Izvestiya Vysshikh Uchebnykh Zavedeniy. Applied Nonlinear Dynamics. 2022;30(6)

Article DOI: 10.18500/0869-6632-003013

On existence of multistability near the boundary of generalized synchronization in unidirectionally coupled systems with complex topology of attractor

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Received 3.06.2022, accepted 6.07.2022, available online 14.10.2022, published 30.11.2022

Abstract. *Aim* of this work is to study the possibility of existence of multistability near the boundary of generalized synchronization in systems with complex attractor topology. Unidirectionally coupled Lorentz systems have been chosen as an object of study, and a modified auxiliary system method has been used to detect the presence of the synchronous regime. *Result* of the work is a proof of the presence of multistability near the boundary of generalized synchronization in unidirectionally coupled systems with a complex topology of attractor. For this purpose, the basins of attraction of the synchronous and asynchronous states of interacting Lorenz systems have been obtained for the value of the coupling parameter corresponding to the realization of the intermittent generalized synchronization regime in the system under study, and the dependence of the multistability measure on the value of the coupling parameter has also been calculated. It is shown that in the regime of intermittent generalized synchronization the measure of multistability turns out to be positive, which is an additional confirmation of the presence of multistability in this case.

Keywords: generalized synchronization, multistability, systems with complex topology of attractor, intermittency, auxiliary system approach.

Acknowledgements. This work was financially supported by the Grant Council of the President of the Russian Federation for the state support of young Russian scientists — doctors of sciences (project No. MD-18.2022.1.2).

For citation: Moskalenko OI, Evstifeev EV. On existence of multistability near the boundary of generalized synchronization in unidirectionally coupled systems with complex topology of attractor. Izvestiya VUZ. Applied Nonlinear Dynamics. 2022;30(6):676–684. DOI: 10.18500/0869-6632-003013

Multistability is a universal phenomenon being observed in systems of different nature [1]. Multistability means the coexistence of several attractors in the phase space of dynamical system, the choice of which depends on the initial conditions of this system. For the first time the term «multistability» was introduced in the paper [2], dedicated to visual perception. Later, multistability was discovered in almost all fields of science and technology, including electronics, optics, mechanics and biology.

Currently, the phenomenon of multistability is well studied in relation to the autonomous and non-autonomous dynamics of the systems under study (see, for example, [3–7] and others.). However, the analysis of synchronous dynamics of interacting systems and phenomena near the boundaries of various types of synchronization from the standpoint of multistability has not been carried out in detail until now. There are works aimed at studying multistability in the destruction of synchronous regimes from the point of view of bifurcation analysis in discrete maps, genetic elements, laser systems and ensembles of coupled oscillators (see, for example, [8–12]).

Among the known types of synchronization, the least studied from the point of view of multistability is the generalized chaotic synchronization regime [13–15]. This regime means establishing a connection between the states of interacting systems in the form of a functional and can be observed both in the case of unidirectional and mutual coupling between these systems. In both cases, intermittent behavior is observed near the boundary of generalized synchronization, and the type of intermittency implemented in this case does not depend on the nature of the coupling between the systems, but is determined by the topology of the attractors of interacting systems: in systems with a relatively simple topology (with a ribbon-type attractor), there is an intermittency of the type «on – off» [16, 17], while in oscillators with a complex (sheeted) structure, there is an intermittency of jumps [18,19]. For systems with a relatively simple attractor topology, the presence of multistability in the intermittent generalized synchronization regime has recently been discovered [20, 21], while such studies have not been carried out so far in systems with a more complex attractor structure near the boundary of this regime.

Therefore, the purpose of this work is to study the possibility of the existence of multistability near the boundary of generalized synchronization in systems with a complex attractor topology.

Two unidirectionally coupling Lorenz systems were chosen as the object of research [22]:

$$
\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1),\\
\dot{y}_1 &= r_1 x_1 - y_1 - x_1 z_1,\\
\dot{z}_1 &= -b z_1 + x_1 y_1,\\
\dot{x}_2 &= \sigma(y_2 - x_2) + \varepsilon(x_1 - x_2),\\
\dot{y}_2 &= r_2 x_2 - y_2 - x_2 z_2,\\
\dot{z}_2 &= -b z_2 + x_2 y_2,
\end{aligned}
\tag{1}
$$

(where $\mathbf{x}_{1,2} = (x_{1,2}, y_{1,2}, z_{1,2})$ are vectors of states of interacting master and slave systems, $\sigma = 10$, $b = 2$, $r_1 = 40$ и $r_2 = 35$ are control parameters, ε is coupling parameter) located near the boundary of generalized synchronization. In this case, the method of the auxiliary system is traditionally used to diagnose generalized synchronization [23], the essence of which boils down to the introduction of an additional, so-called auxiliary system $\mathbf{x}_3 = (x_3, y_3, z_3)$, identical in control parameters to the drive system, but starting from other initial conditions belonging to the basin of attraction of the same attractor. The Lorenz auxiliary system has the following form:

$$
\begin{aligned}
\dot{x}_3 &= \sigma(y_3 - x_3) + \varepsilon(x_1 - x_3),\\
\dot{y}_3 &= r_3 x_3 - y_3 - x_3 z_3,\\
\dot{z}_3 &= -b z_3 + x_3 y_3,
\end{aligned}
\tag{2}
$$

and is considered exclusively in conjunction with the equation (1). If there is no generalized synchronization between the master $\mathbf{x}_1$ and the slave $\mathbf{x}_2$ oscillators, then the slave $\mathbf{x}_2$ and the auxiliary $\mathbf{x}_3$ systems will evolve on the same attractor, but at the same time in the fixed time, their behavior will be completely different ($\mathbf{x}_2 \neq \mathbf{x}_3$). In generalized synchronization regime due to the establishment of a functional relationship between the states of the drive $\mathbf{x}_1$ and the drive $\mathbf{x}_2$ systems, as well as the drive $\mathbf{x}_1$ and auxiliary $\mathbf{x}_3$ systems, the states of the drive and auxiliary systems, they should become identical after the transition process is completed (that is, $\mathbf{x}_2 = \mathbf{x}_3$ at any given time).

As the calculations have shown, the generalized synchronization regime in the system (1) occurs when $\varepsilon = 11.5$. Below the boundary of this regime, as noted above, intermittent behavior is observed. In this case, the functional relationship between the states of interacting systems is not always established, but only at certain time intervals, called laminar phases or phases of synchronous behavior of systems. At the remaining moments of time, called turbulent bursts, the generalized synchronization regime is not observed. Thus, in the regime of intermittent generalized synchronization, there is an alternation of phases of synchronous and asynchronous behavior of interacting systems, and it is also possible to use the auxiliary system method to diagnose this regime. To do this, it is necessary to analyze the differences between the states of the slave and auxiliary systems and determine the duration of the characteristic phases of behavior.

At the same time, as the studies show, with fixed initial conditions of the master and auxiliary systems and different initial conditions of the slave system in the system (1)–(2) at the same moment, both the same and different phases of behavior (synchronous or asynchronous) can be observed. This indicates the presence of multistability near the boundary of generalized synchronization. To illustrate arguments mentioned, Fig. 1 shows the basin of attraction of the slave Lorenz system (1), received at various points in time with the value of the coupling parameter $\varepsilon = 8.8$, which corresponds to the intermittent generalized synchronization regime. Initial conditions for the drive (1) and auxiliary (2) systems, as noted above, have always been selected fixed. For the slave system, the coordinate y_2 was fixed, and the coordinates x_2, z_2 varied, as shown in the figure. In Fig. 1 dark color corresponds to phases of synchronous behavior (in the sense of generalized synchronization), light — asynchronous. The white color corresponds to the jump of the image point to infinity. It can be seen from the figures that at all the considered moments of time near the boundary of generalized synchronization, multistability takes place in the system under study.

To quantify the degree of multistability and diagnose generalized synchronization, taking into account this feature, it is necessary to consider an ensemble of drive Lorentz oscillators under the influence of the same master system:

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1),\\
\dot{y}_1 &= r_1 x_1 - y_1 - x_1 z_1,\\
\dot{z}_1 &= -b z_1 + x_1 y_1,\\
\dot{x}_2^i &= \sigma(y_2^i - x_2^i) + \varepsilon(x_1 - x_2^i),\\
\dot{y}_2^i &= r_2 x_2^i - y_2^i - x_2^i z_2^i,\\
\dot{z}_2^i &= -b z_2^i + x_2^i y_2^i,
\end{aligned} \tag{3}$$

with the same values of control parameters as for the system (1)–(2), and differing values of the

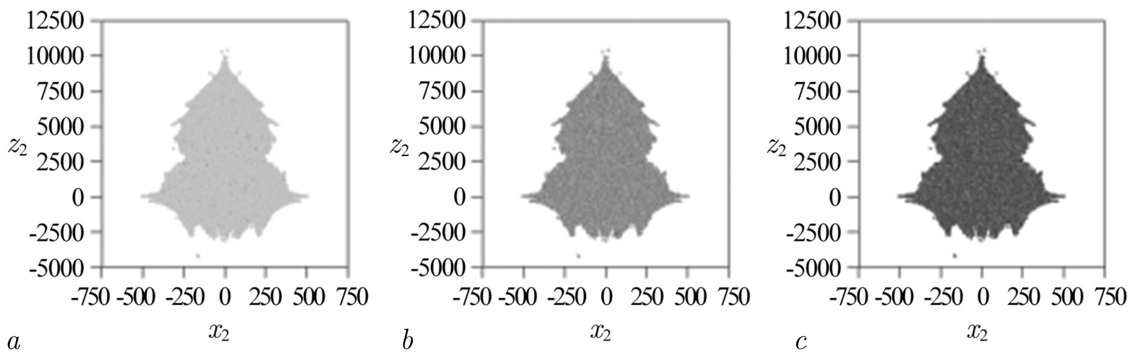

Fig. 1. Basins of attraction of synchronous and asynchronous states of the response Lorenz system (1) being un the intermittent generalized synchronization regime with the drive system for the coupling parameter value $\varepsilon = 8.8$ on the plane of initial conditions (x_2, z_2) $(y_2 = 1.1)$ obtained in different moments of time: $t = 20000$ (a), 40000 (b), 70000 (c). Dark color corresponds to the realization the generalized synchronization regime in system (1) for a fixed moment of time, light color refers to the asynchronous regime. White color corresponds to the going the representation point to infinity

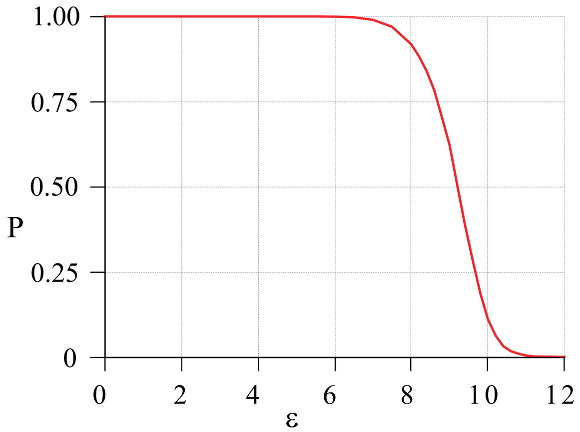

Fig. 2. Dependence of the multistability measure P on the coupling parameter ε obtained by means of the modified auxiliary system method for the system (3)

initial conditions of the drive systems, uniformly distributed over the attractors of these systems. Here $i = 1 \dots N$, $N = 4000$ is the number of elements in the ensemble, $\mathbf{x}_1 = (x_1, y_1, z_1)$ and $\mathbf{x}_2^i = (x_2^i, y_2^i, z_2^i)$ are vectors of states of interacting master and slave systems, respectively. In this case, it is advisable to carry out diagnostics of generalized synchronization using the modified method of the auxiliary system proposed in the work [21], according to which it is necessary to compare the states of the slave systems with each other (in fact, to compare the states of the slave and auxiliary systems under different initial conditions) and calculate the so-called multistability measure depending on the magnitude of the coupling parameter. As a measure of multistability by analogy with work [21] the probability of observing asynchronous regime, calculated as

$$P_a = 1 - \sum_{i=1}^{N} \frac{n(\mathbf{x}_2^i)}{N(N-1)}, \tag{4}$$

where $n(\mathbf{x}_2^i)$ is the number of systems whose states at a given time coincide with the state of the i th oscillator. The coincidence of the states of the two slave systems with each other, as noted above, according to the classical method of the auxiliary system [23], means that they are in generalized synchronization regime with the master system. Then it is clear that if all the slave systems are in generalized synchronization regime with the master system, then $P_a = 0$. Similarly, if asynchronous behavior is observed for all systems at a given time, then $P_a = 1$. An intermediate variant is of interest, when only a part of the systems demonstrates synchronous behavior, and the rest is in asynchronous regime. In this case, $P_a \in (0, 1)$, and near the boundary of generalized synchronization, multistability takes place.

Fig. 2 shows the dependence of the time-averaged probability of observing asynchronous regime

$$\mathrm{P} = \int_0^T P_a(t)dt, \tag{5}$$

received for the system (3), from the coupling parameter ε. It can be seen that as the coupling parameter increases, the measure of multistability monotonically decreases from 1 to 0, reflecting the transition from the asynchronous state to the generalized synchronization regime. Near the boundary of the occurrence of synchronous regime, this measure is different from zero, which is an additional confirmation of the presence of multistability near the boundary of generalized synchronization in the system under study.

Thus, in this paper, using the example of unidirectionally coupled Lorentz systems, it is shown that multistability takes place at the boundary of generalized synchronization in systems with a complex attractor topology. The conclusions are confirmed by constructing maps of the attraction basins of synchronous and asynchronous states of interacting systems, as well as by calculating the measure of multistability depending on the magnitude of the coupling parameter. It is established that, by analogy with systems with a relatively simple attractor topology in the regime of intermittent generalized synchronization of systems with a relatively complex structure, the measure of multistability turns out to be positive, which proves the presence of multistability in this case.

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