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New Lagrangian view of vorticity evolution in two-dimensional flows of liquid and gas

G. B. Sizykh

Moscow Institute of Physics and Technology, Russia
E-mail: o1o2o3@yandex.ru
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Abstract. *Purpose* of the study is to obtain formulas for such a velosity of imaginary particles that the circulation of the velosity of a (real) fluid along any circuit consisting of these imaginary particles changes (in the process of motion of imaginary particles) according to a given time law. (Until now, only those velositys of imaginary particles were known, at which the mentioned circulation during the motion remained unchanged). *Method.* Without implementation of asymptotic, numerical and other approximate methods, a rigorous analysis of the dynamic equation of motion (flow) of any continuous fluid medium, from an ideal liquid to a viscous gas, is carried out. Plane-parallel and nonswirling axisymmetric flows are considered. The concept of motion of imaginary particles is used, based on the K. Zoravsky criterion (which is also called A. A. Fridman's theorem). *Results.* Formulas for the velosity of imaginary particles are proposed. These formulas include the parameters of the (real) flow, their spatial derivatives and the function of time, which determines the law of the change in time of the (real fluid) velocity circulation along the contours moving together with the imaginary particles. In addition, it turned out that for a given function of time (and, as a consequence, for a given law of change in circulation with respect to time), the velosity of imaginary particles is determined ambiguously. As a result, a method is proposed to change the velosity and direction of motion of imaginary particles in a certain range (while maintaining the selected law of changes in circulation in time). For a viscous incompressible fluid, formulas are proposed that do not include pressure and its derivatives. *Conclusion.* A new Lagrangian point of view on the vorticity evolution in two-dimensional flows of fluids of all types is proposed. Formulas are obtained for the velocity of such movement of contours, at which the real fluid velocity circulation along any contour changes according to a given time law. This theoretical result can be used in computational fluid dynamics to limit the number of domains when using a gridless method for calculating flows of a viscous incompressible fluid (the method of viscous vortex domains).

Keywords: velocity of contours motion, circulation of velocity, velocity of imaginary particles, Zoravsky criterion, Friedmann's theorem, method of viscous vortex domains.

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Introduction

The mesh-free discrete vortex method (used to calculate vortex flows of an ideal incompressible fluid [1–3]) was extended to two-dimensional flows of a viscous incompressible fluid after the appearance of analogues of Helmholtz's vortex theorems for such flows [4, 5]. In articles [4, 5], velocities (not coinciding with the velocity of the liquid) of such a motion of the contours were found, in which the circulation of the velocity of the liquid along the contours remains constant. For two-dimensional, that is, for plane-parallel and non-twisted axisymmetric flows of a viscous incompressible fluid, expressions for the transfer rate of contours can be represented by one general formula: $\mathbf{U} = \mathbf{V} - \nu[\boldsymbol{\Omega} \times \mathrm{curl}\boldsymbol{\Omega}]/\Omega^2$, where $\mathbf{V}$ — fluid velocity, $\boldsymbol{\Omega} = \mathrm{curl}\mathbf{V}$ — vorticity, ν — kinematic viscosity coefficient, $\times$ — sign of the vector product. Using these velocities, the so-called viscous vortex domain (VVD) method was developed to calculate the flows of a viscous incompressible fluid [6]. In popular science form, the essence of the VVD method is summarized in the introduction to the article [7], in which the existence of the velocity $\mathbf{U}$ for spatial fluid flows of any type is proved. One of the difficulties that arises when implementing the VVD method is the "generation" of new vortex domains at each time step. This leads to the need to apply different approaches for the redistribution of domains and their intensity in order to limit the total number of domains located in the flow area [8–10]. Preserving the circulation of the fluid velocity along the contour (preserving the intensity of the vortex domain), which moves at a velosity of $\mathbf{U}$, is not necessary for the VVD method. It is enough that the law of change of this circulation in time is known. Therefore, if we find a velosity $\mathbf{U}$ at which the intensity of the domain tends to zero fairly quickly with increasing time, the presence of each domain can be neglected after a finite number of time steps. As a result, the number of domains taken into account will be limited "naturally".

Thus, computational fluid dynamics has set an important task for theoretical fluid dynamics. It is necessary to find such a velocity $\mathbf{U}$ that the intensity of the domain varies according to a predetermined time law. The purpose of this article is to find an expression for such a velocity in terms of flow parameters, their derivatives, and an arbitrarily given law of domain intensity variation over time.

Since the possibility of extending the discrete vortex method to other types of flows is not excluded, the search for the velocity $\mathbf{U}$ is carried out for all types of liquids (from an ideal liquid to a viscous gas). In this case, only such areas of the vortex ($\boldsymbol{\Omega} \neq 0$) are considered fluid flows in which all hydrodynamic parameters and velocity $\mathbf{U}$ are twice continuously differentiable in spatial coordinates and time.

1. The Zoravsky Criterion

We follow [11–17] and for the formulation of statements we will use the representation of the movement of imaginary particles inside the liquid, which is proposed in [11].

Let the spatial domain G be located inside a fluid with a velocity field $\mathbf{V}(x, y, z, t)$ and in it this field is a vortex ($\boldsymbol{\Omega} = \mathrm{curl}\,\mathbf{V} \neq 0$) for some open period of time. In the region of G, we also consider the flow of an imaginary fluid whose particles move at a velosity of $\mathbf{U}(x, y, z, t)$. The particles of an imaginary liquid do not interact with the particles of a real liquid and do not affect its movement. Let in the domain G during the time interval (t_1, t_2) the vorticity of a real fluid $\boldsymbol{\Omega}$ and the velocity of an imaginary fluid $\mathbf{U}$ are related by the equation

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \mathrm{curl}\,(\boldsymbol{\Omega} \times \mathbf{U}) = 0. \tag{1}$$

In this case, from the Zoravsky criterion [18, 19], which is also called Friedman's theorem [20], it follows that in the interval (t_1, t_2) segments of vortex lines and vortex tubes move together with particles of an imaginary medium that move at a velosity of $\mathbf{U}$. At the same time, the intensity of the vortex tubes (circulation of Γ velocity $\mathbf{V}$ along the contour once encircling the tube) is preserved as long as these particles are inside the G region.

This consequence of the Zoravsky criterion will be used below.

2. The general case of continuous fluid

The dynamic equation of motion of a continuous fluid can always be represented as $\partial \mathbf{V}/\partial t + \Omega \times \mathbf{V} = \mathbf{F}_0 - \nabla(\mathbf{V}^2/2)$, where $\mathbf{F}_0$ — the density of the distribution of the resultant of all forces applied to a liquid or gas, related to the density of the liquid or gas. By liquid we will understand both liquid and gas (liquid can be compressible). Sometimes it is convenient to isolate the potential component $\mathbf{F}_0$ and present the dynamic equation as

$$\frac{\partial \mathbf{V}}{\partial t} + \Omega \times \mathbf{V} = \mathbf{F} - \nabla f, \tag{2}$$

where f — some scalar field (gradient ∇f includes $\nabla(\mathbf{V}^2/2)$). For example, the dynamic Navier equation–Stokes equation for a viscous incompressible fluid is represented as follows:

$$\frac{\partial \mathbf{V}}{\partial t} + \Omega \times \mathbf{V} = \mathbf{F} - \nabla \left[\frac{p}{\rho} + \frac{\mathbf{V}^2}{2} + \Pi\right], \quad \mathbf{F} = -\nu \operatorname{curl} \Omega, \tag{3}$$

where p — pressure, ρ — density, Π — potential of mass forces.

Let's turn to the consideration of two-dimensional flows. Let $\alpha(t)$ — any smooth function of time $t \in (t_1, t_2)$, ∇g — gradient of any twice continuously differentiable in time and space function g. For plane-parallel flow, the vectors ∇g, ∇f and $\mathbf{F}$ lie in the flow plane, and for axisymmetric — have a zero circumferential component.

Let's use the orthogonality property of the velocity vectors $\mathbf{V}$ and vorticity Ω in two-dimensional flows. We expand three double vector products (taking into account $\Omega \cdot \mathbf{V} = (\Omega \cdot \nabla g) = (\Omega \cdot \mathbf{F}) = 0$) and we get that the equation

$$\partial \mathbf{V}/\partial t + \Omega \times \mathbf{U} = -\alpha(t)\mathbf{V} - \nabla(f + g), \tag{4}$$

where

$$\mathbf{U} = \mathbf{V} + [\Omega \times \mathbf{F}]/\Omega^2 + \alpha[\Omega \times \mathbf{V}]/\Omega^2 + [\Omega \times \nabla g]/\Omega^2, \tag{5}$$

equivalent to the equation (2).

Along with the velocity $\mathbf{U}$, consider the velocity of another imaginary fluid $\tilde{\mathbf{V}} = \mathbf{V} \exp\{\int_{t_1}^{t} \alpha(\tau) d\tau\}$. The vorticity of $\tilde{\Omega}$ velocity $\tilde{\mathbf{V}}$ is equal to $\tilde{\Omega} = \Omega \exp\{\int_{t_1}^{t} \alpha(\tau) d\tau\}$. Substitute the expressions $\mathbf{V} = \tilde{\mathbf{V}} \exp\{-\int_{t_1}^{t} \alpha(\tau) d\tau\}$ and $\Omega = \tilde{\Omega} \exp\{-\int_{t_1}^{t} \alpha(\tau) d\tau\}$ into the equation (4). After canceling and multiplication by $\exp\{\int_{t_1}^{t} \alpha(\tau) d\tau\}$ we get

$$\frac{\partial \tilde{\mathbf{V}}}{\partial t} + \tilde{\Omega} \times \mathbf{U} = -\exp\left\{\int_{t_1}^{t} \alpha(\tau) d\tau\right\} \nabla(f + g). \tag{6}$$

Application of the operator to equation (6) leads to an equation of the form (1). The latter means (see the text after the formula (1)) that the circulation of $\tilde{\Gamma}$ velocity $\tilde{\mathbf{V}}$ along the contours that move along with the particles of the imaginary fluid at a velosity (5), persists over time. Since the velocity $\mathbf{V}$ is related to the velocity $\tilde{\mathbf{V}}$ by the ratio $\mathbf{V} = \tilde{\mathbf{V}} \exp\{-\int_{t_1}^{t} \alpha(\tau) d\tau\}$, then the circulation Γ of the velocity of the (real) liquid $\mathbf{V}$ along each contour that moves at a velosity (5) changes over time according to the law

$$\Gamma(t) = \Gamma(t_1) \exp\left\{-\int_{t_1}^{t} \alpha(\tau) d\tau\right\}. \tag{7}$$

The formulas (5) and (7) are the main result of this article. The proper choice of the function $\alpha(t)$ allows you to set the law of change of circulation over time. The choice of the function g — allows you to change the magnitude and direction of the velocity of imaginary particles $\mathbf{U}$. At the same time, as noted in [7], different $\alpha(t)$ and g will correspond to different equal in rights points of view on the evolution of vorticity.

3. Viscous incompressible liquid

For a viscous incompressible fluid, the equation of motion has the form (3). Therefore $\mathbf{U} = \mathbf{V} - \nu[\Omega \times \operatorname{curl}\Omega]/\Omega^2 + \alpha[\Omega \times \mathbf{V}]/\Omega^2 + [\Omega \times \boldsymbol{\nabla} g]/\Omega^2$. When the contours (domains) move at this velosity, their intensity Γ will change according to (7). When applying this velosity in the VVD method, the functions $\alpha(t)$ and g during each step must have smoothness, which is described after the formula (3). However, they can be discontinuous when moving from one time step to the next. Such discontinuities are allowed because they correspond to transitions from one Lagrangian point of view on the evolution of vorticity to another. For example, the function $\alpha(t)$ can be considered a constant at each time step. In this case, the value of the constant may be different at different steps. To use the values that are calculated in any case when implementing the VVD method, you can put $\boldsymbol{\nabla} g = \beta \boldsymbol{\nabla}|\Omega|$ or $\boldsymbol{\nabla} g = \beta \boldsymbol{\nabla} \ln|\Omega|$, where $\beta(t)$ is any step function of time constant at each calculation step (for example, $\beta(t) \equiv 0$).

Conclusion

A new Lagrangian point of view on the evolution of vorticity in plane-parallel and non-twisted flows of liquids of all types is proposed. Formulas are obtained for the velocity of such a displacement of the contours, at which the circulation of the velocity of the liquid along any moving contour changes according to a given time law. This theoretical result can be used in computational fluid dynamics to limit the number of domains when using a mesh-free method for calculating viscous incompressible fluid flows (the viscous vortex domain method).

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