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Criteria for internal fixed points existence of discrete dynamic Lotka–Volterra systems with homogeneous tournaments

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Abstract. *Purpose* of the work is to study the dynamics of the asymptotic behavior of trajectories of discrete Lotka–Volterra dynamical systems with homogeneous tournaments operating in an arbitrary $(m-1)$-dimensional simplex. It is known that a dynamic system is an object or a process for which the concept of a state is uniquely defined as a set of certain quantities at a given time, and a law describing the evolution of initial state over time is given. Mainly in questions of population genetics, biology, ecology, epidemiology and economics, systems of nonlinear differential equations describing the evolution of the process under study often arise. Since the Lotka –Volterra equations often arise in life phenomena, the main purpose of the work is to study the trajectories of discrete dynamical Lotka–Volterra systems using elements of graph theory. *Methods.* In the paper cards of fixed points are constructed for quadratic Lotka–Volterra mappings, that allow describing the dynamics of the systems under consideration. *Results.* Using cards of fixed points of a discrete dynamical system, criteria for the existence of fixed points with odd nonzero coordinates are given in a particular case, and these results on the location of fixed points of Lotka–Volterra systems are generalized accordingly in the case of an arbitrary simplex. The main results are theorems 5–9, which allow us to describe the dynamics of these systems arising in a number of genetic, epidemiological and ecological models. *Conclusion.* The results obtained in the paper give a detailed description of the dynamics of the trajectories of Lotka–Volterra maps with homogeneous tournaments. The map of fixed points highlights a specific area in the simplex that is most important and interesting for studying the dynamics of these maps. The results obtained are applicable in environmental problems, for example, to describe and study the cycle of biogens.

Keywords: quadratic Lotka–Volterra mapping, simplex, graph, tournament, homogeneous tournament, fixed point, fixed point map, cyclic triple, transitive triple and skew-symmetric matrix.

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Introduction

A dynamic system as a mathematical object serves as a model for various kinds of natural systems. In matters of economics, population genetics, in particular in epidemiology, evolution is described by systems of nonlinear differential equations. In these sections of natural phenomena, the Lotka–Volterra equations are often used.

Consider the mapping $V : \mathbb{R}^m \to \mathbb{R}^m$, given by the equalities [1,2]

$$x_k^{'} = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right), \quad k = \overline{1, m},$$

where $Vx = (x_1^{'}, ..., x_m^{'})$ and $A = (a_{ki})$ — skew-symmetric matrix. This mapping under the condition $|a_{ki}| \leqslant 1$ is called the Lotka–Volterra mapping [3,4].

1. Methodology

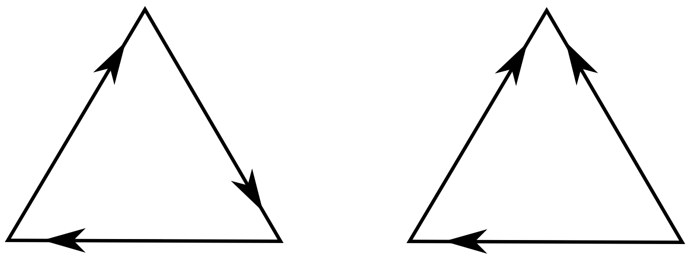
Let Y be a finite nonempty set, and M be a set of disordered pairs (x, y), where $x, y \in Y$, and $x \neq y$. Then the pair (Y, M) is called a graph.

The elements of Y are called vertices, but if $(x, y) \in M$, then (x, y) is called an edge of the graph (Y, M), the vertices x and y in this case are called adjacent.

Two graphs (Y_1, M_1) and (Y_2, M_2) are called isomorphic if there exists a bijection Y_1 on Y_2 preserving the adjacency of vertices.

A graph is — complete if any two distinct vertices are adjacent. If each edge is provided with a direction, then the graph is called oriented. The tournament — is a complete directed graph.

The works [5,6] are devoted to the classification of tournaments with a given number of vertices up to isomorphisms. For example, up to isomorphism, there are only two tournaments with three vertices (Fig. 1).



a *b*

Fig. 1. Types of tournaments at $m = 3$: a — cyclic triple, b — transitive triple

Let x_1, x_2 be the vertices of the tournament. The entry $x_1 \to x_2$ means that the edge connecting x_1 and x_2, directed from x_1 to x_2. The final sequence of vertices is $x_1 \to x_2 \to ... \to x_p$ is called a route if $x_i \neq x_j$ at $i \neq j$. A loop — is a closed route, that is $x_p = x_1$.

A tournament is called strong if for any vertices $x, y \in Y$ there is a route with the beginning of x and the end y.

It is known [7] that a tournament is — strong if and only if there exists a cycle of length $|Y|$ ($|Y|$ — number of elements Y).

A tournament that has no cycles is called transitive. The concept of a sub-tournament is naturally defined. (In the definitions we follow the terminology adopted in the works [3,8,9].)

Definition 1. *A tournament is called homogeneous if any of its sub-tournaments is either strong or transitive.*

Obviously, with $|Y| \leqslant 3$, any tournament is homogeneous.

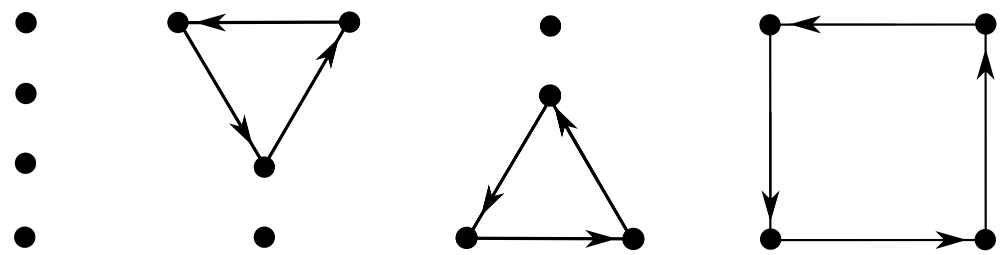


Fig. 2. This illustration of the tournaments is given in [7]. Here, in all unspecified edges, the directions go from top to bottom: a — transitive tournament, b — strong tournament. c and d are not strong, not transitive

It is known [9] that for $|Y| = 4$ there are four pairwise non-isomorphic tournaments, the form of which is shown in Fig. 2.

Therefore, any tournament containing an isomorphic sub-tournament (either c or d) cannot be homogeneous.

Theorem 1. *Let $|Y| \geqslant 4$. Any tournament with vertices Y, containing no sub-tournaments isomorphic to c or d, is homogeneous.*

Proof. If the tournament is not strong, then there is a sub-tournament in it that is neither strong nor transitive. In this case [6] the vertices of the sub-tournament can be divided into two disjoint and non-empty classes so that all arrows (edges) are directed from one class to another, and at least one of the classes forms a strong sub-tournament. A strong sub-tournament always contains a cyclic triple. Then this cyclic triple, together with any vertex from another class, forms a four-vertex sub-tournament isomorphic to either c, or d. □

Let $I = \{1, 2, ..., m\}$ and $\alpha \subset I$ be a nonempty subset I.

Definition 2. *Two tournament sub-tournaments with vertices from $\alpha \subset I$ and $\beta \subset I$ are called adjacent if $|\alpha| = |\beta|$, and the intersection of these sub-tournaments has the number of vertices equal to $|\alpha| - 1$.*

Let $e_k = (\delta_{1k}, \delta_{2k}, ..., \delta_{mk})$, $k = 1, ..., m$, where δ_{ij} is the Kronecker symbol, there is a standard basis in $\mathbb{R}^m$. Then

$$S^{m-1} = co\{e_1, ..., e_m\} = \{x = (x_1, ..., x_m) : \sum_{i=1}^{m} x_i = 1, x_i \geqslant 0\}$$

it is called a simplex. It is clear that S^{m-1} is a convex compact of dimension $m - 1$.

For any non-empty $\alpha \subset I$, put

$$\Gamma_\alpha = co\{e_i : i \in \alpha\}.$$

Γ_α is called $(|\alpha| - 1)$ is the dimensional face of the simplex S^{m-1}. Obviously, any face of S^{m-1} is also a simplex.

The concept of adjacency for the faces of S^{m-1} is defined in the same way as for the tournament sub-tournaments. Two faces having equal dimensions are considered adjacent if their intersection has a dimension 1 less than the original ones. For example, two edges are adjacent only if they have a common vertex.

Let $A = (a_{ij})$, $i, j = 1, ..., m$ be a real skew-symmetric matrix acting in $\mathbb{R}^m$. Then Ax and x are orthogonal, that is, $(Ax, x) = 0$ for all $x \in \mathbb{R}^m$. It is easy to prove that the converse statement is also true. If $(Ax, x) = 0$ for all $x \in \mathbb{R}^m$, then the matrix A is skew-symmetric.

For $\alpha \subset I$, we put $A_\alpha = (a_{ij})$, where $i, j \in \alpha$. In this case, A_α is called the main submatrix of the matrix. It is clear that A_α is also skew-symmetric. Let $|A_\alpha|$ be the determinant of the matrix A. Obviously, $|A_\alpha| = 0$ if $|\alpha|$ — is odd, and $|A_\alpha| \geqslant 0$ if $|\alpha|$ — is even.

If $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ — points from $\mathbb{R}^m$, then $x \geqslant y$ means that $x_i \geqslant y_i$ for all $i = \overline{1, m}$.

Theorem 2. *If A is a skew-symmetric matrix, then*

$$P = \{x \in S^{m-1} : Ax \geqslant 0\}$$

— a nonempty convex polyhedron.

Proof. We reduce the proof to Sperner's lemma [8], which states that if the closed sets $F_1, ..., F_m$ are such that

$$\Gamma_\alpha \subset \bigcup_{i \in \alpha} F_i$$

for all $\alpha \subset I$, then $\bigcap\limits_{i=1}^{m} F_i \neq \varnothing$.

Let $F_k = \{x \in S^{m-1} : \sum\limits_{i=1}^{m} a_{ki} x_i \geqslant 0\}$, $k = 1, ..., m$ and $f_k(x) = \sum\limits_{i=1}^{m} a_{ki} x_i$.

It is clear that F_k — closed convex sets. Since A is skew-symmetric, then

$$f_k(e_k) = a_{kk} = 0.$$

Therefore, $e_k \in F_k$ when $k = 1, ..., m$.

Let $\alpha = \{1, 2, ..., t\}$ and $x \in \Gamma_\alpha$ be represented as $x = \sum\limits_{i=1}^{t} \lambda_i e_i$, where $\lambda_i \geqslant 0$ and $\sum\limits_{i=1}^{t} \lambda_i = 1$. Then

$$\begin{cases} f_1(x) = \lambda_2 a_{12} + \lambda_3 a_{13} + ... + \lambda_t a_{1t}, \\ f_2(x) = \lambda_1 a_{21} + \lambda_3 a_{23} + ... + \lambda_t a_{2t}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ f_t(x) = \lambda_1 a_{t1} + \lambda_2 a_{t2} + ... + \lambda_{t-1} a_{tt-1}, \end{cases} \tag{1}$$

as $a_{kk} = 0$.

Multiplying in the system (1) the first equality by λ_1, the second by λ_2, etc., then summing up the resulting equalities, due to the skew symmetry of A_α, we get

$$\sum_{i=1}^{t} \lambda_i f_i(x) = 0. \tag{2}$$

Since $\lambda_i \geqslant 0$ and at least one λ_i is positive, it follows from (2) that at least one of the numbers $f_1(x), ..., f_t(x)$ is non-negative. Therefore,

$$\Gamma_\alpha \subset \bigcup_{i=1}^{t} F_i.$$

Thus, $P = \bigcap_{i=1}^{m} F_i$— a nonempty set.

The fact that P is a convex polyhedron obviously follows from the fact that F_k is a closed part of a half-space, and S^{m-1} is a convex polyhedron. □

Corollary 1. *$Q = \{x \in S^{m-1} : Ax \leqslant 0\}$ — a nonempty convex polyhedron.*

Indeed, in Theorem 2 it is enough to replace the matrix A with $-A$.

Example 1. *If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $Q = (1, 0)$ and $P = (0, 1)$.*

Example 2. *If $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, then $P = Q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.*

Example 3. *If $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, then $P = (0, \lambda, 1-\lambda)$, where $0 \leqslant \lambda \leqslant 1$ and $Q = (1, 0, 0)$.*

Example 4. *Let $A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 0.5 & -1 \\ 1 & -0.5 & 0 & 0.5 \\ -1 & 1 & -0.5 & 0 \end{pmatrix}$,*

then $P = Q = (0.2\lambda; 0.15\lambda + 0.25; 0.5 - 0.1\lambda; 0.25 - 0.25\lambda)$, where $0 \leqslant \lambda \leqslant 1$.

Definition 3. *$A = (a_{ki})$ is called a skew-symmetric matrix of general position if $|A_\alpha| > 0$ for all $\alpha \subset I$ such that $|\alpha|$ — an even number.*

It is easy to notice that in the examples 3 and 4, the matrix A is not a matrix of general position.

Theorem 3. *The set of all skew-symmetric matrices of general position is open and everywhere dense in the set of all skew-symmetric matrices.*

Proof. The theorem is proved in the work [1]. □

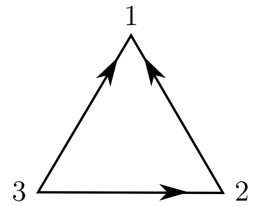
In particular, if $|\alpha| = 2$, $\alpha = \{k, i\}$, then $A_\alpha = \begin{pmatrix} 0 & a_{ki} \\ -a_{ki} & 0 \end{pmatrix}$. Therefore, $|A_\alpha| > 0$ means that $a_{ki} \neq 0$ for all $k \neq i$, which allows you to build a tournament corresponding to the matrix A.

We can introduce the concept of a tournament along with a skew-symmetric matrix corresponding to the mapping Lotka–Volterra [1,4]. We mark the elements of the set $I = \{1, ..., m\}$ as points and connect the point k with the point i with an arrow directed from k to i if $a_{ki} < 0$, and vice versa if $a_{ki} > 0$. The resulting oriented graph is called a tournament [3,8,9].

For example, the skew-symmetric matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ corresponds to the tournament

It is clear that in this example A is a general position matrix, and the tournament is a transitive triple.

Theorem 4. *If A is a skew-symmetric matrix of general position, then the sets P and Q consist of a single point.*



Proof. Let P have more than one point, hence P is an infinite set. Since the number of faces S^{m-1} is of course some face Γ_α has at least two points from P, and these points are internal to Γ_α.

a) Let these points belong to the interior of S^{m-1}, that is, all their coordinates are positive. Since

$$Ax \geqslant 0 \quad \text{and} \quad (Ax, x) = 0,$$

then $Ax = 0$, since all $x_i > 0$, $i = \overline{1, m}$.

Similarly, $Ay = 0$, where $y \in P$, and all coordinates of y are positive. It is clear that two different points from the simplex are linearly independent. Therefore, $\dim\mathrm{Ker}A \geqslant 2$. Since A is a general position matrix, then $\dim\mathrm{Ker}A \leqslant 1$. We get a contradiction.

b) If P is contained in some face of Γ_α, then instead of A consider A_α. It is clear that A_α is also a skew-symmetric matrix of general position of dimension $|\alpha| \times |\alpha|$. If $x \in \Gamma_\alpha$, then x_α the same point x, but only with coordinates from α. Then $Ax \geqslant 0$ follows $A_\alpha x_\alpha \geqslant 0$ provided that $x \in \Gamma_\alpha$. Hence, the case of b) reduces to the case a). □

2. Results

Let $A = (a_{ij})$ be an arbitrary skew-symmetric matrix with the condition $a_{ij} \neq 0$ for $i \neq j$. Consider the mapping $V : \mathbb{R}^m \to \mathbb{R}^m$, given by equalities

$$x_k^{'} = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k = \overline{1, m}, \tag{3}$$

where $Vx = (x_1^{'}, ..., x_m^{'})$.

It is known [1] that for $V : S^{m-1} \to S^{m-1}$ it is necessary and sufficient that $|a_{ki}| \leqslant 1$ for all $k, i = 1, ..., m$, and V is a homeomorphism of S^{m-1} on itself. Since $x_k = 0$ implies that $x_k^{'} = 0$, then any face Γ_α of the simplex S^{m-1} is invariant, that is, $V(\Gamma_\alpha) = \Gamma_\alpha$, in particular, all vertices S^{m-1} — fixed points.

Mapping (3) at $|a_{ki}| \leqslant 1$ is called mapping Lotka–Volterra. Next, we consider $V : S^{m-1} \to S^{m-1}$ only as a mapping that translates the probability distribution of the system from m types also into the probability distribution. Since $a_{ij} \neq 0$ for $i \neq j$, we construct a tournament T corresponding to the matrix A.

Theorem 5. *If the face Γ_α of the simplex S^{m-1} has an internal (relative to the face) fixed point, then the tournament sub-tournament T with vertices from α is strong.*

Proof. Since all faces of S^{m-1} are invariant with respect to V, we can assume that $\Gamma_\alpha = S^{m-1}$, that is, $\alpha = I = \{1, ..., m\}$. Let's say that T — is not a strong tournament. Then [1,3] the set I can be divided into two non-empty classes so that an edge connecting two vertices from different classes is always directed from the first class to the second.

Let $I_1 = \{1, 2, ..., t\}$ and $I_2 = \{t+1, t+2, ..., m\}$, then $a_{ij} < 0$ for all $i \in I_1$ and $j \in I_2$.

According to (3), we write out the first t coordinates:

$$\begin{cases} x_1^{'} = x_1 \left(1 + \sum\limits_{i=1}^{t} a_{1i}x_i + \sum\limits_{i=t+1}^{m} a_{1i}x_i\right), \\ x_2^{'} = x_2 \left(1 + \sum\limits_{i=1}^{t} a_{2i}x_i + \sum\limits_{i=t+1}^{m} a_{2i}x_i\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ x_t^{'} = x_1 \left(1 + \sum\limits_{i=1}^{t} a_{ti}x_i + \sum\limits_{i=t+1}^{m} a_{ti}x_i\right). \end{cases} \tag{4}$$

Summing up these equalities, we get

$$\sum_{j=1}^{t} x_j^{'} = \sum_{j=1}^{t} x_j + \sum_{j=1}^{t}\sum_{i=1}^{t} a_{ji}x_jx_i + \sum_{j=t+1}^{m} a_{ji}x_ix_j, \tag{5}$$

where the second term in the right part is zero, since the submatrix of the matrix A corresponding to I_1 is skew-symmetric. For all interior points $x_i > 0$, so

$$\sum_{j=t+1}^{m} a_{ji}x_ix_j < 0.$$

Hence, from (5) we get

$$\sum_{j=1}^{t} x_j^{'} < \sum_{j=1}^{t} x_j \tag{6}$$

for all internal points of the simplex S^{m-1}. Then $V : S^{m-1} \to S^{m-1}$ cannot have internal fixed points. We get a contradiction. □

Corollary 2. *If the T_α sub-tournament corresponding to the Γ_α face is transitive, then V on Γ_α has no fixed points except the vertices of this face.*

Theorem 6. *For the existence of a fixed point V with three positive coordinates, say x_i, x_j, x_k, it is necessary and sufficient that the sub-tournament T with vertices i, j and k is isomorphic to a cyclic triple.*

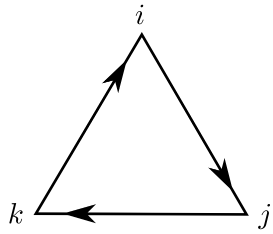
Proof. Necessity follows from Theorem 5. To prove sufficiency, we take the narrowing of V to the edge Γ_α

$$\begin{aligned} x_i^{'} &= x_i(1 - a_{ij}x_j + a_{ik}x_k), \\ x_j^{'} &= x_j(1 + a_{ij}x_j - a_{jk}x_k), \\ x_k^{'} &= x_k(1 - a_{ik}x_i + a_{jk}x_j), \end{aligned}$$

where $\alpha = \{i, j, k\}$, and a_{ij}, a_{ik}, a_{jk} are positive.

Then the mapping V to Γ_α has a single internal fixed point with coordinates

$$\left(\frac{a_{jk}}{a_{ij} + a_{ik} + a_{jk}}, \frac{a_{ik}}{a_{ij} + a_{ik} + a_{jk}}, \frac{a_{ij}}{a_{ij} + a_{ik} + a_{jk}}\right).$$

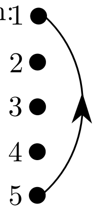


$\square$

Let $|Y| = 5$, then the dynamic system looks like this:

$$\begin{cases} x_1' = x_1(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 + a_{15}x_5), \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 - a_{25}x_5), \\ x_3' = x_3(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - a_{35}x_5), \\ x_4' = x_4(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5), \\ x_5' = x_5(1 - a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4). \end{cases}$$

The corresponding tournament has the form:



There are three strong sub-tournaments with three vertices — these are $\overline{125}$, $\overline{135}$ and $\overline{145}$. All these sub-tournaments are — strong and adjacent, since the intersection of any of the two is— a one-dimensional edge.

Consider for the faces $\alpha = \{1, 2, 5\}$ and $\beta = \{1, 3, 5\}$, $\gamma = \alpha \cup \beta = \{1, 2, 3, 5\}$ matrix narrowing A:

$$A_\gamma = \begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{15} \\ a_{12} & 0 & -a_{23} & -a_{25} \\ a_{13} & a_{23} & 0 & -a_{35} \\ -a_{15} & a_{25} & a_{35} & 0 \end{pmatrix}.$$

Calculating the determinant of the matrix, we get $|A_\gamma| = (a_{13}a_{25} - a_{12}a_{35} + a_{15}a_{23})^2$, expressions in parentheses are denoted by $\Delta_1 = a_{13}a_{25} - a_{12}a_{35} + a_{15}a_{23}$.

The mapping corresponding to the above matrix A_γ has the form (narrowing the mapping to a face Γ_γ):

$$\begin{cases} x_1' = x_1(1 - a_{12}x_2 - a_{13}x_3 + a_{15}x_5), \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{25}x_5), \\ x_3' = x_3(1 + a_{13}x_1 + a_{23}x_2 - a_{35}x_5), \\ x_5' = x_5(1 - a_{15}x_1 + a_{25}x_2 + a_{35}x_3). \end{cases}$$

On this face Γ_γ there are fixed points with coordinates

$$x_\alpha = \left(\frac{a_{25}}{a_{12} + a_{25} + a_{15}}; \frac{a_{15}}{a_{12} + a_{25} + a_{15}}; 0; \frac{a_{12}}{a_{12} + a_{25} + a_{15}}\right),$$

$$x_\beta = \left(\frac{a_{35}}{a_{13} + a_{35} + a_{15}}; 0; \frac{a_{15}}{a_{13} + a_{35} + a_{15}}; \frac{a_{13}}{a_{13} + a_{35} + a_{15}}\right).$$

In order to find the sets P and Q, we find $A_\gamma x_\alpha$ and $A_\gamma x_\beta$:

$$A_\gamma x_\alpha = \frac{1}{a_{12}+a_{25}+a_{15}} \begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{15} \\ a_{12} & 0 & -a_{23} & -a_{25} \\ a_{13} & a_{23} & 0 & -a_{35} \\ -a_{15} & a_{25} & a_{35} & 0 \end{pmatrix} \begin{pmatrix} a_{25} \\ a_{15} \\ 0 \\ a_{12} \end{pmatrix} = \frac{1}{a_{12}+a_{15}+a_{25}} (0; 0; \Delta_1; 0)\,,$$

$$A_\gamma x_\beta = \frac{1}{a_{13}+a_{15}+a_{35}} (0; -\Delta_1; 0; 0)\,.$$

Here A_γ is a general position matrix, since $\Delta_1 \neq 0$.

If $\Delta_1 > 0$ then $A_\gamma x_\alpha \geqslant 0$ and $A_\gamma x_\beta \leqslant 0$, means fixed points x_α and x_β make up a pair of (P, Q) on the face of Γ_γ. This means that the direction is set from the point $P = x_\alpha$ to the point $Q = x_\beta$. If $\Delta_1 < 0$, then we get the opposite.

Now let's move on to other adjacent faces. Let $\alpha = \{1, 2, 5\}$ and $\beta = \{1, 4, 5\}$, then $\gamma = \alpha \cup \beta = \{1, 2, 4, 5\}$. Here, the narrowing of the matrix A by Γ_γ has the form:

$$A_\gamma = \begin{pmatrix} 0 & -a_{12} & -a_{14} & a_{15} \\ a_{12} & 0 & -a_{24} & -a_{25} \\ a_{14} & a_{24} & 0 & -a_{45} \\ -a_{15} & a_{25} & a_{45} & 0 \end{pmatrix}.$$

We also calculate $|A_\gamma| = (a_{14}a_{25} - a_{12}a_{45} + a_{15}a_{24})^2$ and $\Delta_2 = a_{14}a_{25} - a_{12}a_{45} + a_{15}a_{24}$.

The narrowing of the mapping on this face $\Gamma_\gamma = \Gamma_{1245}$ has two fixed points $x_\alpha = x_{125}$ and $x_\beta = x_{145}$.

For each of these points, we determine their character, for this we find

$$A_\gamma x_\alpha = \frac{1}{a_{12}+a_{15}+a_{25}} (0; 0;\ \Delta_2; 0)\,,$$

$$A_\gamma x_\beta = \frac{1}{a_{14}+a_{15}+a_{45}} (0; -\Delta_2; 0; 0)\,.$$

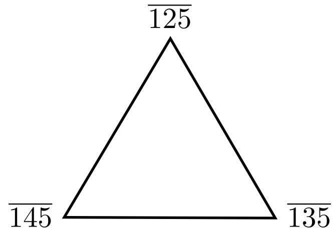
If $\Delta_2 > 0$, then $P = x_\alpha$ and $Q = x_\beta$, and, inversely, if $\Delta_2 < 0$, then $P = x_\beta$ and $Q = x_\alpha$. Let's move on to the last one, let $\alpha = \{1, 3, 5\}$, $\beta = \{1, 4, 5\}$, then $\gamma = \alpha \cup \beta = \{1, 3, 4, 5\}$. Having done the same calculations, we get

$$A_\gamma x_\alpha = \frac{1}{a_{13}+a_{15}+a_{35}} (0; 0; \Delta_3; 0)\,,$$

$$A_\gamma x_\beta = \frac{1}{a_{14}+a_{15}+a_{45}} (0; -\Delta_3; 0; 0)\,.$$

Here $\Delta_3 = a_{14}a_{35} - a_{13}a_{45} + a_{15}a_{34}$ — determinant of matrix narrowing A_γ. If $\Delta_3 > 0$, then $P = x_\alpha$, $Q = x_\beta$ and inversely, if $\Delta_3 < 0$, then $P = x_\beta$, $Q = x_\alpha$.

As a result, for a complete study of the picture of the trajectories of the internal points of the simplex, we obtained a sub-tournament, which we will call the map of fixed points. Here the map of fixed points has the form,



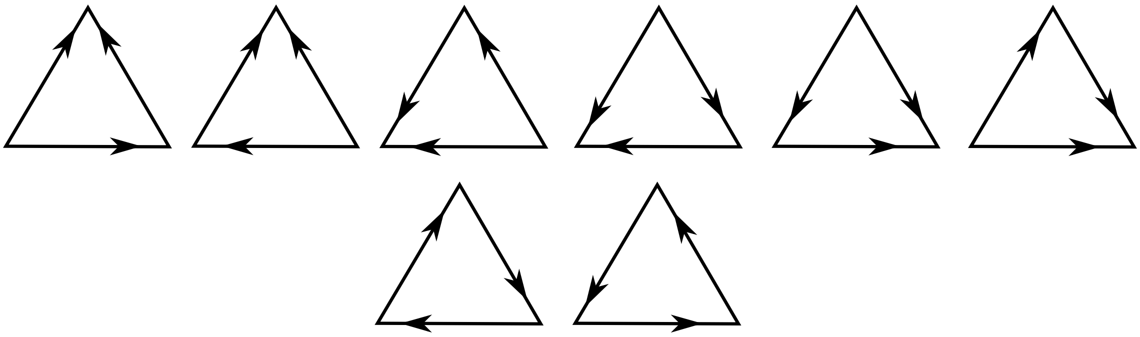
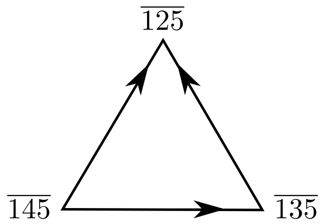


Fig. 3. All types of cards at $m = 3$

in which the directions on the edges are defined by signs Δ_i, $i = 1, 2, 3$.

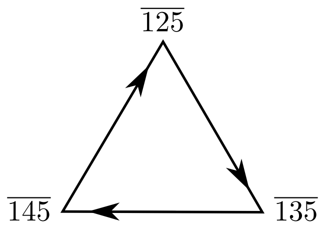
Here we get only $2^3 = 8$ cases of fixed point maps (Fig. 3), among which there are isomorphic [6].

From the figure we see that the first 6 cases are isomorphic; these triples are called transitive. For these six cases , we will focus on the following form:



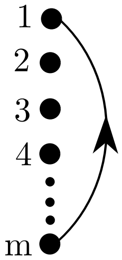
Lemma 1. *If Δ_i $(i = 1, 2, 3)$ have different signs, then a transitive triple is formed in the map of fixed points, and the simplex S^4 has no internal fixed points.*

Let's move on to the last two cases from Fig. 3. These two cases are isomorphic, so we will focus on any of them, for example,



Lemma 2. *If the signs of all Δ_i $(i = 1, 2, 3)$ coincide, then a Hamiltonian cycle (strong triple) is formed in the map, and in the simplex S^4 there is an internal fixed point.*

Let's generalize what was obtained in the previous section to $|Y| = m$. The corresponding tournament, according to [7], has the form



Further, in the skew-symmetric matrix A of the general position, we write only positive $a_{ij}(i \neq j)$, and we will place the signs in front of them in accordance with the tournament T_m.

For example, the studied tournament T_m corresponds to the matrix

$$A = \begin{pmatrix} 0 & -a_{12} & -a_{13} & \dots & -a_{1m-1} & a_{1m} \\ a_{12} & 0 & -a_{23} & \dots & -a_{2m-1} & -a_{2m} \\ a_{13} & a_{23} & 0 & \dots & -a_{3m-1} & -a_{3m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -a_{1m} & a_{2m} & a_{3m} & \dots & a_{m-1m} & 0 \end{pmatrix}.$$

So, the mapping V, constructed by T_m, has m fixed points with one and $(m-2)$ fixed points with three nonzero coordinates for any a_{ij}. The existence of fixed points with 5, 7, 9, etc.,etc. nonzero coordinates belonging to S^{m-1} depends on some inequalities from the coefficients of the skew-symmetric matrix.

To clarify, consider two strong sub-tournaments T_m with three vertices, for example, $\alpha = \{1, 2, m\}$ and $\beta = \{1, 3, m\}$. As noted above, they are adjacent.

Let $\gamma = \alpha \cup \beta = \{1, 2, 3, m\}$, then the narrowing of V_γ by Γ_γ has the form

$$\begin{cases} x_1' = x_1(1 - a_{12}x_2 - a_{13}x_3 + a_{1m}x_m), \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{2m}x_m), \\ x_3' = x_3(1 + a_{13}x_1 + a_{23}x_2 - a_{3m}x_m), \\ x_m' = x_m(1 - a_{1m}x_1 + a_{2m}x_2 + a_{3m}x_3). \end{cases}$$

Then on Γ_γ we have two fixed points:

$$x_\alpha = \left(\frac{a_{2m}}{a_{12} + a_{2m} + a_{1m}}, \frac{a_{1m}}{a_{12} + a_{2m} + a_{1m}}, 0, \frac{a_{12}}{a_{12} + a_{2m} + a_{1m}}\right),$$

$$x_\beta = \left(\frac{a_{3m}}{a_{13} + a_{3m} + a_{1m}}, 0, \frac{a_{1m}}{a_{13} + a_{3m} + a_{1m}}, \frac{a_{13}}{a_{13} + a_{3m} + a_{1m}}\right)$$

with carriers α and β, respectively.

For $A_\gamma x_\alpha$ and $A_\gamma x_\beta$ we have

$$A_\gamma x_\alpha = \frac{1}{a_{12} + a_{2m} + a_{1m}} \begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{1m} \\ a_{12} & 0 & -a_{23} & -a_{2m} \\ a_{13} & a_{23} & 0 & -a_{3m} \\ -a_{1m} & a_{2m} & a_{3m} & 0 \end{pmatrix} \begin{pmatrix} a_{2m} \\ a_{1m} \\ 0 \\ a_{12} \end{pmatrix} =$$

$$= \frac{1}{a_{12} + a_{2m} + a_{1m}} \left(0, 0, a_{13}a_{2m} + a_{23}a_{1m} - a_{12}a_{3m}, 0\right),$$

$$A_\gamma x_\beta = \frac{1}{a_{13} + a_{1m} + a_{3m}} \left(0, a_{12}a_{3m} - a_{23}a_{1m} - a_{2m}a_{13}, 0, 0\right).$$

Calculating $|A_\gamma|$, we find that

$$|A_\gamma| = (a_{13}a_{2m} + a_{23}a_{1m} - a_{12}a_{3m})^2. \tag{7}$$

Since A is a general position matrix, then

$$a_{13}a_{2m} + a_{23}a_{1m} - a_{12}a_{3m} \neq 0.$$

So if $a_{13}a_{2m} + a_{23}a_{1m} - a_{12}a_{3m} > 0$, then $A_\gamma x_\alpha \geqslant 0$ and $A_\gamma x_\beta \leqslant 0$. Therefore, x_α is a P point, and x_β —Q is a point on the face Γ_γ.

If $a_{13}a_{2m}+a_{23}a_{1m}-a_{12}a_{3m} < 0$, then x_α and x_β are swapped. We formulate these arguments in the form of a theorem.

Theorem 7. *Any two cyclic triples in T_m are adjacent, and of the fixed points defined, one is a – P point, and the other is a– Q point for the face containing them.*

Proof. The theorem can be proved based on the above reasoning. □

Based on Theorem 7, we represent all cyclic triples T_m as points and connect $\overline{1im}$ with $\overline{1jm}$ with an arrow going from P point to Q point. Thus, we get a new tournament of $(m-1)$ points, which we denote by G_{m-1} and call the map of fixed points.

As we have shown above, there are only $2^3 = 8$ possible cases of fixed point maps, among which we have considered two non-isomorphic [3], and the arrows are placed in accordance with theorem 7.

Theorem 8. *If G_{m-1} is a transitive tournament, then the mapping V does not have fixed points with five or more nonzero coordinates in the simplex S^{m-1}.*

Proof. If S^{m-1} has an internal fixed point ($(m-1)$— odd) of the mapping V, say, x, then by all means $Ax = 0$, that is, x is both P, and at the same time Q with a dot. Let x be a fixed point with five nonzero coordinates and belong to the face Γ_γ, where $|\gamma| = 5$. It corresponds to a strong sub-tournament T_γ (Theorem 5). A strong T_m sub-tournament with five vertices has only vertices $1, i, j, k, m$, that is, $\gamma = \{1, i, j, k, m\}$ where $1 < i, j, k < m$ and i, j, k — are different.

Therefore, it has three sub-tournaments $\overline{1im}$, $\overline{1jm}$ and $\overline{1km}$. Since G_{m-1} is transitive, it is one of them that is the P point for the face Γ_γ. Since P is the only point for each face, since A is a matrix of general position, the latter contradicts the fact that a fixed point with five non-zero coordinates is a P point for Γ_γ. □

Theorem 9. *If there is a cyclic triple in G_{m-1}, then there is a fixed point with five non-zero coordinates.*

Proof. Let $\overline{1im}$, $\overline{1jm}$ and $\overline{1km}$ form a cyclic triple in G_{m-1} and $\gamma = \{1, i, j, k, m\}$. Then the fixed points defined by $\overline{1im}$, $\overline{1jm}$ and $\overline{1km}$ cannot be P points for Γ_γ. Therefore, the face Γ_γ must have an internal fixed point. □

Corollary 3. *The number of fixed display points V with five non-zero coordinates is equal to the number of cyclic triples in the map G_{m-1}.*

Conclusion

It is known that dynamical systems originate in mechanics from the works of Henri Poincare, in which it is stated that some systems after some finite time will return to a state very close to the original [10]. In 1988 A.Lyapunov developed methods to determine the stability of ordinary differential equations. In many branches of science, for example, natural sciences and engineering disciplines, the rule of evolution of dynamical systems is described either by a differential or difference equation.

In these systems, given the location of the starting point, it is possible to determine the state in the future — this is a set of points known as a trajectory or orbit, which is why we are interested in finding the equilibrium states of the system.

Quadratic maps of the simplex can be applied to problems of population genetics, epidemiology, ecology, economics. In this paper, the asymptotic behavior of the trajectories of quadratic maps was investigated Lotka–Volterra, operating in a $(m-1)$-dimensional simplex with homogeneous tournaments. These systems with homogeneous tournaments describe the process of ecological circulation, in particular, the model under consideration allows us to more adequately describe the process of the cycle of biogens [11]. Along with discrete dynamical systems, the elements of graph theory were considered in the work, that is, these systems were associated with concepts such as tournaments. The concept of a map of fixed points is introduced. According to the state of the nature of the maps of fixed points, the criteria for the existence of such fixed points are determined, with the help of which the flow of trajectories that allow describing the evolution of the biosphere is described [11].

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