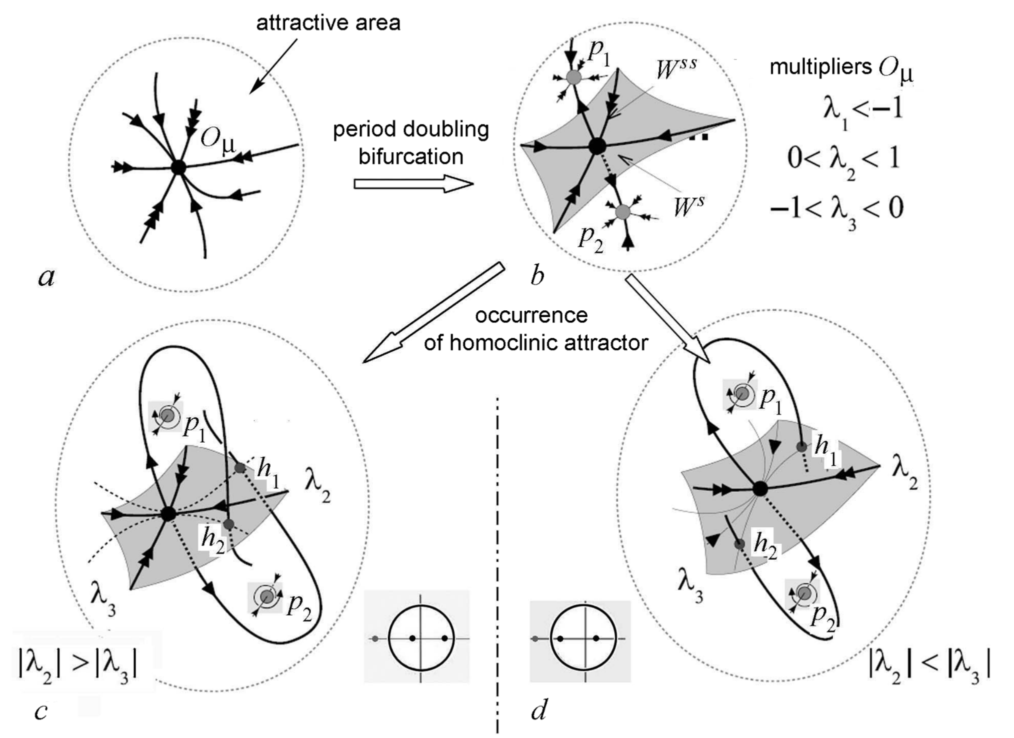


Fig. 5. Two phenomenological scenarios of occurrence of discrete homoclinic attractors of Lorenz type, track $(a) \to (b) \to (c)$; or figure-eight type, track $(a) \to (b) \to (d)$

and $\lambda_3 < 0$ (here also $\lambda_1 < -1$). Then W^u is divided by the point O into two connected separatrix components W^{u+} and W^{u-}, invariant with respect to f^2, such that $f(W^{u+}) = W^{u-}$ and $f(W^{u-}) = W^{u+}$. Let further suppose that W^{u+} intersects $W^s_{\text{loc}}(O)$ in the point h_1, then automatically W^{u-} intersects $W^s_{\text{loc}}(O)$ in the point $h_2 = f(h_1)$. The map f in the constraint on $W^s_{\text{loc}}(O)$ is very simple: it has here a stable fixed point O of nonorientable node type, because $\lambda_2\lambda_3 < 0$.

In the case when $|\lambda_2| > |\lambda_3|$ (see Fig. 5 *c*), on $W^s_{\text{loc}}(O)$ there exists a strongly stable manifold W^{ss} – a curve, invariant with respect to f, which is (at the point O) tangent to the its eigen direction corresponding to the negative multiplier λ_3. The curve W^{ss} separates the plane $W^s_{\text{loc}}(O)$ into two components W^s_1 and W^s_2. Since $\lambda_2 > 0$ (and $|\lambda_2| > |\lambda_3|$), then every of the components is invariant with respect to f, i.e. the points from W^s_1 cannot get to W^s_2 during iterations f, and vice versa. On W^s_{loc} there exists also a continuous family of smooth invariant curves, which all enter the point O, touching the leading eigen direction corresponding to positive multiplier λ_2. Let the point h_1 belongs to one of these curves, for example l_1. Then the curve $l_2 = f(l_1)$ will be also an invariant curve from this family and $h_2 \in l_2$. Curves l_1 and l_2 lie exactly in one component W^s_1 or W^s_2, and enter the point O, forming the configuration of a "wedge with a zero angle". Accordingly, the configuration of unstable separatrices of the point O (see Fig. 5, *c*) will resemble the one that is characteristic for unstable separatrices of the Lorenz attractor. Therefore, the attractor, arising here, was named in [39] "discrete Lorenz attractor".

The similar simple geometric reasoning for the case $|\lambda_2| < |\lambda_3|$ (see Fig. 5, *d*) shows that here the configuration of unstable separatrices of the point O will be completely different. It is rather similar to the configuration of separatrices in the attractor of Poincaré map of periodically perturbed two-dimensional system with a homoclinic "figure eight" saddle [30]. Therefore, the attractor, arising in this case, was in [38] called "discrete figure eight attractor" (Fig. 1 and the following Fig. 7 give an idea of the typical shape of such an attractor).

Note that for both such attractors the condition $\sigma > 1$ (here $\sigma = |\lambda_1\lambda_2|$ for Lorenz attractor and $\sigma = |\lambda_1\lambda_3|$ for the figure-eight attractor, respectively) is very important, because it is quite necessary for pseudohyperbolicity of the studied attractor. Otherwise, it will be either quasiattractors of the Lorenz or figure-eight type; or – that's another variant – from the resulting homoclinic configuration in the case $\sigma < 1$ there can be "born off" a large stable closed invariant curve (torus), enveloping this configuration. This torus, in turn, can collapse, and chaos of a completely different nature may arise in its place (for example, "torus-chaos"). Both these possibilities are well observed in computer experiments (see, for example, [37]).

These obvious observations tell us that in the cases of saddle fixed points of other types one can also expect the existence of homoclinic attractors, which configuration will depend significantly on the eigenvalues of these points and, first of all, from their signs. In the case when among their eigenvalues there are complex conjugate ones, one can also expect the existence of discrete attractors of spiral type.

Remark 2. However, our "discrete" Lorenz and figure-eight attractors are significantly different from their analogues obtained in Poincaré maps of periodically perturbed three-dimensional flows. So, with a small periodic perturbation of the system with Lorenz attractor we obtain pseudohyperbolic attractor [50], which has a saddle fixed point with all positive multipliers, and in the "holes" of the attractor there lie fixed points. Possibly "discrete figure eight attractors" have no flow analogues at all. It is connected with the fact that if the corresponding system has a homoclinic "figure eight" saddle, then either this "figure eight" is stable (attractor), but then the resulting attractor will have $\sigma < 1$, or it is unstable, then there is no attractor at all. It allows one to say that both the "discrete Lorenz attractor" and the "discrete figure-eight attractor" are new.

The *problem of studying and classifying homoclinic attractors of three-dimensional diffeomorphisms* was first formulated in [37], although the first results on this topic were obtained back in [45], in which discrete Lorenz attractors were discovered in three-dimensional Hénon maps. Note that the possibility of the appearance of such attractors in local bifurcations of triply degenerate fixed points, for example, having multipliers $+1; -1; -1$, was investigated in [64]. Since the Hénon map (1) has three parameters, it contains such a point, and moreover, as shown in [45], the conditions from [64] are satisfied for it. Thus, the main idea of our work [45] consists in applying knowledge about the properties of degenerate local bifurcations to a specific situation. Obviously this approach may be also used in the study of various other models containing at least three parameters.

Another idea was proposed in [37], based on the implementation of phenomenological scenarios of the appearance of strange homoclinic attractors, which are possible in one-parameter families of three-dimensional maps. Scenarios such as those shown in Fig. 5, look quite feasible in specific systems and very convenient for computer research – here, for example, one does not need to know all the subtleties of global bifurcation leading to the emergence of homoclinic structures, but it's enough to calculate/build the basic simple characteristics (phase portrait, fixed point multipliers, Lyapunov exponents, etc.). The very idea of studying strange attractors with phenomenological scenarios, including two main bifurcation stages – loss of stability of a simple attractor (equilibrium state, limit cycle, fixed point, etc.) and the emergence of a homoclinic attractor – was first proposed in the work of L.P. Shilnikov [36], in which such a scenario was used to explain the phenomenon of spiral chaos in the case of multidimensional flows.

In this paper, we illustrate how these ideas can be applied to the study of strange homoclinic attractors in specific models.

2.1. Method of saddle charts. The fact that the configuration of such attractors significantly depends on the eigenvalues of their fixed points was used in [40] for the purposes of their classification in the case of orientable three-dimensional maps. If we confine ourselves with pseudohyperbolic homoclinic attractors only, then this problem turns out to be completely solvable, if we distinguish attractors by types of their homoclinic structures. In this case, as shown in [40], there are probably 5 different types of such pseudohyperbolic attractors. They all relate to the case, when the fixed point is a saddle (all multipliers are real) with a one-dimensional unstable manifold. Two of these types – discrete Lorenz attractor and figure-eight attractor – are observed when the unstable multiplier λ_1 is negative, $\lambda_1 < -1$, and the other attractors ("double figure-eight", "super figure-eight" and "super Lorenz") relate to the case when $\lambda_1 > 1$ [40].

To find such attractors in specific models, [40] proposed the so-called "saddle chart method". We illustrate the essence of this method. On the example of *three-dimensional generalized Hénon map* like

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = Bx + Ay + Cz + f(y, z), \tag{3}$$

where function f depends from coordinates y and z only, and also $f(0,0) = 0$, $f'_y(0,0) = f'_z(0,0) = 0$. The map (3) depends from three parameters A, B and C and has a constant Jacobian equal to B. Obviously, one can reduced to the form (3) any map like $\bar{x} = y$, $\bar{y} = z$, $\bar{z} = M_1 + Bx + g(y,z)$, which has a fixed point (for example, the map (1) with $(1 + B - M_2)^2 + 4M_1 > 0$), if this point is shifted to the origin.

The point $O(0,0,0)$ is fixed for the map (3), in which it has a characteristic equation of the form

$$\chi(\lambda) \equiv \lambda^3 - A\lambda^2 - C\lambda - B = 0. \tag{4}$$

It is important here that the eigenvalues of the point O no longer depend on the nonlinearities of $f(y,z)$, but are functions only of the parameters A, B and C. Therefore for the map (3), one can construct a partition of the set of parameter values into regions corresponding to different types of eigenvalues of the point O. Such a partition with a fixed value of the Jacobian, i.e. in this case, with a fixed B, is called a "saddle chart" [40][8]. We note that on this map we also distinguish between the

[8] In the case of three-dimensional flows, a similar "saddle chart" for equilibrium states was proposed in the form of a table in [65, Appendix C.2.]

regions corresponding to $\sigma > 1$ and $\sigma < 1$ in the case when the unstable manifold of the point O is one-dimensional. Examples of such saddle charts are shown in Fig. 6.

On these maps one can see the highlighted area IV, the so-called "stability triangle" (the area $\{C > B^2 - 1 - BA\} \cap \{C < A + B + 1\} \cap \{C < 1 - B - A\}$), when the fixed point $O(0,0,0)$ is asymptotically stable. For all other values of the parameters A and C (except for bifurcation curves), the point $O(0,0,0)$ is a saddle point – it has multipliers both inside and outside the unit circle (their location is also shown in Fig. 6). The boundaries of the regions on the saddle charts are 7 main curves. These are, firstly, three bifurcation curves, on which, with certain parameters values, the point $O(0,0,0)$ has multipliers on the unit circle:

- curve L_+: $C = 1 - B - A$ (multiplier $\lambda = +1$);
- curve L_-: $C = 1 + B + A$ (multiplier $\lambda = -1$);
- curve L_φ: $C = B^2 - 1 - BA$ for $-2 < A - B < 2$ (multipliers $e^{\pm i\varphi}$).

Note that the curve $C = B^2 - 1 - BA$ actually enters the boundaries of the regions entirely, but with $|A - B| \geq 2$ it is not a bifurcation curve: here with $A - B \leq -2$ the point M has multipliers of the form $(B, -|\lambda|, -|\lambda|^{-1})$, and with $A - B \geq 2$ – of the form $(B, |\lambda|, |\lambda|^{-1})$). Secondly, there are 4 additional curves on the saddle chart:

- "resonance curve" $AC + B = 0, \;\; A < 0$ (when $\lambda_1 = -\lambda_2$),
- curve "$\sigma = 1$" $C = 1 + B^2 + AB$ (when $\lambda_1\lambda_2 = -1$)

and two curves of "equal roots"

- S^- (when $\lambda_1 = \lambda_2 < 0$),
- S^+ (when $\lambda_1 = \lambda_2 > 0$),

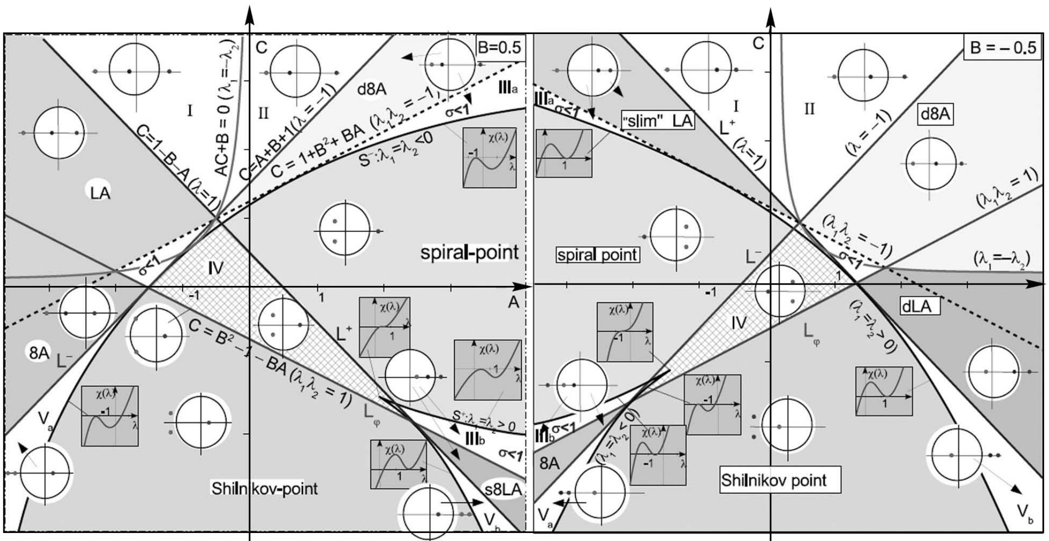
which separate areas with points of "node" type and "focus" type, as well as "saddle" and "saddle-focus". The last curves are given by the formula

$$S^\pm: \quad (\lambda_\pm)^3 - A(\lambda_\pm)^2 - C\lambda_\pm - B = 0,$$

where

$$\lambda_\pm = \frac{A \pm \sqrt{A^2 + 3C}}{3}$$

with $A^2 + 3C > 0$ (i.e. $\lambda_\pm$ are the roots of the equation $3\lambda^2 - 2A\lambda - C = 0$).



a *b*

Fig. 6. An example of a saddle map for the map (3) for various values of B: *a* – 0.5, *b* – (-0.5)

For the purposes of our article (study of pseudohyperbolic attractors) of greatest interest are 4 areas, with the value of the parameters of which the fixed point is $O(0,0,0)$ of the map (3) is a saddle with one-dimensional unstable manifold and saddle value $\sigma > 1$. These are areas LA, 8A, d8A and sL8A in orientable case – Fig. 6, *a*; and also areas "slim" LA, 8A, d8A и dLA in nonorientable case – Fig. 6, *b*. In other areas, except for the "stability triangle", the point is either a saddle-focus or a saddle with a two-dimensional unstable manifold, or has $\sigma < 1$. As we assume, if the map has a homoclinic attractor (containing the point O) with parameter values from these last areas, then it is, in our opinion, a quasiattractor (see *Remark 1*).

The use of saddle charts in numerical studies is a very convenient auxiliary tool along with the Lyapunov diagram method. However, here we are also slightly modify the last method. Typically, it consists in constructing maps of Lyapunov exponents, in which in different colours (in black and white figures – by numbers) there are indicated areas of parameters corresponding to different spectra of Lyapunov exponents are indicated $\Lambda_1 > \Lambda_2 > \Lambda_3$. In particular, we use green (number "1") for indicating the areas of stable periodic regimes ($\Lambda_1 < 0$); blue (number "2") – for quasiperiodic ($\Lambda_1 = 0$); yellow (number "3") – when $\Lambda_1 > 0,\ \Lambda_2 < 0$; red (number «4»), when $\Lambda_1 > 0,\ \Lambda_2 \sim 0$; dark blue (number «5»), when $\Lambda_1 > \Lambda_2 > 0$ – for strange attractors.[9] To these five colours we added one more – dark grey (number "6") to indicate areas with homoclinic attractors, when the numerically obtained points on the attractor approach the point O at a distance which is less than 10^{-4}.

3. Examples of strange attractors in generalized Hénon maps

In this section, we will illustrate the possibility of efficiently using "saddle charts" (such as in Fig. 6) to search homoclinic attractors of various types in generalized Hénon maps of the form (3).

Note that there are many different ways to study chaotic dynamics in specific models. One of the regular and reasonable approaches to this problem is associated with the construction of dynamical regimes charts and/or diagrams of Lyapunov exponents. In this way (using diagrams of Lyapunov exponents) discrete Lorenz attractors in the three-dimensional Hénon map were found in [45] (see also Fig. 1). Now we can find such an attractor, as they say, "purposefully" using our approach. For this, we consider map (1) in the "reduced" form

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = Bx + Az + Cy - z^2 \tag{5}$$

and take its saddle chart which is actually already given, such as in Fig. 6, *a*, but built with the desired fixed B, in this case for $B = 0.7$. Further, on the background of this chart, we build numerically a diagram of Lyapunov exponents. As a result, we get a picture such as in Fig. 7, when, in particular, the area of chaos in dark grey colour (in the black-and-white image, the number «6») intersects the area LA. This suggests that here, for the corresponding values of the parameters A an C, a discrete Lorenz attractor can be observed. The numerical results shown in the figure, confirm this.

Obviously, in the case of map (3), the form of the saddle chart does not depends on its nonlinear terms. At the same time, the Lyapunov diagram is precisely determined by these terms. On modern computers building diagrams of Lyapunov exponents, especially for the case of maps, does not take much time (and the saddle map for the map (3) is built instantly), especially at the "search stage". In addition, as our experience have showed, when changing nonlinearities one can see certain trends, in particular, in changing location of "dark grey spot" (when the attractor is homoclinic). If desired, it can be "driven" inside any of the saddle map areas (except "stability triangle") and, accordingly, we

[9]The red areas (number "4" in our case), $\Lambda_1 > 0$, $\Lambda_2 \sim 0$, were specially marked back in [45] to indicate areas where the value Λ_2 was either fluctuating in very narrow boundaries around zero, or differed from zero by a value (of the order of 10^{-5} or 10^{-6}), comparable with the calculation accuracy of exponents. Surprisingly, these areas turned out to be quite large, and this phenomenon (apparently related to the fact that the map on the attractor turned out to be very close to discretization of a certain flow, for example, with the Lorenz attractor) was discussed in [45].

can discover the attractor that interests us. On this way in the work [40] there was found a variety of homoclinic attractors of map (3). Some of them (for values of the parameters A and C from the areas LA, 8A, d8A и sL8A), when the necessary condition (2) was fulfilled), were presented as candidates for pseudohyperbolic attractors. Examples of such attractors are shown in Fig. 8.

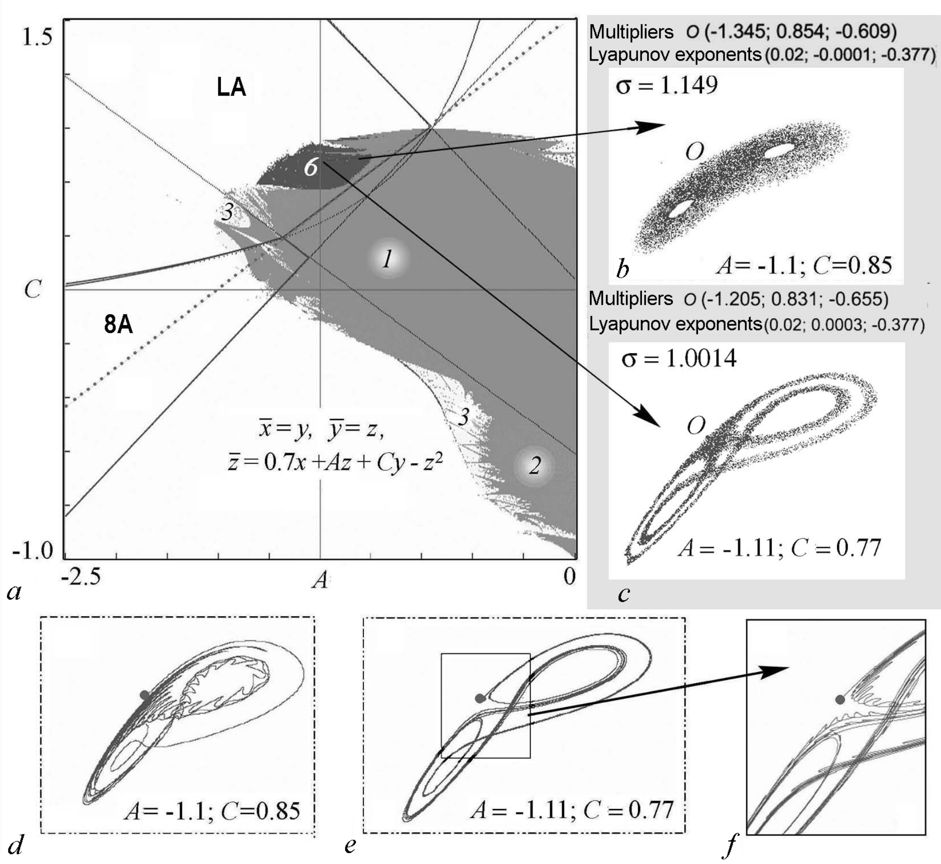


Fig. 7. a – diagram of Lyapunov exponents on the background of the “saddles chart” on the plane of the parameters A and C for the map (5) with $B = 0.7$; b and c – phase portraits of attractors (approximately 10^4 iterations of one point); d and e – numerically constructed one of the unstable separatrices of the point O (another separatrix will look symmetrically, since $\lambda_1 < -1$), presence of characteristic “strollers” on the separatrix indicates that the attractor is wild hyperbolic, i.e. contains homoclinic tangencies; f – enlarged fragment of Fig. e. The figure is taken from the work [40]

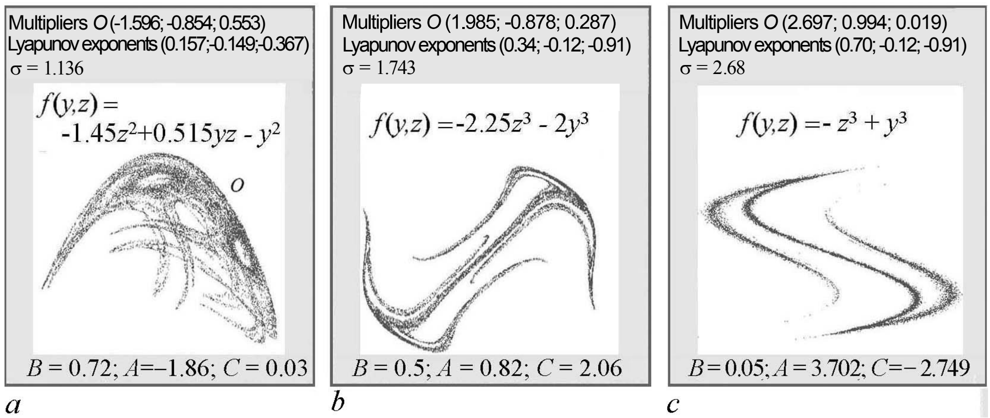


Fig. 8. Examples of discrete homoclinic attractors, corresponding to generalized Hénon maps: a – figure-eight attractor from area 8A; b – double figure-eight attractor from area d8A; c – super figure-eight attractor from area sL8A

3.1. Examples of pseudohyperbolic attractors in three-dimensional nonorientable Hénon maps. It is natural to expect that in the case of *nonorientable* three-dimensional maps (diffeomorphisms) there can be realized scenarios of occurrence of strange attractors, similar to that shown in Fig. 5. However, due to the nonorientability of the maps, as shown in [41], they all have their own specifics.

Fig. 9 schematically shows 3 such scenarios, that can be implemented in one-parameter families. These scenarios are very similar to what happens in an orientable case: they all start with a stable fixed point O, for which a period-doubling bifurcation then occurs, and the point O becomes a saddle point with one unstable multiplier $\lambda_1 < -1$ etc. The difference is that the map f in the restriction on $W^s_{\text{loc}}(O)$ is orientable, and here for $f|_{W^s}$, the point O is either a node-plus ($\lambda_2 > 0,\ \lambda_3 > 0$), or a node-minus ($\lambda_2 < 0,\ \ \lambda_3 < 0$), or a focus ($\lambda_{2,3} = \rho e^{\pm i\varphi},\ \ 0 < \rho < 1$).Therefore, the configuration of the obtained homoclinic attractors becomes different. We called such discrete nonorientable attractors, respectively, “thin Lorenz”, “figure-eight” and “spiral”. The first two can be, in principle, pseudohyperbolic, the last is always a quasiattractor.

Remark 3. Nonorientable “figure-eight” attractor is very similar to orientable one (compare Fig. 9, *d* and Fig. 5, *d*). “Thin Lorenz” attractor is not quite similar to its orientable analogue (compare Fig. 9, c and Fig. 5, c). This is due to the fact that in the case of a “thin” attractor the homoclinic point h_1 on $W^s_{\text{loc}}(O)$ and all its further iterations lie on the same invariant curve ℓ in O (this is the cause of the term “thin”), and in the orientable attractor the iterations of the points h_1 lie alternately on two different curves ℓ_1 and ℓ_2, converging in O, touching each other.

These attractors were found in [41] for nonorientable three-dimensional Henon maps of the form (3) for $B < 0$. For such maps, both a saddle map can also be constructed, such as in Fig. 6, *b*, and a modified diagram of Lyapunov exponents. Then, using the fact that a thin Lorenz attractor can exist for

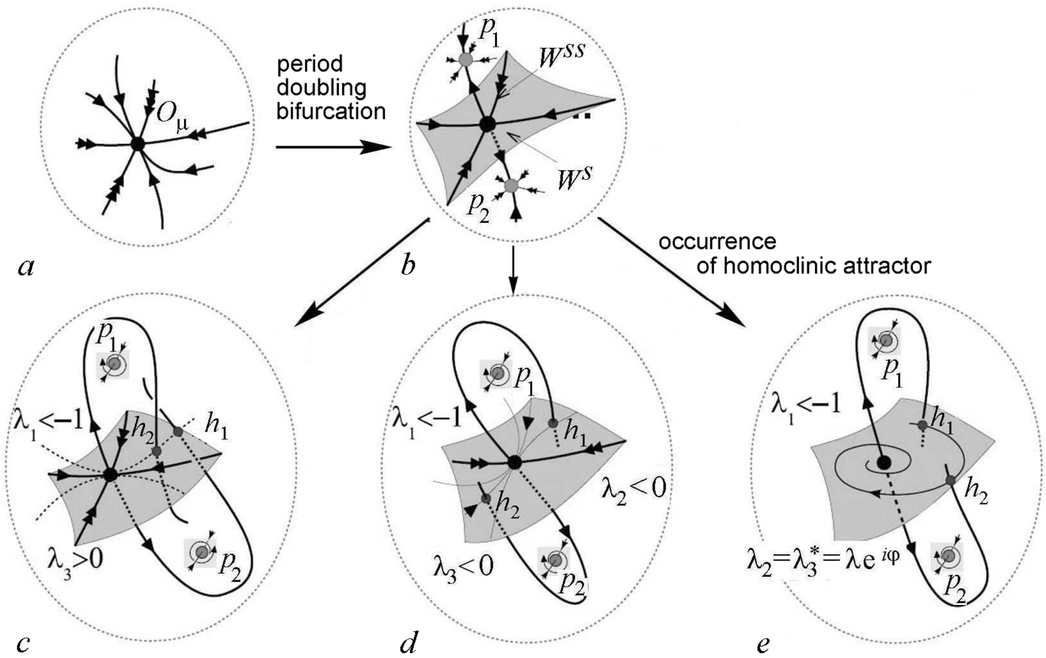
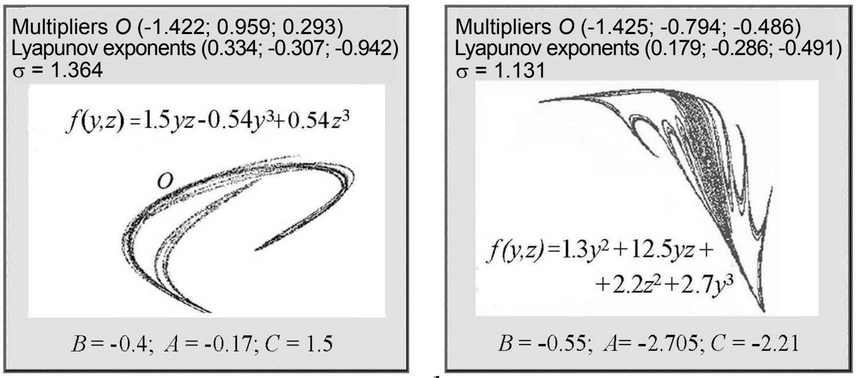


Fig. 9. Illustration of bifurcation scenarios leading to the emergence of discrete nonorientable homoclinic attractors: “thin” Lorenz attractor (track *a*⇒*b*⇒*c*); “figure-eight” attractor (track *a*⇒*b*⇒*d*); spiral attractor (track *a*⇒*b*⇒*d*). Shown here are the points h_1 and h_2 belonging to the same homoclinic orbit, such that $h_i \in W^s(O_\mu) \cap W^u(O_\mu)$ и $h_2 = f(h_1)$. These points are located on one side of $W^{ss}_{\text{loc}}(O_\mu)$ in the Lorenz case (*c*), on different sides of $W^{ss}_{\text{loc}}(O_\mu)$ in the case of “figure-eight” attractor (*d*), and homoclinic points lie in $W^{ss}_{\text{loc}}(O_\mu)$ on a spiral, spinning around O_μ, in the case of a spiral attractor. (*e*)



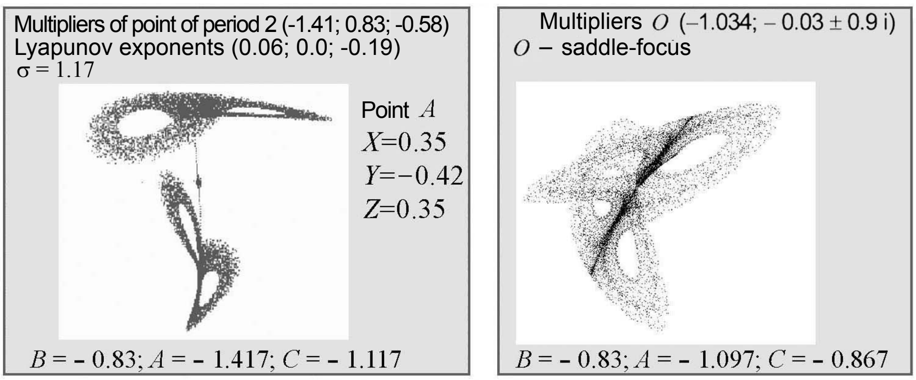
a *b*

Fig. 10. Examples of nonorientable homoclinic attractors, corresponding to three-dimensional Hénon maps: *a* – thin Lorenz attractor; *b* – "figure-eight" attractor (for this attractor $\Lambda_1 + \Lambda_2 = -0.107 < 0$, that is why it is quasiattractor)

parameter values A and C from the region "slim" LA, and a nonorientable "figure-eight" attractor – for values from region 8A, one can already purposefully find such attractors. The examples of thin Lorenz and "figure eight" attractors are shown in Fig. 10.

Interestingly, three-dimensional nonorientable maps may have the same strange homoclinic attractors, as in orientable case, but with points of period 2. Such attractors consist of two components, each of which contains point of a cycle of period 2. Besides, each component of the attractor is invariant with respect to f^2. The examples of discrete (orientable) Lorenz attractors of period 2) and "figure eight" attractor in the case of map (5) are shown in Fig. 11, *a* and *b*, respectively.

In principle, these attractors arise "inside" the nonorientable scenario (see Fig. 9), but according the orientable scenario (see Fig. 5) for the map f^2, in which each point of the cycle of the period 2 is fixed. It is clear that the moment of occurrence of attractors of the period 2, shown in Fig. 11, is some intermediate stage of the scenario of the formation of a homoclinic attractor with fixed point O. One of the final stages of this scenario is shown in Fig. 11, *c*, which shows the "four-eyed" attractor that occurs when a Lorenz attractor of period 2 merges with a fixed point O of saddle-focus type. It seems that map orientability is not a necessary condition for the occurrence of such attractors. In addition, there is an interesting question of the existence of such attractors with a fixed point of "saddle" type, which could give examples of discrete pseudohyperbolic attractors of new types.



a *b*

Fig. 11. Examples of strange attractors in nonorientable map (5): *a* – discrete Lorenz attractor of period 2; *b* – discrete spiral-Lorenz attractor

On the definition of pseudohyperbolicity of maps

Let is consider a diffeomorphism f, defined in $\mathbb{R}^m$, and let Df be its differential.[10] An open region $\mathcal{D} \subset \mathbb{R}^m$ is called *absorbing region* for f, if $f(\overline{\mathcal{D}}) \subset \mathcal{D}$.

Definition 1. Diffeomorphism f is called *pseudohyperbolic* on $\mathcal{D}$, if the following conditions are fulfilled.

1) Every point in $\mathcal{D}$ has two transversal linear subspaces N_1 and N_2, which have additional dimensions ($\dim N_1 = k \geq 1, \dim N_2 = m - k \geq 2$), continuously depend from the point, are invariant with respect to Df, i.e.

$$Df(N_1(x)) = N_1(f(x)), \;\; Df(N_2(x)) = N_2(f(x)),$$

and such that for every orbit $L : \{x_i \mid x_{i+1} = f(x_i),\ i = 0, 1, ...;\ x_0 \in \mathcal{D}\}$ the maximal Lyapunov exponent corresponding to the subspace N_1 is strictly less than the minimal Lyapunov exponent corresponding to the subspace N_2, i.e. the the following inequality is fulfilled:

$$\limsup_{n\to\infty} \frac{1}{n} \ln \Big(\sup_{\substack{u \in N_1(x_0) \\ \|u\| = 1}} \|Df^n(x_0)u\| \Big) < \liminf_{n\to\infty} \frac{1}{n} \ln \Big(\inf_{\substack{v \in N_2(x_0) \\ \|v\| = 1}} \|Df^n(x_0)v\| \Big), \tag{6}$$

where Df^n is a matrix with dimension $m \times m$, defined by the relation

$$Df^n = Df_{x_{n-1}} \cdot \ldots \cdot Df_{x_1} \cdot Df_{x_0},$$

and $\limsup\limits_{n\to\infty}$, $\liminf\limits_{n\to\infty}$ are the upper and the lower limits, respectively.

2) The diffeomorphism f in the restriction to N_1 is uniformly contracting, i.e. there exist constants $\lambda > 0$ and $C_1 > 0$, such that

$$\|Df^n(N_1)\| \leq C_1 e^{-\lambda n}. \tag{7}$$

3) The diffeomorphism f in the restriction to N_2 stretches $(m-k)$-dimensional volumes exponentially, i.e. there exist such constants $\sigma > 0$ and $C_2 > 0$, that[11]

$$|\det Df^n(N_2)| \geq C_2 e^{\sigma n}. \tag{8}$$

From *Definition 1* it immediately folows that

1* all the orbits in $\mathcal{D}$ are unstable: each orbit has a positive maximal Lyapunov exponent

$$\Lambda_{max}(x) = \limsup_{n\to\infty} \frac{1}{n} \ln \|Df^n(x)\| > 0.$$

Note that the conditions of pseudohyperbolicity mean that on $(m-k)$-dimensional subspaces N_2, the $(m-k)$-dimensional volumes are stretched. This does not prohibit the existence of contracting directions on N_2, but the contraction along them should be uniformly not as strong as any contraction in N_1. Therefore, the conditions of pseudohyperbolicity are weaker than the conditions of uniform hyperbolicity, which means that $\|Df^{-n}(N_2)\| < Ce^{-\sigma n}$, i.e. uniform stretching should take place in all directions on N_2. However, as for hyperbolic systems, here the following result is standardly proved [66,42]:

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[10] We recall that the differential in a point x_0 of map $f : \mathbb{R}^m \to \mathbb{R}^m$ is a linear operator $A = \left.\dfrac{\partial f}{\partial x}\right|_{x=x_0}$, which transforms the vector ℓ_{x_0} at the point x_0 into the vector $\ell_{x_1} = A\ell_{x_0}$ at the point $x_1 = f(x_0)$.

[11] If $\dim N_2 = 1$ then we obtain the usual definition of uniform hyperbolicity; therefore, we require in the definition that $\dim N_2 \geq 2$.

2* The conditions of pseudohyperbolicity are preserved for all sufficiently small C^r-perturbations of the system. Moreover, the spaces N_1 and N_2 are changing continuously in this case.

It follows from the statement 1* that if the diffeomorphism f has an attractor in $\mathcal{D}$, then this attractor is strange and it does not contain stable periodic orbits, which, as follows from the condition 2*, also do not appear with small smooth perturbations. In other words, pseudohyperbolic attractors are genuine attractors.

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