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Hunt for chimeras in fully coupled networks of nonlinear oscillators

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Abstract. *The purpose* of this work is to study the dynamic properties of solutions of special systems of ordinary differential equations, called fully connected networks of nonlinear oscillators. *Methods.* A new approach to obtain periodic regimes of the chimeric type in these systems is proposed, the essence of which is as follows. First, in the case of a symmetric network, a simpler problem is solved of the existence and stability of quasi-chimeric solutions — periodic regimes of two-cluster synchronization. For each of these modes, the set of oscillators falls into two disjoint classes. Within these classes, full synchronization of oscillations is observed, and every two oscillators from different classes oscillate asynchronously. *Results.* On the basis of the proposed methods, it is separately established that in the transition from a symmetric system to a general network, the periodic regimes of two-cluster synchronization can be transformed into chimeras. *Conclusion.* The main statements of the work concerning the emergence of chimeras were obtained analytically on the basis of an asymptotic study of a model example. For this example, the notion of a canonical chimera is introduced and the statement about the existence and stability of solutions of chimeric type in the case of asymmetry of the network is proved. All the results presented are extended to a continuous analogue of the corresponding system. The obtained results are illustrated numerically.

Keywords: fully coupled network of nonlinear oscillators, periodic modes of two-cluster synchronization, hunting for chimeras, stability, buffering.

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1. The general strategy of hunting chimeras

A fully connected network of nonlinear oscillators or simply a fully connected network is a system of the form

$$\dot{x}_j = F_j(x_j, u_j), \quad j = 1, 2, \ldots, m. \tag{1}$$

Here $m \geqslant 2, x_j = x_j(t) \in \mathbb{R}^n,\ n \geqslant 2,$ dot — differentiation by t,

$$u_j = \sum_{\substack{s=1 \\ s\neq j}}^{m} G_s(x_s), \tag{2}$$

and the vector functions $F_j(x,u)$, $G_j(x), j = 1,2,\ldots,m$, with values in $\mathbb{R}^n$ are infinitely differentiable by their variables $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Each of the corresponding networks of (1), (2) partial systems

$$\dot{x} = F_j(x,0), \quad j = 1,2,\ldots,m \tag{3}$$

admits an exponentially orbitally stable cycle, that is, it is a nonlinear oscillator. We consider the situation when m of oscillators (3) interact with each other according to the principle «everyone with everyone».

In the particular case when

$$F_j(x,u) = F_j(x) + D_j(x)u, \quad j = 1,2,\ldots,m,$$

where $D_j(x)$ — square matrices of size $n\times n$, system(1), (2) takes the form

$$\dot{x}_j = F_j(x_j) + D_j(x_j)\sum_{\substack{s=1\\ s\neq j}}^{m} G_s(x_s), \quad j = 1,2,\ldots,m. \tag{4}$$

This situation deserves special mention due to the fact that it is the systems (4) most commonly found in applications.

Chains and lattices of connected identical nonlinear oscillators are used as mathematical models in various fields of natural science: biophysics, ecology, optics, chemical kinetics, neurodynamics, genetic engineering, etc. At the same time, in many works devoted to systems of coupled nonlinear oscillators, special stationary modes of the mentioned systems are studied — the so-called chimeras. There is no universal mathematical definition of a chimera. At the heuristic level of rigor, one of the possible situations in which we are dealing with a chimera is as follows. Let m be the number of oscillators in the system, and $m_1, m_2 : m_1 < m_2 < m$ — some fixed natural numbers. Then, in the case of a chimeric stationary regime, there are two coherence zones: for any two components x_j with numbers $1 \leqslant j \leqslant m_1$ or $m_2 < j \leqslant m$, synchronization (possibly approximate) is observed. By the term «synchronization», in order to avoid misunderstandings, we mean the fulfillment of equalities of the form $x_1 = x_2 = \ldots = x_{m_1}, x_{m_2+1} = x_{m_2+2} = \ldots = x_m$. Any two oscillators with the numbers $j_1, j_2 : m_1 < j_1 < j_2 \leqslant m_2$ oscillate asynchronously, that is, $x_{j_1} \not\equiv x_{j_2}$ (this is the so-called incoherence zone). Chimeras are characteristic of the case $m, m_1, m_2 \gg 1$ and are identified by numerical analysis of the corresponding mathematical models.

Since its discovery in 2002 (see [1]), chimeras have become an actively studied object of research [2, 3]. They can occur both in globally connected [4, 5] and in locally connected [6] networks of nonlinear oscillators. For example, chimeras were found in the Stewart – Landau model [7, 8], in the related van der Pol systems [9], FitzHugh−Nagumo [10], in Hodgkin models−Huxley [11], Hindmarsh−Rose [12], etc. This is not a complete bibliographic list. A more detailed bibliography on chimeras can be found in the book [13].

In this paper, we propose a general approach to finding chimeric stationary regimes, which is appropriately called a "chimera hunt". The mentioned approach consists of two stages.

At the first stage, we consider a similar (1), (2) symmetric fully connected network

$$\dot{x}_j = F(x_j, u_j), \quad u_j = \sum_{\substack{s=1\\ s\neq j}}^{m} G(x_s), \quad j = 1,2,\ldots,m, \tag{5}$$

where $F(x,u) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, $G(x) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. We will be interested in the periodic modes of the (5) system, which can be qualified as quasi-chimers. In the situation described above, they correspond to the case $m_1 = m_2$,, when there is no zone of incoherent behavior of oscillators, and there are only two coherence zones. This type of periodic solutions is naturally called two-cluster synchronization modes. The simplest of them has the form

$$C: \quad x_1 = x_2 = \ldots = x_k = v_{(k)}(t), \quad x_{k+1} = x_{k+2} = \ldots = x_m = w_{(k)}(t), \tag{6}$$

where $k : 1 \leqslant k \leqslant m$ — some fixed natural number, and $v_{(k)}(t)$, $w_{(k)}(t)$ — periodic vector functions. Suppose that this cycle is in the system (5) exists and is exponentially orbitally stable.

At the second stage , we will move from (5) to an asymmetric network

$$\dot{x}_j = F_j(x_j, u_j, \varepsilon), \quad u_j = \sum_{\substack{s=1 \\ s \neq j}}^{m} G_s(x_s, \varepsilon), \quad j = 1, 2, \ldots, m, \tag{7}$$

where

$$\begin{gathered} F_j(x,u,\varepsilon) = F(x,u) + \varepsilon \Delta_{1,j}(x,u), \quad G_j(x,\varepsilon) = G(x) + \varepsilon \Delta_{2,j}(x), \\ \Delta_{1,j}(x,u) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n), \quad \Delta_{2,j}(x) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n), \quad j = 1, 2, \ldots, m, \end{gathered} \tag{8}$$

$\varepsilon > 0$ — small parameter. It is clear that the system (7) admits asymptotically close to (6) stable cycle $C(\varepsilon)$, which, under suitable perturbations $\Delta_{1,j}$, $\Delta_{2,j}$ of (8) it can become a chimera.

Indeed, we fix natural $m_1, m_2 : m_1 < k < m_2 < m$ and suppose that

$$\begin{gathered} \Delta_{1,1} = \Delta_{1,2} = \ldots = \Delta_{1,m_1}, \quad \Delta_{1,m_2+1} = \Delta_{1,m_2+2} = \ldots = \Delta_{1,m}, \\ \Delta_{2,1} = \Delta_{2,2} = \ldots = \Delta_{2,m_1}, \quad \Delta_{2,m_2+1} = \Delta_{2,m_2+2} = \ldots = \Delta_{2,m}, \end{gathered} \tag{9}$$

and for $m_1 + 1 \leqslant j \leqslant m_2$, the additives $\Delta_{1,j}$, $\Delta_{2,j}$ do not trivially depend on the index j (a strict meaning to this concept will be given below when considering a specific example of the network (7)). Then the system(7) admits so-called partially homogeneous solutions (not necessarily periodic) for which similar relations (9) hold

$$x_1 = x_2 = \ldots = x_{m_1}, \quad x_{m_2+1} = x_{m_2+2} = \ldots = x_m. \tag{10}$$

And it follows that the equalities (10) are also valid for the cycle $C(\varepsilon)$ (since it obviously has partial uniformity at $\varepsilon = 0$). As for the components x_j of this cycle at $m_1 + 1 \leqslant j \leqslant m_2$ in general they exhibit incoherent behavior, that is, each pair of them oscillates asynchronously.

So, in order to implement our proposed chimera hunting strategy, it is necessary to know the periodic modes of two-cluster synchronization in a symmetric network (5). Unlike chimeras, these modes allow a strict mathematical description, which allows us to analyze questions about their existence and stability. The relevant theory is outlined below.

We fix an arbitrary natural $k : 1 \leqslant k \leqslant m-1$ and suppose that the set of indices $1 \leqslant j \leqslant m$ is divided into two disjoint sets $\mathscr{A}$ and $\mathscr{B}$, which consist of k and $m-k$ elements.That is

$$\{1, 2, \ldots, m\} = \mathscr{A} \cup \mathscr{B}. \tag{11}$$

Then the system (5) allows solutions with components

$$x_j = v(t) \text{ with } j \in \mathscr{A}, \quad x_j = w(t) \text{ with } j \in \mathscr{B}, \tag{12}$$

where the variables v,w satisfy the auxiliary system

$$\dot{v} = F(v, u_{k,1}), \quad \dot{w} = F(w, u_{k,2}), \tag{13}$$

in which

$$u_{k,1} = (k-1)G(v) + (m-k)G(w), \quad u_{k,2} = k\,G(v) + (m-k-1)G(w). \tag{14}$$

And since we are interested in periodic solutions of a system (5) of the form (12), we assume the following to be fulfilled.

Condition 1. *The system* (13), (14) *has a non-constant periodic solution*

$$C_k: \quad (v, w) = (v_{(k)}(t), w_{(k)}(t)) \tag{15}$$

of the period $T_{(k)} > 0$, *satisfying the requirement of inhomogeneity*

$$v_{(k)}(t) \not\equiv w_{(k)}(t). \tag{16}$$

The formulated condition guarantees that the original system (5) has an entire family of cycles $\mathscr{U}_k$. All cycles from this family are given by the equalities (11), (12) with

$$v = v_{(k)}(t), \quad w = w_{(k)}(t), \tag{17}$$

and their number is equal to C_m^k. Due to the heterogeneity condition (16) they are two-cluster synchronization modes.

To study the stability of the system (5) cycles (11), (12), (17) we will need a series of $T_{(k)}$-periodic on t matrices $A_s(t)$, $B_s(t)$, $s = 1, 2, 3$ of size $n \times n$. We will set the mentioned matrices by equalities

$$A_1(t) = \frac{\partial F}{\partial x}(x, u)\bigg|_{x=v_{(k)}(t),\, u=u_{k,1}(t)}, \tag{18}$$

$$A_2(t) = \frac{\partial F}{\partial u}(x, u)\bigg|_{x=v_{(k)}(t),\, u=u_{k,1}(t)} \cdot G'_x(v_{(k)}(t)), \tag{19}$$

$$A_3(t) = \frac{\partial F}{\partial u}(x, u)\bigg|_{x=v_{(k)}(t),\, u=u_{k,1}(t)} \cdot G'_x(w_{(k)}(t)), \tag{20}$$

$$B_1(t) = \frac{\partial F}{\partial x}(x, u)\bigg|_{x=w_{(k)}(t),\, u=u_{k,2}(t)}, \tag{21}$$

$$B_2(t) = \frac{\partial F}{\partial u}(x, u)\bigg|_{x=w_{(k)}(t),\, u=u_{k,2}(t)} \cdot G'_x(v_{(k)}(t)), \tag{22}$$

$$B_3(t) = \frac{\partial F}{\partial u}(x, u)\bigg|_{x=w_{(k)}(t),\, u=u_{k,2}(t)} \cdot G'_x(w_{(k)}(t)), \tag{23}$$

where

$$u_{k,1}(t) = (k-1)G(v_{(k)}(t)) + (m-k)G(w_{(k)}(t)),$$
$$u_{k,2}(t) = k\,G(v_{(k)}(t)) + (m-k-1)G(w_{(k)}(t)).$$

Let us introduce linear systems into consideration

$$\begin{aligned} \dot c_1 &= (A_1(t) + (k-1)A_2(t))c_1 + (m-k)A_3(t)c_2, \\ \dot c_2 &= k\,B_2(t)c_1 + (B_1(t) + (m-k-1)B_3(t))c_2, \end{aligned} \tag{24}$$

$$\dot c \;= (A_1(t) - A_2(t))c, \quad \dot c = (B_1(t) - B_3(t))c, \tag{25}$$

where $c_1, c_2, c \in \mathbb{R}^n$.

It follows from the formulas (18)–(23) that the system (24) is a linearization of the auxiliary system (13), (14) on a cycle(15). Obviously, it has one multiplier. In the following, we believe that this multiplier is simple. Assume that the following condition 2 is met.

Condition 2. *All multipliers of systems* (24), (25) (*except for a simple unity in the case of a system* (24)) *absolute value strictly less than one.*

The formulated constraints allow us to prove the following basic result.

Theorem 1. *Under the conditions* 1, 2 *system* (5) *admits a family of $\mathscr{U}_k$ of periodic modes of two-cluster synchronization, the number of which is equal to C_m^k. All these cycles are exponentially orbitally stable.*

Proof. The existence of family of cycles $\mathscr{U}_k$ in the system (5) is guaranteed by the condition 1. Therefore, let's go straight to the question of the stability of these cycles. In this regard, we will make two useful observations.

First, the system (5) is invariant with respect to the replacement of

$$(x_1, x_2, \dots, x_m) \to (x_{j_1}, x_{j_2}, \dots, x_{j_m}), \tag{26}$$

where $(j_1, j_2, \dots, j_m)$ is an arbitrary permutation of the index set $(1, 2, \dots, m)$. Secondly, periodic modes from the $\mathscr{U}_k$ family allow encoding using binary vectors

$$(\vartheta_1, \vartheta_2, \dots, \vartheta_m): \ \vartheta_j = 1 \quad \text{or} \quad 0, \quad j = 1, 2, \dots, m. \tag{27}$$

The j-th coordinate of the vector (27) is equal to 1 or 0 for $j \in \mathscr{A}$ or $j \in \mathscr{B}$, respectively. In this case, there is a one-to-one correspondence between the vectors (27), which contain k ones and $m-k$ zeros, and the cycles of the $\mathscr{U}_k$ family.

Any two cycles from $\mathscr{U}_k$ pass into each other under the action of substitutions (26) and have the same stability properties. The problem of stability of all modes of the family $\mathscr{U}_k$ is reduced to the study of the stability of only one cycle, which corresponds to the binary vector

$$(\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{m-k}).$$

We linearize the original system (5) on the specified cycle and come to a linear system of the form

$$\dot{h}_j = A_1(t)h_j + \sum_{\substack{s=1\\ s\neq j}}^{k} A_2(t)h_s + \sum_{s=k+1}^{m} A_3(t)h_s, \quad 1 \leqslant j \leqslant k, \tag{28}$$

$$\dot{h}_j = B_1(t)h_j + \sum_{s=1}^{k} B_2(t)h_s + \sum_{\substack{s=k+1\\ s\neq j}}^{m} B_3(t)h_s, \quad k+1 \leqslant j \leqslant m, \tag{29}$$

where $A_s(t)$, $B_s(t)$, $s = 1, 2, 3$ are matrices (18)–(23), $h_j \in \mathbb{R}^n$, $j = 1, 2, \dots, m$.

With a suitable change of variables, the system (28), (29) obtains a block structure: it splits into one $2n$-dimensional subsystem and $m-2$ subsystems of dimension n.

Let us introduce into consideration a series of matrices of size $nm \times n$, which are given in the form of matrix columns

$$e_{1,0} = (\underbrace{I, I, \dots, I}_{k}, \underbrace{O, O, \dots, O}_{m-k})^*, \quad e_{2,0} = (\underbrace{O, O, \dots, O}_{k}, \underbrace{I, I, \dots, I}_{m-k})^*, \tag{30}$$

$$e_{1,s} = (I, O, \ldots, O, \underset{s+1}{-I}, O, \ldots, O)^*, \quad s = 1, \ldots, k-1, \tag{31}$$

$$e_{2,s} = (O, \ldots, \underset{k}{O}, \underset{k+1}{I}, O, \ldots, O, \underset{k+1+s}{-I}, O, \ldots, O)^*, \quad s = 1, \ldots, m-k-1. \tag{32}$$

Here I and O are the identity and zero matrices of size $n \times n$, $*$ is the transpose operation, and the lower labels denote the position numbers those these $n \times n$-blocks occupy in the corresponding matrix columns (30)–(32).

We rely on the above matrices and perform the substitution (28), (29) in the system

$$\operatorname{colon}(h_1, h_2, \ldots, h_m) = e_{1,0}c_{1,0} + e_{2,0}c_{2,0} + \sum_{s=1}^{k-1} e_{1,s}c_{1,s} + \sum_{s=1}^{m-k-1} e_{2,s}c_{2,s}, \tag{33}$$

where $c_{1,0}, c_{2,0}, c_{1,s}, c_{2,s} \in \mathbb{R}^n$ are new variables. As a result, we conclude that the pair of components $(c_{1,0}, c_{2,0}) = (c_{1,0}(t), c_{2,0}(t))$ from (33) satisfies the linear system (24). The components $c_{1,s} = c_{1,s}(t)$, $s = 1, \ldots, k-1$ are solutions of the first system in (25). Similarly, the coordinate group $c_{2,s} = c_{2,s}(t)$, $s = 1, \ldots, m-k-1$ satisfies the second system from (25). And from this and from the Condition 2 the required exponential orbital stability of all cycles of the family $\mathscr{U}_k$ follows. Theorem 1 is proved. □

We complete the description of the general scheme for studying the problems of the existence and stability of periodic regimes of two-cluster synchronization. Let us remind, see [14, 15], that earlier such regimes were found in fully connected neural and gene networks, which contain a time delay. However, the fact of the presence or absence of a delay in this matter is not significant.

2. Analysis of a model example

In this section, the general chimera hunting strategy is illustrated with a specific model example. Namely, we consider such a symmetric fully connected network (5), for which all periodic modes of two-cluster synchronization are given by explicit formulas.

To construct the network of interest to us, we first turn to a two-dimensional system, which in complex form is written as

$$\dot{z} = z - d_0|z|^2 z, \quad z = x + iy, \quad x, y \in \mathbb{R}, \quad d_0 = 1 - ic_0, \quad c_0 = \text{const} > 0. \tag{34}$$

It is easy to check that this system admits an exponentially orbitally stable harmonic cycle $z_0(t) = \exp(ic_0 t)$, i. e. it is the simplest nonlinear oscillator. Consider next a fully connected network of oscillators (34), where $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$, $\nu = \text{const} > 0$, $d_1 = 1 + ic_1$, $c_1 = \text{const} \in \mathbb{R}$.

$$\dot{z}_j = z_j - d_0|z_j|^2 z_j - \frac{\nu}{m} d_1 \overline{z}_j \sum_{\substack{s=1\\ s\neq j}}^{m} z_s^2, \quad j = 1, 2, \ldots, m \tag{35}$$

As it will be shown below, the network (35) has the property of interest to us: all its periodic two-cluster synchronization modes, as well as their stability conditions, are written out explicitly.

We begin our analysis of the system (35) by obtaining conditions for its dissipativity. For this we need a function

$$V(z_1, z_2, \ldots, z_m, \overline{z}_1, \overline{z}_2 \ldots, \overline{z}_m) = \sum_{j=1}^{m} |z_j|^2. \tag{36}$$

The following assertion holds.

Lemma 1. *When the following inequality is satisfied*

$$\nu < m \tag{37}$$

we have the estimate

$$\frac{1}{2}\dot{V} \leqslant V - \left(1 - \frac{\nu}{m}\right)\frac{1}{m}V^2, \tag{38}$$

where $\dot{V}$ is the derivative of the function (36) due to the formulae (35).

Proof. To calculate the derivative $\dot{V}$, we first note that, by virtue due to of (35), for any index $j : 1 \leqslant j \leqslant m$, the following equality holds:

$$\frac{1}{2}\frac{d}{dt}|z_j|^2 = |z_j|^2 - \left(1 - \frac{\nu}{m}\right)|z_j|^4 - \frac{\nu}{2m}\left(d_1\bar{z}_j^2\sum_{s=1}^{m} z_s^2 + \bar{d}_1 z_j^2 \sum_{s=1}^{m}\bar{z}_s^2\right). \tag{39}$$

Summing up the expressions (39) over j, we conclude that

$$\frac{1}{2}\dot{V} = V - \left(1 - \frac{\nu}{m}\right)\sum_{j=1}^{m}|z_j|^4 - \frac{\nu}{m}\left|\sum_{j=1}^{m} z_j^2\right|^2 \leqslant V - \left(1 - \frac{\nu}{m}\right)\sum_{j=1}^{m}|z_j|^4.$$

And from here, taking into account the obvious property

$$\left(\sum_{j=1}^{m}|z_j|^2\right)^2 \leqslant m\sum_{j=1}^{m}|z_j|^4$$

we obtain the required estimate (38). Lemma 1 is proved. □

The established lemma makes it easy to deal with the question of the dissipativity of the (35) system that interests us. Indeed, let the condition (37) be fulfilled. Then by virtue of due to (38) for any fixed $R > m^2/(m - \nu)$, all the trajectories of our system with increasing t flow into the ball

$$\left\{(z_1, z_2, \ldots, z_m) : \quad \sum_{j=1}^{m}|z_j|^2 \leqslant R\right\},$$

that is, the required dissipativity property holds. However, when the strictly opposite to (37) inequality holds, there is no dissipativity.

Indeed, a direct verification shows that the system (35) admits an invariant manifold of the form

$$z_j = z\exp(i\gamma_j), \quad j = 1, 2, \ldots, m,$$

where the real constants γ_j are subject to the condition

$$\sum_{j=1}^{m}\exp(2i\gamma_j) = 0,$$

and the complex variable z satisfies the equation

$$\dot{z} = z - \left(d_0 - \frac{\nu}{m}d_1\right)|z|^2 z. \tag{40}$$

In the case of $\nu > m$, any solution of the equation (40) $z(t) \neq 0$ is defined on a finite half-interval of the form $[0, t_0)$ and $|z(t)| \to +\infty$ for $t \to t_0$.

Let us now turn to questions about the existence and stability of periodic solutions of the system (35) . It is easy to check that the network under consideration admits a homogeneous cycle

$$z_j = \xi_0 \exp(i\omega_0 t), \quad j = 1, 2, \dots, m, \tag{41}$$

where

$$\xi_0 = \sqrt{1/(1+\nu(m-1)/m)}, \quad \omega_0 = \Big(c_0 - \frac{\nu(m-1)}{m}\, c_1\Big)\xi_0^2, \tag{42}$$

as well as periodic modes of two-cluster synchronization

$$z_j = \xi_0 \exp(i\omega_0 t), \quad 1 \leqslant j \leqslant k, \quad z_j = -\xi_0 \exp(i\omega_0 t), \quad k+1 \leqslant j \leqslant m \tag{43}$$

for all $k : 1 \leqslant k \leqslant m-1$. As noted in section 1, each of the cycles (43), in turn, generates a whole family $\mathscr{U}_k$ of two-cluster synchronization modes. Denote by $\mathscr{U}$ the union of cycles in $\bigcup_{k=1}^{m-1} \mathscr{U}_k$ with the homogeneous cycle (41). As it turns out, all these periodic regimes are either stable or unstable at the same time.

Lemma 2. *Any cycle in $\mathscr{U}$ is exponentially orbitally stable under inequalities*

$$\frac{\nu(m+1)}{m} < 1, \quad \frac{\nu}{m}\,(1+c_1^2) + c_0 c_1 - 1 > 0 \tag{44}$$

and unstable under strict violation of at least one of them.

Proof. In substantiating the lemma, we pay attention to two circumstances. First, it is easy to see that the system (35) is invariant under variable substitutions

$$(\theta_1 z_1, \theta_2 z_2, \dots, \theta_m z_m) \to (z_1, z_2, \dots, z_m), \tag{45}$$

where the multipliers θ_j, $j = 1, 2, \dots, m$ independently take the values 1 or -1. Secondly, any two cycles of the family $\mathscr{U}$ pass into each other with some substitution of the form (45). And from here it automatically follows that the stability properties of all cycles from $\mathscr{U}$ are the same. Thus, it suffices to deal with the stability of only one of them, the homogeneous cycle (41), (42).

Put in the equation (35) $z_j = \xi_0(1+h_j)\exp(i\omega_0 t)$, $h_j = h_{1,j} + ih_{2,j}$, $h_{1,j}, h_{2,j} \in \mathbb{R}$ and discard terms that are nonlinear in h_j and $\overline{h}_j$. As a result, we come to the linear system

$$\begin{aligned} \dot h_j = -d_0\xi_0^2(h_j + \overline{h}_j) + \frac{\nu(m+1)}{m}\, d_1\xi_0^2\, h_j - \frac{\nu(m-1)}{m}\, d_1\xi_0^2\, \overline{h}_j - \\ -\frac{2\nu}{m}\, d_1\xi_0^2 \sum_{s=1}^{m} h_s, \quad j = 1, 2, \dots, m. \end{aligned} \tag{46}$$

Note further that the resulting system admits an invariant subspace $h_1 = h_2 = \ldots = h_m$, on which it can be written as

$$\dot h = -\xi_0^2\Big(d_0 + \frac{\nu(m-1)}{m}\, d_1\Big)(h + \overline{h}), \quad h = \frac{1}{m}\sum_{j=1}^{m} h_j,$$

as well as the invariant subspace $(h_1, h_2, \dots, h_m) : \sum_{j=1}^{m} h_j = 0$, on which it takes the form

$$\dot h_j = -\xi_0^2\Big(d_0 - \frac{\nu(m+1)}{m}\, d_1\Big)h_j - \xi_0^2\Big(d_0 + \frac{\nu(m-1)}{m}\, d_1\Big)\overline{h}_j, \quad j = 1, 2, \dots, m.$$

Thus, the stability spectrum of the system (46) coincides with the eigenvalues of the matrix

$$B_1 = -\xi_0^2 \begin{pmatrix} d_0 + \dfrac{\nu(m-1)}{m}\, d_1 & d_0 + \dfrac{\nu(m-1)}{m}\, d_1 \\ \overline{d}_0 + \dfrac{\nu(m-1)}{m}\, \overline{d}_1 & \overline{d}_0 + \dfrac{\nu(m-1)}{m}\, \overline{d}_1 \end{pmatrix} \tag{47}$$

and $m-1$ equal matrices

$$B_2 = -\xi_0^2 \begin{pmatrix} d_0 - \dfrac{\nu(m+1)}{m}\, d_1 & d_0 + \dfrac{\nu(m-1)}{m}\, d_1 \\ \overline{d}_0 + \dfrac{\nu(m-1)}{m}\, \overline{d}_1 & \overline{d}_0 - \dfrac{\nu(m+1)}{m}\, \overline{d}_1 \end{pmatrix}. \tag{48}$$

It remains to note that the matrix (47) has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$, and the matrix (48) under the conditions (44) is Hurwitz . If at least one of these conditions is strictly violated, B_2 admits an eigenvalue in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. Lemma 2 is proved. $\square$

Now let us move on from (35) to the corresponding asymmetric network

$$\dot{z}_j = (1 + i\varepsilon\mu_j)z_j - d_0|z_j|^2 z_j - \frac{\nu}{m}\, d_1\, \overline{z}_j \sum_{\substack{s=1\\ s\neq j}}^{m} z_s^2, \quad j = 1, 2, \ldots, m, \tag{49}$$

where $\varepsilon \in \mathbb{R}$, $|\varepsilon| \ll 1$, $\mu_j = \text{const} \in \mathbb{R}$, $j = 1, 2, \ldots, m$. We assume the existence of natural $m_1, m_2 : m_1 < m_2 < m$ such that

$$\mu_1 = \mu_2 = \ldots = \mu_{m_1}, \quad \mu_{m_2} = \mu_{m_2+1} = \ldots = \mu_m. \tag{50}$$

In the case of $m_1 + 1 \leqslant j \leqslant m_2 - 1$, we consider the dependence of μ_j on the index j to be nontrivial. The latter means that

$$\mu_{j_1} \neq \mu_{j_2} \quad \forall\, j_1, j_2 \in [m_1 + 1, m_2 - 1], \quad j_1 \neq j_2. \tag{51}$$

As it turns out, for the (49) system, the heuristic description of the chimera given in the 1 section 1 can be given a rigorous definition. Namely, a canonical chimera is a cycle of this system whose components $z_j = z_j(t)$ satisfy the requirements

$$z_1(t) \equiv z_2(t) \equiv \ldots \equiv z_{m_1}(t), \quad z_{m_2}(t) \equiv z_{m_2+1}(t) \equiv \ldots \equiv z_m(t), \tag{52}$$

$$z_{j_1}(t) \not\equiv z_{j_2}(t) \quad \forall\, j_1, j_2 \in [m_1 + 1, m_2 - 1], \quad j_1 \neq j_2, \tag{53}$$

where the natural numbers m_1, m_2 are borrowed from (50), (51). As for the segments $1 \leqslant j \leqslant m_1$, $m_2 \leqslant j \leqslant m$ and $m_1 + 1 \leqslant j \leqslant m_2 - 1$, we will call them intervals of coherent and incoherent behavior of oscillators.

When solving the problem of the existence of a canonical chimera, we assume that the parameters c_0, c_1, ν of the system (49) satisfy the requirements

$$0 < \nu < \frac{m}{m+1}, \quad c_0 c_1 > 1. \tag{54}$$

In this case, the following assertion is true.

Theorem 2 (on the existence of a canonical chimera). *Under the above constraints and for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is sufficiently small, the system (49) has an exponentially orbitally stable cycle*

$$z_j = \rho_j(\varepsilon) \exp[i\varphi_j(\varepsilon) + i\omega(\varepsilon)t], \quad j = 1, 2, \ldots, m, \quad \sum_{j=1}^{m} \varphi_j(\varepsilon) = 0, \tag{55}$$

which is a canonical chimera for $\varepsilon \neq 0$. Here the real functions $\rho_j(\varepsilon)$, $\varphi_j(\varepsilon)$, $\omega(\varepsilon)$ depend analytically on ε and, for $\varepsilon \to 0$, admit the asymptotics:

$$\rho_j(\varepsilon) = \rho_0 + \varepsilon\rho_{1,j} + O(\varepsilon^2),\ \varphi_j(\varepsilon) = \varepsilon\varphi_{1,j} + O(\varepsilon^2),\ \omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + O(\varepsilon^2), \tag{56}$$

where

$$\begin{gathered}
\rho_0 = \xi_0, \quad \rho_{1,j} = \frac{\operatorname{Re}\varkappa_j}{\xi_0}, \quad \varphi_{1,j} = \frac{\operatorname{Im}\varkappa_j}{\xi_0^2}, \quad \omega_1 = \frac{1}{m}\sum_{s=1}^{m}\mu_s, \\
\varkappa_j = \frac{\nu c_1 - i(1-\nu/m)}{2\nu[c_0c_1 - 1 + \nu(1+c_1^2)/m]}(\omega_1 - \mu_j), \quad j = 1, 2, \ldots, m,
\end{gathered} \tag{57}$$

and ξ_0, ω_0 are from (42).

Proof. By virtue of due to the conditions (54) and Lemma 2 for $\varepsilon = 0$ the homogeneous cycle (41) of the system (49) is exponentially orbitally stable. Therefore, for small absolute values of ε, it passes into an exponentially orbitally stable cycle of the form

$$z_j = z_j(\varepsilon)\exp[i\omega(\varepsilon)t], \quad j = 1, 2, \ldots, m, \tag{58}$$

where the complex amplitudes $z_j(\varepsilon)$, $z_j(0) = \xi_0$ and the frequency $\omega(\varepsilon)$, $\omega(0) = \omega_0$ depend analytically on ε. Next, we substitute the formulas (58) together with the expansions

$$z_j(\varepsilon) = \xi_0 + \varepsilon\xi_{1,j} + \ldots, \quad \omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \ldots \tag{59}$$

into the system (49), reduce the result by $\exp[i\omega(\varepsilon)t]$ and equate the coefficients of ε in the left and right parts of the resulting relations. As a result, for the constants $\xi_{1,j} \in \mathbb{C}$, $\omega_1 \in \mathbb{R}$ from (59) we arrive at a linear inhomogeneous system

$$\begin{aligned}
i\omega_1 = i\mu_j - d_0\xi_0(\xi_{1,j} + \bar{\xi}_{1,j}) + \frac{\nu(m+1)}{m}\, d_1\xi_0\xi_{1,j} - \\
-\frac{\nu(m-1)}{m}\, d_1\xi_0\bar{\xi}_{1,j} - \frac{2\nu}{m}\, d_1\xi_0\sum_{s=1}^{m}\xi_{1,s}, \quad j = 1, 2, \ldots, m.
\end{aligned}$$

The latter, as it is easy to check, with the value of ω_1 defined by the corresponding formula from (57), has the solution

$$\xi_{1,j} = \frac{\varkappa_j}{\xi_0}, \quad j = 1, 2, \ldots, m. \tag{60}$$

From the established relations (59), (60) it follows that the cycle (58) is transformed to the required form (55)–(57). It is only necessary to note that the fulfillment of the equality $\sum_{j=1}^{m}\varphi_j(\varepsilon) = 0$ can always be achieved by replacing t with $t + c$, $c \in \mathbb{R}$.

Finally, we add that for $\varepsilon \neq 0$ the cycle (55) is a canonical chimera. Indeed, equalities of the form (52) are valid for it for the same reasons as similar equalities (10) for the cycle $C(\varepsilon)$ of the system (7), see the corresponding place in section 1. As for the requirements (53), their validity for $\varepsilon \neq 0$ is guaranteed by the condition (51) and the formulas (56), (57). Theorem 2 is proved. □

Let us pay attention to the following circumstance. Since for $\varepsilon \neq 0$ the system (49) is still invariant under the substitutions (45), the canonical chimera (55) generates a whole family $\mathscr{U}$ consisting of $2^m - 1$ exponentially orbitally stable cycles. All these cycles, which are obtained from one cycle (55) by means of the above changes of variables, will be called chimera-like structures. It is also clear that if we put

$$\mu_j = \mu(s)|_{s=j/m}, \quad j = 1, 2, \ldots, m, \tag{61}$$

where $\mu(s)$ is some continuous function on the segment $0 \leqslant s \leqslant 1$, then under the conditions similar to (54)

$$0 < \nu < 1, \quad c_0 c_1 > 1 \tag{62}$$

and as $\varepsilon \to 0$, $m \to +\infty$ converge, the number of stable cycles coexisting in the (49) system grows without limit. In other words, the well-known phenomenon of buffering is observed. As shown in monographs [16, 17], this phenomenon is typical for a wide class of dynamical systems from various fields of natural science.

Concluding the discussion of the chimera hunting strategy as applied to the system (49), we note that due to the appropriate choice of the μ_j coefficients, we can guarantee the existence of a chimera with pre-planned properties. Indeed, suppose that the index array $1 \leqslant j \leqslant m$ is arbitrarily divided into k, $k \geqslant 2$ segments, i. e.

$$\{1, 2, \ldots, m\} = [1, m_1] \cup [m_1 + 1, m_2] \cup \ldots \cup [m_s + 1, m_{s+1}] \cup \ldots \cup [m_{k-1} + 1, m]. \tag{63}$$

We further assume that on each of these segments the sequence μ_j is either constant or depends nontrivially on j, and segments of this kind alternate. In this case Theorem 2 remains valid provided that we modify the definition of canonical chimera appropriately. Namely, in this case, it is appropriate to call it a chimera, in which the intervals of coherent and incoherent behavior of oscillators coincide with segments from (63) of constancy and inconstancy of the coefficients μ_j.

3. Continuous chimeras

In this section, the above results for the model system (35) are extended to its continuous counterpart. Namely, we study the evolution equation of the form

$$\dot{z} = z - d_0|z|^2 z - \nu d_1 \overline{z} \int_0^1 z^2(t, s)\, ds, \tag{64}$$

resulting from (35) as $m \to +\infty$. Here, as usual, the dot denotes the derivative with respect to t,

$$\begin{gathered} z = x(t, s) + iy(t, s),\ t \geqslant 0,\ s \in [0, 1],\ x(t, s), y(t, s) \in \mathbb{R}, \quad d_0 = 1 - ic_0, \\ c_0 = \text{const} > 0, \quad \nu = \text{const} > 0, \quad d_1 = 1 + ic_1, \quad c_1 = \text{const} \in \mathbb{R}. \end{gathered} \tag{65}$$

Let us first dwell on the question of the solvability of the Cauchy problem for the equation (64) with an initial condition from an appropriate phase space. As such, we take the real Banach space E whose elements are the vectors

$$\xi = (x(s), y(s)): \quad x(s), y(s) \in L_\infty(0, 1). \tag{66}$$

As for the norm of the element (66), we define it by the formula

$$||\xi|| = \operatorname*{ess\,sup}_{0 \leqslant s \leqslant 1} \sqrt{x^2(s) + y^2(s)}, \tag{67}$$

where ess sup is the essential least upper bound. Next, we introduce an abstract function $\xi(t) = (x(t,s), y(t,s))$ with values in E, where $x(t,s)$, $y(t,s)$ — functions from (65). As a result, the equation (64) is written in the abstract form

$$\dot{\xi} = \mathscr{F}(\xi), \tag{68}$$

where, as is easy to see, the nonlinear operator $\mathscr{F} : E \to E$ is the sum of the identity operator and a term of the form $\mathscr{G}(\xi,\xi,\xi)$, where $\mathscr{G}(\cdot,\cdot,\cdot)$ is a continuous cubic form from $E \times E \times E$ to E.

Elements of the general theory of abstract equations of the form (68) with bounded and Fréchet smooth right-hand side are contained, for example, in the monograph [18]. This theory implies, in particular, the following assertion.

Lemma 3. *Given any fixed bounded set $\Omega \subset E$, one can specify $t_0 = t_0(\Omega) > 0$ such that for $\forall\ \xi_0 \in \Omega$ the solution $\xi = \xi(t)$ of the equation (68) with initial condition*

$$\xi|_{t=0} = \xi_0 \tag{69}$$

is uniquely defined on the segment $0 \leqslant t \leqslant t_0$.

In what follows, we will need one more standard assertion from the theory of abstract equations of the form (68). Before stating it, for the solution $\xi = \xi(t)$ of the Cauchy problem (68), (69), we define the maximum existence half-interval $[0, t_{\max})$, where

$$t_{\max} = \sup\{t_0 : \ \ \xi(t) \ \text{ exists on the segment } \ [0, t_0]\}. \tag{70}$$

According to the Lemma 3, the set of values t_0, by which sup is taken in (70), is certainly not empty. Moreover, any solution $\xi(t)$ can be uniquely extended to its maximal half-interval. If it turns out that $t_{\max} < \infty$, then the following lemma is true.

Lemma 4. *Suppose that for some solution $\xi(t)$ of the equation (68) the quantity (70) is finite. Then we have the limit equality*

$$\lim_{t \to t_{\max}-0} ||\xi(t)|| = +\infty, \tag{71}$$

where hereinafter $||\cdot||$ is the norm (67).

Proof of the formulated lemma is carried out by contradiction. Indeed, suppose that the relation (71) does not hold. In this case, there is a sequence of times t_n, $t_n \nearrow t_{\max}$ as $n \to +\infty$, and a constant $M > 0$ such that $||\xi(t_n)|| \leqslant M$. Further, under the conditions of the Lemma 3, as a bounded set Ω we take $\{\xi(t_n),\ n \geqslant 1\}$. Then it is obvious that the solution $\xi(t)$ will be defined on segments of the form $[t_n, t_n + t_0]$, $n \geqslant 1$, where $t_0 = t_0(\Omega) > 0$. It is also clear that for sufficiently large n the inclusion $t_{\max} \in (t_n, t_n + t_0)$ holds. The latter contradicts the definition of $t_{\max}$ (see (70)) □

Let us now turn to the question of the dissipativity of the equation (68). As it turns out, in contrast to the discrete case (35), where the dissipativity condition has the form (37), the continuous system (64) is dissipative for every $\nu > 0$. More precisely, the next lemma is valid

Lemma 5. *There is a constant $R_0 > 0$ with the following properties. Given any bounded set $\Omega \subset E$, there exists $t_* = t_*(\Omega) > 0$ such that for each $\xi_0 \in \Omega$ the solution $\xi(t)$ of the corresponding Cauchy problem (68), (69) is defined on the semiaxis $t \geqslant 0$, and for all $t \geqslant t_*$ satisfies the inequality*

$$||\xi(t)|| \leqslant R_0. \tag{72}$$

Proof. We fix an arbitrarily bounded set Ω from E and an element $\xi_0 \in \Omega$. Next, denote by $\xi(t)$ the solution to the Cauchy problem (68), (69) defined on its maximum half-interval $[0, t_{\max})$, where $t_{\max}$ — value (70). In what follows, we will also need the representation

$$\xi(t) = (x(t,s), y(t,s)): \quad x(t,s), y(t,s) \in L_\infty(0,1) \text{ по } s \text{ при } \forall\, t \in [0, t_{\max}) \tag{73}$$

and the associated function $z(t,s) = x(t,s) + iy(t,s)$ satisfying for all values of $t \in [0, t_{\max})$ the equation (64) (for almost all $s \in [0,1]$).

The justification of the Lemma is divided into two stages. On the first of them, an upper bound will be set for the function

$$V(t) = \int\limits_0^1 |z(t,s)|^2 ds, \quad 0 \leqslant t < t_{\max}. \tag{74}$$

Obtaining the required estimate is carried out as follows. First, we supplement the equation (64) with a complex conjugate equation. We then multiply the first of these equations by $\overline{z}$, and the second by z. As a result, after adding the resulting expressions and then integrating over $s \in [0,1]$, we arrive at the equality

$$\dot{V} = 2\Bigg(V(t) - \int\limits_0^1 |z(t,s)|^4 ds - \nu\Bigg|\int\limits_0^1 z^2(t,s)ds\Bigg|^2\Bigg). \tag{75}$$

Further, from the relation (75) and from the formula (74) it obviously follows that

$$\dot{V} \leqslant 2\Bigg(V(t) - \int\limits_0^1 |z(t,s)|^4 ds\Bigg) \leqslant 2(V(t) - V^2(t)), \quad V(0) \leqslant V_0,$$

where

$$V_0 \overset{\text{def}}{=} \sup_{\xi_0 \in \Omega} ||\xi_0||^2 > 0$$

(the case $V_0 = 0$ is trivial, because then $\Omega = \{0\}$ and $\xi(t) \equiv 0$). In turn, from here we have

$$V(t) \leqslant V_*(t), \quad 0 \leqslant t < t_{\max}, \tag{76}$$

where $V_*(t), t \geqslant 0$ is a solution of the Cauchy problem

$$\dot{V} = 2(V - V^2), \quad V|_{t=0} = V_0. \tag{77}$$

Let us add that

$$V_*(t) \leqslant R_1 \ \ \forall\, t \geqslant 0, \quad \lim_{t \to +\infty} V_*(t) = 1, \tag{78}$$

where $R_1 = R_1(\Omega)$ is some positive constant.

The second stage of the justification of the Lemma is connected with obtaining, for almost all values of $s \in [0,1]$, an upper estimate for the function

$$W(t,s) = |z(t,s)|^2, \quad 0 \leqslant t < t_{\max}. \tag{79}$$

To do this, we use the analogous (75) to equality

$$\dot{W}(t,s) = 2(W(t,s) - W^2(t,s)) - 2\nu\, \mathrm{Re}\,[d_1\, \alpha(t)\, \overline{z}^2(t,s)], \tag{80}$$

where, as usual, the point is the differentiation with respect to t, and the function $\alpha(t)$ has the form

$$\alpha(t) = \int\limits_0^1 z^2(t,s)ds, \quad 0 \leqslant t < t_{\max}.$$

Further, combining the formulas (79), (80) with the obvious fact $|\alpha(t)| \leqslant V(t)$ and the estimate (76) already established, we come to the conclusion that (almost everywhere in s)

$$\begin{gathered} \dot{W}(t,s) \leqslant 2[(1+\nu\,|d_1|V_*(t))W(t,s) - W^2(t,s)], \quad 0 \leqslant t < t_{\max}, \\ W(0,s) \leqslant W_0 \overset{\text{def}}{=} \sup_{\xi_0\in\Omega} ||\xi_0||^2. \end{gathered} \tag{81}$$

And finally, from (81), by virtue of due to the theorem on differential inequalities, for almost all $s \in [0,1]$ we have

$$W(t,s) \leqslant W_*(t), \quad 0 \leqslant t < t_{\max}, \tag{82}$$

where $W_*(t), t \geqslant 0$ is a solution of the Cauchy problem similar to (77)

$$\dot{W} = 2[(1+\nu\,|d_1|V_*(t))W - W^2], \quad W|_{t=0} = W_0.$$

We also note that, due to (78), the solution $W_*(t)$ of this problem has the properties

$$W_*(t) \leqslant R_2 \;\; \forall\, t \geqslant 0, \quad \lim_{t\to+\infty} W_*(t) = 1 + \nu\,|d_1|, \tag{83}$$

where $R_2 = R_2(\Omega) > 0$.

Summing up, we pass in the inequality (82) to an essential least upper bound in $s \in [0,1]$. As a result, taking into account the way of specifying the norm in the space E (see (67)), we obtain the estimate for the function (73)

$$||\xi(t)||^2 \leqslant W_*(t), \quad 0 \leqslant t < t_{\max}. \tag{84}$$

Further, (84) automatically implies that $t_{\max} = +\infty$. Otherwise, the limit equality (71) takes place, which contradicts the estimate (84). It remains to note that, due to (83), (84), all trajectories of the system (68) flow into the ball $\{\xi \in E : ||\xi|| \leqslant R_0\}$ for any fixed $R_0 > \sqrt{1+\nu\,|d_1|}$. Thus, with an appropriate choice of $t_* = t_*(\Omega) > 0$, the required inequality (72) is certainly satisfied on the semiaxis $t \geqslant t_*$. Lemma 5 is proved. □

Questions about the existence and stability of the equation (64) of periodic regimes of two-cluster synchronization require separate consideration. We note right away that, as in the discrete case (35), the continuous model (64) has a homogeneous cycle

$$z = \xi_0 \exp(i\omega_0 t), \tag{85}$$

where

$$\xi_0 = 1/\sqrt{1+\nu}, \quad \omega_0 = (c_0 - \nu\, c_1)\xi_0^2. \tag{86}$$

As it turns out, this cycle generates a whole family of cycles of the equation (64).

Indeed, based on the representation

$$[0,1] = \mathscr{A} \cup \mathscr{B}, \tag{87}$$

where $\mathscr{A}$, $\mathscr{B}$ are arbitrary Lebesgue measurable disjoint subsets of the segment $[0,1]$ of positive measure, consider the function

$$I_{\mathscr{A},\mathscr{B}}(s) = \begin{cases} 1 & \text{при } \; s \in \mathscr{A}, \\ -1 & \text{при } \; s \in \mathscr{B}. \end{cases} \tag{88}$$

Note further that since the equation (64) is invariant under the change

$$I_{\mathscr{A},\mathscr{B}}(s)z \to z, \tag{89}$$

then along with the cycle (85), (86) it admits the cycle

$$z(t,s) = I_{\mathscr{A},\mathscr{B}}(s)\xi_0 \exp(i\omega_0 t), \tag{90}$$

which we will call the periodic regime of two-cluster synchronization. All cycles (90) corresponding to any partitions of the form (87), as well as the homogeneous cycle (85), (86), are combined into the family $\mathscr{U}$ (identifying cycles that differ from each other in s only on a set of zero measure). The analogue of Lemma 2 in this case is the following assertion.

Lemma 6. *Under the conditions (62) each cycle of the family $\mathscr{U}$ is exponentially orbitally stable. If at least one inequality $\nu > 1$ or $c_0 c_1 < 1$ holds, then all cycles of the given family are unstable.*

Proof. As in the justification of Lemma 2, we first pay attention to the fact that any two cycles of the family $\mathscr{U}$ are transformed into each other under the action of some replacement from the class (88), (89). Thus, their stability properties are the same, which means that we can restrict ourselves to considering only the homogeneous cycle (85), (86). We also add that since the Andronov–Witt theorem on the stability of the cycle in the first approximation is obviously valid for the abstract equation (68), the problem of the stability of the mentioned periodic regime is reduced to the analysis of the corresponding system in variations.

Let (64) $z = \xi_0(1 + h(t,s))\exp(i\omega_0 t)$, where

$$h(t,s) = h_1(t,s) + ih_2(t,s), \quad h_1(t,s), h_2(t,s) \in \mathbb{R},$$

discarding terms that are nonlinear in h and $\overline{h}$, we conclude that the equation of interest to us in variations has the form

$$\dot{h} = -d_0\xi_0^2(h + \overline{h}) + \nu\, d_1\xi_0^2(h - \overline{h}) - 2\nu\, d_1\xi_0^2 \int\limits_0^1 h(t,s)ds. \tag{91}$$

Note further that the phase space E of the equation (91), consisting of pairs

$$(h_1(s), h_2(s)): \quad h_j(s) \in L_\infty(0,1), \quad j = 1,2,$$

expands into a direct sum of closed linear subspaces E_1, E_2. The first of them is two-dimensional and is given by the equality

$$E_1 = \{(h_1(s), h_2(s)): \quad h_j(s) \equiv h_j, \quad h_j \in \mathbb{R}, \quad j = 1,2\}, \tag{92}$$

and the second has the form

$$E_2 = \left\{(h_1(s), h_2(s)): \quad \int\limits_0^1 h_j(s)ds = 0, \quad j = 1,2\right\}. \tag{93}$$

Moreover, it is easy to check that the subspaces (92), (93) are invariant for the trajectories of the equation (91).

Denote by $L : E \to E$ the linear bounded operator generated by the right side of the equation (91) and acting according to the rule:

$$(h_1(s), h_2(s)) \mapsto (\operatorname{Re}\mathscr{L}(h,\overline{h}), \operatorname{Im}\mathscr{L}(h,\overline{h})),$$

where

$$\mathscr{L}(h,\overline{h}) = -d_0\xi_0^2(h+\overline{h}) + \nu\, d_1\xi_0^2(h-\overline{h}) - 2\nu\, d_1\xi_0^2\int\limits_0^1 h(s)ds, \quad h(s) = h_1(s) + ih_2(s).$$

It follows from the above facts that, firstly, $LE_j \subset E_j$, $j = 1, 2$; secondly, the spectra of the restrictions $L|_{E_j}$, $j = 1, 2$ coincide with the eigenvalues of the matrices B_1 and B_2, respectively, where

$$B_1 = -\xi_0^2\begin{pmatrix} d_0 + \nu\, d_1 & d_0 + \nu\, d_1 \\ \overline{d}_0 + \nu\,\overline{d}_1 & \overline{d}_0 + \nu\,\overline{d}_1 \end{pmatrix}, \quad B_2 = -\xi_0^2\begin{pmatrix} d_0 - \nu\, d_1 & d_0 + \nu\, d_1 \\ \overline{d}_0 + \nu\,\overline{d}_1 & \overline{d}_0 - \nu\,\overline{d}_1 \end{pmatrix}. \tag{94}$$

It remains to add that the first of the matrices (94) has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -2$, and the second matrix is Hurwitz under the conditions (62). In the case of $\nu > 1$ or $c_0c_1 < 1$, B_2 has an eigenvalue with a positive real part. Lemma 6 is proved. □

In connection with the established Lemma, let us pay attention to the following purely infinite-dimensional effect. Although the family $\mathscr{U}$ has the cardinality of the continuum, in the phase space E any two distinct cycles of this family are located at a positive distance from each other, equal to $2\xi_0$. In the finite-dimensional case, and even in the case of an infinite-dimensional separable phase space, such a situation is obviously impossible.

Now let us move on to an asymmetric continuous network

$$\dot{z} = (1 + i\varepsilon\mu(s))z - d_0|z|^2 z - \nu d_1\overline{z}\int\limits_0^1 z^2(t,s)ds, \tag{95}$$

obtained from (49) under the conditions (61) and $m \to +\infty$. Here, as in the discrete case, $\varepsilon \in \mathbb{R}$, $|\varepsilon| \ll 1$. Further, by analogy with the requirements of (50), (51), we assume the existence of such $s_1, s_2 : 0 < s_1 < s_2 < 1$ that the continuous real function $\mu(s)$ is constant on the segments $[0, s_1]$ and $[s_2, 1]$, and on the segment $[s_1, s_2]$ nontrivially depends on s. The latter means that for $\forall\, c \in \mathbb{R}$, the set $\{s \in [s_1, s_2] : \mu(s) = c\}$ has zero Lebesgue measure.

As in the case of the (49) system, the definition of a continuous canonical chimera can be formulated for the equation (95). In this situation, a canonical chimera is a periodic solution $z = z(t,s)$ of this equation that satisfies the relations (almost everywhere in s)

$$z(t,s) \equiv z_1(t) \ \text{при} \ s \in [0, s_1], \quad z(t,s) \equiv z_2(t) \ \text{при} \ s \in [s_2, 1], \tag{96}$$

where $z_1(t), z_2(t)$ are some complex-valued functions. In the case of the segment $s_1 \leqslant s \leqslant s_2$, we assume that for $\forall\, c \in \mathbb{C}$, $\forall\, t \in \mathbb{R}$ the set of values $\{s \in [s_1, s_2] : z(t,s) = c\}$ has measure zero. If we consider $z = z(t,s)$ as a continuum array of nonlinear oscillators, then due to (96) it is natural to call the intervals $0 \leqslant s \leqslant s_1$ and $s_2 \leqslant s \leqslant 1$ coherence zones . As for the segment $s_1 \leqslant s \leqslant s_2$, we will call it the zone of incoherence.

An analogue of Theorem 2 for the continuous model (95) is the following theorem

Theorem 3 (on the continuous canonical chimera). *Under the conditions formulated above on the function* $\mu(s)$*, under the conditions* (62) *and for all sufficiently small values of* ε*, the* *equation* (95) *has an exponentially orbitally stable cycle*

$$z = \rho(s,\varepsilon)\exp[i\varphi(s,\varepsilon) + i\omega(\varepsilon)t], \qquad \int_0^1 \varphi(s,\varepsilon)ds \equiv 0, \tag{97}$$

which is a canonical chimera for $\varepsilon \neq 0$. *Here the real functions* $\rho(s,\varepsilon)$, $\varphi(s,\varepsilon)$, $\omega(\varepsilon)$, *which are continuous in their variables and analytic in* ε *as* $\varepsilon \to 0$ *admit asymptotics:*

$$\begin{gathered}\rho(s,\varepsilon) = \rho_0 + \varepsilon\rho_1(s) + O(\varepsilon^2), \quad \varphi(s,\varepsilon) = \varepsilon\varphi_1(s) + O(\varepsilon^2),\\ \omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + O(\varepsilon^2),\end{gathered} \tag{98}$$

where

$$\begin{gathered}\rho_0 = \xi_0, \quad \rho_1(s) = \frac{\operatorname{Re}\varkappa(s)}{\xi_0}, \quad \varphi_1(s) = \frac{\operatorname{Im}\varkappa(s)}{\xi_0^2}, \quad \omega_1 = \int_0^1 \mu(s)ds,\\ \varkappa(s) = \frac{\nu c_1 - i}{2\nu(c_0c_1 - 1)}(\omega_1 - \mu(s)),\end{gathered} \tag{99}$$

and ξ_0, ω_0 *are from* (86).

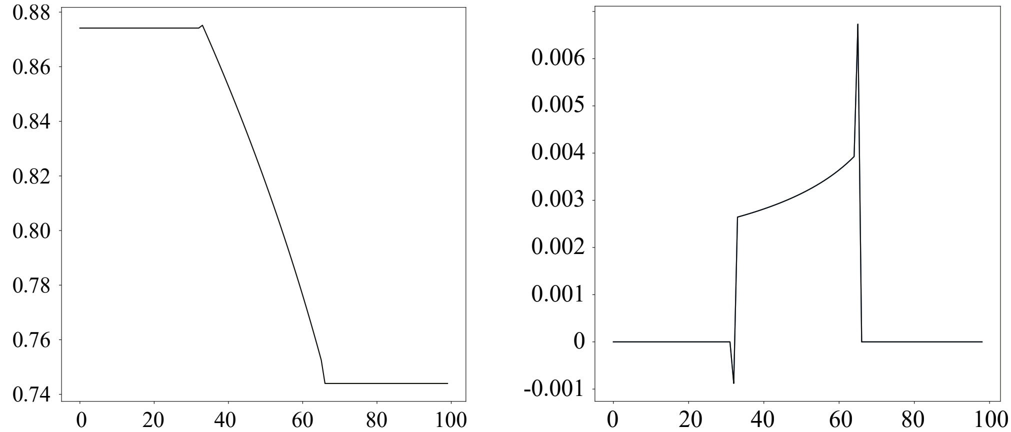
We do not dwell on the proof of Theorem 3, since in its conceptual part it is identical to the justification of Theorem 2. We only note that an analog of this theorem is also valid in the case when the function $\mu(s)$ has several alternating intervals of constancy and nontrivial dependence on s. It is only necessary to adjust the definition of a canonical chimera appropriately.

Concluding the consideration of the continuous model (95), we note that for $\varepsilon \neq 0$ it is still invariant under the change of variables (88), (89). Therefore, the canonical chimera (97) generates a continuum family $\mathscr{U}$ of exponentially orbitally stable cycles obtained from (97) under the indicated substitutions. As in the discrete case, it is appropriate to call these periodic regimes chimera-like structures.

Conclusion

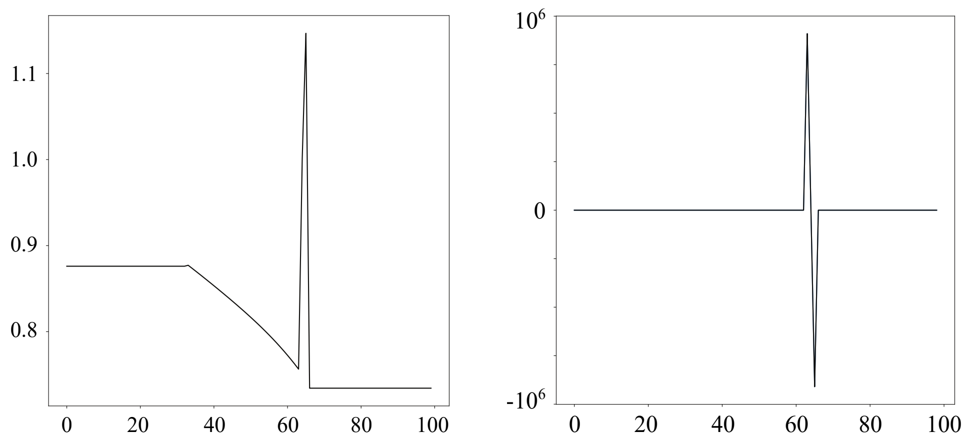
This section presents the results of a numerical analysis of the (49) system. The purpose of this analysis is to identify possible types of chimera-like regimes that exist in this system for fixed parameters ν, c_1, c_2, satisfying the conditions (62), and for different values of the control parameter $\varepsilon > 0$.

Before proceeding directly to the description of numerical experiments, let us dwell on the characteristic features of the system under study. We call two of its cycles equivalent if they pass into each other under the action of a change of variables of the form (45). It is clear that the properties of their stability and the possible bifurcations that occur with them are the same. In this regard, it is appropriate to group all cycles equivalent to each other into the corresponding families. One of such families is the set of chimera-like structures $\mathscr{U}$, the number of which is equal to $2^m - 1$ (which for $m = 100$ has the order of 10^{30}). Recall that these chimeras are obtained from the canonical chimera (55) under the action of the substitutions (45). There are other families of equivalent periodic regimes, different from $\mathscr{U}$ and also consisting of at least $2^m - 1$ elements. All this leads to the realization of the so-called phenomenon of fluctuation chaos. The essence of the phenomenon mentioned is that, due to the narrowness of the pools of attraction of individual stable cycles, the system "slides" over stable regimes. With a slight change in the initial conditions, a transition from one attractor to another occurs. It is clear that in such a situation it is not possible to trace the evolution of any one cycle or even a separate family of equivalent cycles by the parameter ε. In this regard, below we restrict ourselves to only some classification of



a *b*

Fig. 1. a — ρ_j, b — $\varphi_{j+1} - \varphi_j$; $\varepsilon = 0.07$



a *b*

Fig. 2. a — ρ_j, b — $\varphi_{j+1} - \varphi_j$; $\varepsilon = 0.08$, $t = 1000000$

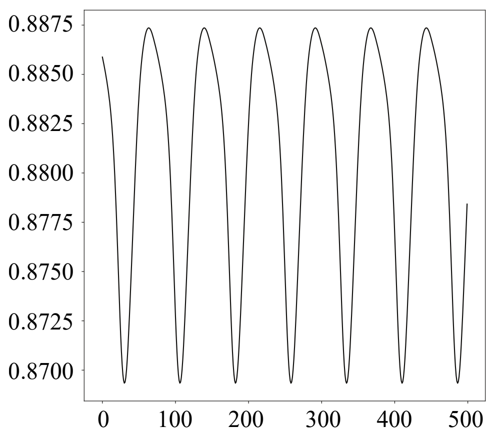
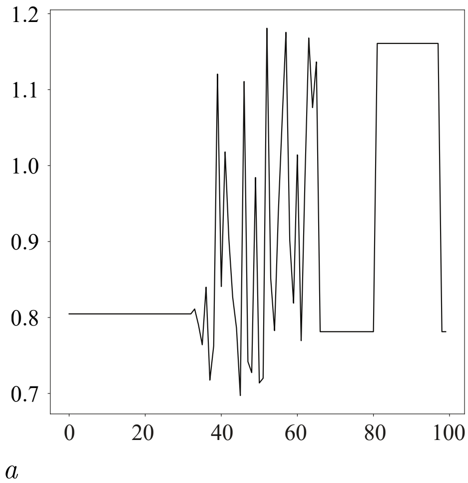


Fig. 3. Graph of $\rho_5(t)$ for $\varepsilon = 0.08$



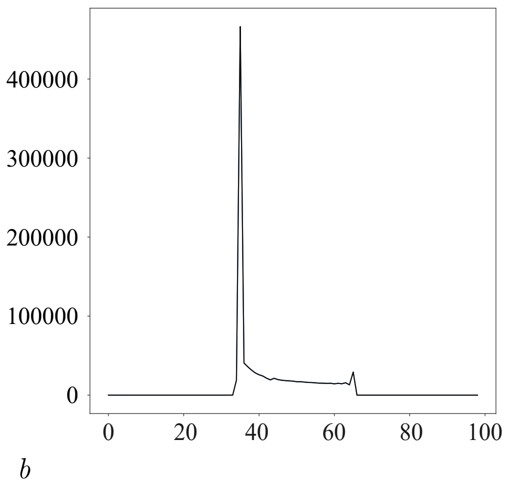


Fig. 4. a — ρ_j, b — $\varphi_{j+1} - \varphi_j$; $\varepsilon = 0.18$, $t = 1000000$

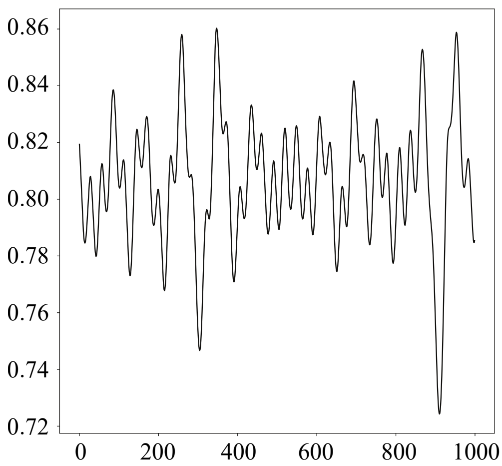


Fig. 5. Graph of $\rho_5(t)$ for $\varepsilon = 0.18$

chimera-like structures realized in the system (49).

Before the numerical integration of the system (49), a transition to polar coordinates was made in it

$$z_j = \rho_j \exp(i\varphi_j), \quad \rho_j > 0, \quad \varphi_j \in \mathbb{R}, \quad j = 1, 2, \ldots, m.$$

Further, with fixed parameters $c_0 = 1$, $c_1 = 3$, $\nu = 0.5$, $m = 100$ and

$$\mu_j = \mu(s)|_{s=j/m}, \quad \mu(s) = \begin{cases} -1 & \text{at } 0 \leqslant s \leqslant 1/3, \\ 3(2s-1) & \text{at } 1/3 \leqslant s \leqslant 2/3, \\ 1 & \text{at } 2/3 \leqslant s \leqslant 1 \end{cases}$$

and for different values of $\varepsilon > 0$, the segment of the trajectory of the resulting system for ρ_j, φ_j was calculated corresponding to the time interval $0 \leqslant t \leqslant 1000000$ and the initial conditions

$$\rho_j(0) = 0.2\Big(1 + 0.01 \cdot \frac{j}{j+1}\Big), \quad \varphi_j(0) = 0, \quad j = 1, \ldots, m.$$

At this way, the following types of chimera-like structures were identified.

For $0 < \varepsilon \lesssim 0.077$, stable cycles of the chimeric type are observed, which are continuations in parameter ε of cycles of the family $\mathscr{U}$. On fig. 1, a and b at $\varepsilon = 0.07$ for one of the chimeric cycles the dependencies of the variables $\rho_j(t)$ and $\varphi_{j+1}(t) - \varphi_j(t)$ from index $1 \leqslant j \leqslant m$ are shown. We emphasize that since this cycle has a self-similar form

$$z_j = z_j^0 \exp(i\omega t), \quad z_j^0 = \text{const} \in \mathbb{C}, \quad \omega = \text{const} \in \mathbb{R}, \quad j = 1, 2, \ldots, m, \tag{100}$$

then these variables do not depend on time. It is appropriate to call such a chimera stationary. Let us also add that for the given ε all chimeras from $\mathscr{U}$ are stationary.

With a subsequent increase in the parameter ε, one succeeds in discovering the so-called two-dimensional self-similar tori of the form

$$z_j = z_j^0(t) \exp(i\omega t), \quad z_j^0(t) \in \mathbb{C}, \quad \omega = \text{const} \in \mathbb{R}, \quad j = 1, 2, \ldots, m, \tag{101}$$

where $z_j^0(t)$ are some periodic complex-valued functions with period $T > 0$. We also add that tori (101) are certainly not reduced to the form (100). Chimera-like structures of this type will be called quasi-periodic. The figure 2, a, b shows the dependencies on $1 \leqslant j \leqslant m$ corresponding to one of these function tori $\rho_j(t)$ and $\varphi_{j+1}(t) - \varphi_j(t)$ for $\varepsilon = 0.08$ and fixed $t = 1000000$. We emphasize that, in contrast to the previous case, this chimeric regime is no longer stationary. In particular, due to (101) the corresponding polar radii $\rho_j(t)$ are periodic in t with period T. Dependence on t on the interval $999500\ t \leqslant 1000000$ of one of these radii, namely $\rho_5(t)$, is shown in the figure 3 (point zero on the horizontal axis corresponds to value $t = 999500$).

A further increase in the parameter ε leads to yet another complication of the dynamics. Namely, stable chimera-like regimes appear with a non-periodic dependence of the $\rho_j(t)$ components on t. We call such chimeras turbulent. On fig. 4, a, b for one of the turbulent chimeras in the case of $\varepsilon = 0.18$, the dependences $\rho_j(t)$ and $\varphi_{j+1}(t) - \varphi_j(t)$ on j at $t = 1000000$ are shown, and Fig. 5 shows $\rho_5(t)$ on the interval $999000 \leqslant t \leqslant 1000000$ (point zero on the horizontal axis corresponds to the value $t = 999000$).

In conclusion, we add that all the above characteristic features of the dynamics associated with the complication of chimeric regimes with increasing parameter ε are also preserved for the continuous model (95).

References

1. Kuramoto Y, Battogtokh D. Coexistence of coherence and incoherence in nonlocally coupled phase oscillators. Nonlinear Phenomena in Complex Systems. 2002;5(4):380–385.
2. Abrams DM, Strogatz SH. Chimera states for coupled oscillators. Phys. Rev. Lett. 2004;93(17): 174102. DOI: 10.1103/PhysRevLett.93.174102.
3. Panaggio MJ, Abrams DM. Chimera states: coexistence of coherence and incoherence in networks of coupled oscillators. Nonlinearity. 2015;28(3):R67. DOI: 10.1088/0951-7715/28/3/R67.
4. Sethia GC, Sen A. Chimera states: The existence criteria revisited. Phys. Rev. Lett. 2014;112(14): 144101. DOI: 10.1103/PhysRevLett.112.144101.
5. Schmidt L, Krischer K. Clustering as a prerequisite for chimera states in globally coupled systems. Phys. Rev. Lett. 2015;114(3):034101. DOI: 10.1103/PhysRevLett.114.034101.
6. Laing CR. Chimeras in networks with purely local coupling. Phys. Rev. E. 2015;92(5):050904. DOI: 10.1103/PhysRevE.92.050904.
7. Laing CR. Chimeras in networks of planar oscillators. Phys. Rev. E. 2010;81(6):066221. DOI: 10.1103/PhysRevE.81.066221.
8. Zakharova A, Kapeller M, Schöll E. Chimera death: Symmetry breaking in dynamical networks. Phys. Rev. Lett. 2014;112(15):154101. DOI: 10.1103/PhysRevLett.112.154101.

9. Omelchenko I, Zakharova A, Hövel P, Siebert J, Schöll E. Nonlinearity of local dynamics promotes multi-chimeras. Chaos. 2015;25(8):083104. DOI: 10.1063/1.4927829.
10. Omelchenko I, Omelchenko OE, Hövel P, Schöll E. When nonlocal coupling between oscillators becomes stronger: Patched synchrony or multichimera states. Phys. Rev. Lett. 2013;110(22):224101. DOI: 10.1103/PhysRevLett.110.224101.
11. Sakaguchi H. Instability of synchronized motion in nonlocally coupled neural oscillators. Phys. Rev. E. 2006;73(3):031907. DOI: 10.1103/PhysRevE.73.031907.
12. Hizanidis J, Kanas V, Bezerianos A, Bountis T. Chimera states in networks of nonlocally coupled Hindmarsh–Rose neuron models. International Journal of Bifurcation and Chaos. 2014;24(3): 1450030. DOI: 10.1142/S0218127414500308.
13. Zakharova A. Chimera Patterns in Networks: Interplay between Dynamics, Structure, Noise, and Delay. Berlin: Springer; 2020. 233 p. DOI: 10.1007/978-3-030-21714-3.
14. Glyzin SD, Kolesov AY, Rozov NK. Self-excited relaxation oscillations in networks of impulse neurons. Russian Mathematical Surveys. 2015;70(3):383–452. DOI: 10.1070/RM2015v070n03ABEH004951.
15. Glyzin SD, Kolesov AY, Rozov NK. Periodic two-cluster synchronization modes in completely connected genetic networks. Differential Equations. 2016;52(2):157–176. DOI: 10.1134/S0012266116020038.
16. Kolesov AY, Rozov NK. Invariant Tori of Nonlinear Wave Equations. Moscow: Fizmatlit; 2004. 408 p. (in Russian).
17. Mishchenko EF, Sadovnichii VA, Kolesov AY, Rozov NK. Autowave Processes in Nonlinear Media with Diffusion. Moscow: Fizmatlit; 2010. 400 p. (in Russian).
18. Daleckii JL, Krein MG. Stability of Solutions of Differential Equations in Banach Space. Providence: American Mathematical Society; 2002. 386 p.