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Dynamics of solutions of nonlinear functional differential equation of parabolic type

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Abstract. *Purpose* of this work is to study the initial-boundary value problem for a parabolic functional-differential equation in an annular region, which describes the dynamics of phase modulation of a light wave passing through a thin layer of a nonlinear Kerr-type medium in an optical system with a feedback loop, with a rotation transformation (corresponds the involution operator) and the Neumann conditions on the boundary in the class of periodic functions. A more detailed study is made of spatially inhomogeneous stationary solutions bifurcating from a spatially homogeneous stationary solution as a result of a bifurcation of the "fork" type and time-periodic solutions of the "traveling wave" type. *Methods.* To represent the original equation in the form of nonlinear integral equations, the Green's function is used. The method of central manifolds is used to prove the theorem on the existence of solutions of the indicated equation in a neighborhood of the bifurcation parameter and to study their asymptotic form. Numerical modeling of spatially inhomogeneous solutions and traveling waves was carried out using the Galerkin method. *Results.* Integral representations of the considered problem are obtained depending on the form of the linearized operator. Using the method of central manifolds, a theorem on the existence and asymptotic form of solutions of the initial-boundary value problem for a functional-differential equation of parabolic type with an involution operator on an annulus is proved. As a result of numerical modeling based on Galerkin approximations, in the problem under consideration, approximate spatially inhomogeneous stationary solutions and time-periodic solutions of the traveling wave type are constructed. *Conclusion.* The proposed scheme is applicable not only to involutive rotation operators and Neumann conditions on the boundary of the ring, but also to other boundary conditions and circular domains. The representation of the initial-boundary value problem in the form of nonlinear integral equations of the second kind allows one to more simply find the coefficients of asymptotic expansions, prove existence and uniqueness theorems, and also use a different number of expansion coefficients of the nonlinear component in the right-hand side of the original equation in the neighborhood of the selected solution (for example, stationary). Visualization of the numerical solution confirms the theoretical calculations and shows the possibility of forming complex phase structures.

Keywords: optical system, Kerr-type nonlinear medium, parabolic nonlinear equation, involution operator, stability solved.

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Introduction

In the last few decades, mathematical models of nonlinear optics have attracted the attention of researchers. They have rich dynamics of self-organizing systems, and variation of parameters allows you to control such dynamics and observe experimentally a wide range of changes in the light field. Characteristic is an optical system consisting of a thin layer of a nonlinear Kerr-type medium and a variously organized external contour of two-dimensional feedback [1–3]. Depending on the feedback implementation, models are considered that are described by ordinary differential equations or parabolic functional differential equations with the transformation of spatial variables of the desired function [4,5]. More general is the case of taking into account the delay in the system [4,6–8]. In this case, the functional differential equation

$$\tau_1 \frac{\partial u(x,t)}{\partial t} + u(x,t) = \mu \triangle u(x,t) + K(1+\gamma \cos Qu(x,t-\tau)), \quad x \in S, \quad t \geqslant 0$$

describes the phase modulation of a light wave $u(x,t)$ in a thin layer of a nonlinear Kerr-type medium within the aperture of $S \subset \mathbb{R}^2$. The specified equation is supplemented by boundary conditions at the boundary ∂S, as well as initial conditions for $(x,t) \in S \times [-\tau, 0]$. In the equation $\triangle$ — Laplace operator, $\mu > 0$ — diffusion coefficient of particles of a nonlinear medium, $K > 0$ — the nonlinearity coefficient proportional to the intensity of the input field, γ $(0 < \gamma < 1)$ — visibility (contrast) of the interference pattern, $Qu(x,t) = u(q(x),t)$, $q(x)$ — smooth reversible transformation of a spatial variable (for example, reflection, rotation).

The study of functional differential equations has a long history, starting with the works of A.,D.Myshkis [9], R. Bellman, K., Cook [10], the classical work of J. Hale [11], the cycle of works by A., L. Skubachevsky and his students [12], V. M. Varfolomeeva [13,14], A. B. Muravnik [15,16], works by A. V. Razgulin and his students [17–19], E. P. Belana [20,21] and his students [22–24], O. B. Lykova [25] and other authors.

Identification of traveling waves, rotating waves, fronts is of practical interest. Andronov's bifurcation−Hopf leads to the birth of rotating waves on the circumference in the case of the rotation transformation of spatial arguments [5,18,26,27]. Their interaction on a circle was studied in works [20,21], and two-dimensional rotating waves in a circle with rotation transformation were considered in [25].

The task of modeling the phenomena of structure formation manifested in experimental experiments, such as traveling (rotating) waves, is far from complete. The aperture of the region, the parameters of the problem, the organization of feedback, boundary conditions, as well as the choice of the bifurcation parameter play an essential role here. Unlike many studies in this paper, such a parameter is the diffusion coefficient μ. In the works of S. D. Glyzin, A. Yu. Kolesov, N. H. Pink (in particular, [28]) for dynamic systems of the reaction type−diffusion under the condition of a decrease in the diffusion coefficient, the phenomenon of multimode diffusion chaos is considered. The study of these problems is relevant both from the standpoint of the theory of nonlinear functional differential equations of the parabolic type, and in connection with various applications in nonlinear optics.

1. Problem statement

On the ring $S = \{(r,\theta) |\, 0 < r_1 \leqslant r \leqslant r_2;\ 0 \leqslant \theta \leqslant 2\pi\}$ the equation is considered

$$\frac{\partial u}{\partial t} + u = \mu \triangle u + K(1+\gamma \cos Qu), \quad (r,\theta) \in S, \quad t \geqslant 0, \tag{1}$$

where $u = u\,(r,\theta,t)$ with transformation of rotation by angle h, $Qu = u(r,\theta+h,t)$, for example, $h = (2\pi)/p$ $(p \in \mathbb{N})$, with Neumann conditions on the boundary

$$\frac{\partial u(r_1,\theta,t)}{\partial r} = \widetilde{g}_1(\theta,t), \quad \frac{\partial u(r_2,\theta,t)}{\partial r} = \widetilde{g}_2(\theta,t), \tag{2}$$

the initial condition

$$u\,(r,\theta,0) = u_0\,(r,\theta)\,, \tag{3}$$

periodicity condition

$$u(r, \theta + 2\pi, t) = u(r, \theta, t). \quad (4)$$

Here $\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ — Laplace operator in the polar coordinate system, $\mu > 0$ — diffusion coefficient of particles of a nonlinear medium, angle rotation transformation $h = (2\pi)/p$ defines the operator $Q = Q_h$, which is the involution operator $Q^p = I$ [29], $K > 0$ — a coefficient proportional to the intensity of the incoming flow, $\gamma\,(0 < \gamma < 1)$ — coefficient of visibility (contrast) of the interference pattern.

When investigating the problem (1)–(4), the following spaces are used: functional space $H = L_2^r(r_1, r_2) \times (0, 2\pi)$ — the space of functions of L^2 quadratically integrable with weight r, with scalar product and norm, respectively

$$\langle u, v\rangle_H = \int_0^{2\pi}\int_{r_1}^{r_2} u(r,\theta)\overline{v(r,\theta)}r dr d\theta, \quad \|u\|_H^2 = \int_0^{2\pi}\int_{r_1}^{r_2} |u(r,\theta)|^2 r dr d\theta;$$

functional space H^2 — Sobolev space of complex-valued functions of two real variables with scalar product and norm, respectively

$$\langle u, v\rangle_{H^2} = \langle u, v\rangle_H + \langle -\triangle u, -\triangle v\rangle_H, \quad \|u\|_{H^2}^2 = \sqrt{\langle u, u\rangle_{H^2}};$$

function space $H_{2\pi}^2 = \{u | u(\theta + 2\pi) = u(\theta)\}$ —closed space of 2π-periodic functions from H^2.

The correctness of the initial boundary value problem (1)–(4) for the ring S can be proved by analogy with the problem for the circle $0 < r < r_1$, proved earlier in the paper [30].

The problem of finding an approximate spatially inhomogeneous solution of the problem (1)–(4) bifurcating from its spatially homogeneous solution is considered. The diffusion coefficient is chosen as the bifurcation parameter μ.

2. Integral representations of the equation

Let w be one of the solutions to the problem (1)–(4). Let's replace $u = w+v$, where $v(r, \theta, t)$ — a new unknown function. Then taking into account $\cos(w + v) = \cos w \cos v - \sin w \sin v = \cos w(\cos v - 1)$ −
$- \sin w \sin v + \cos w$, we get

$$K\,[1 + \gamma \cos Q_h u] = K[1 + \gamma Q_h \cos(w + v)] =$$
$$= K\,[1 + \gamma \cos Q_h w] - K\gamma \sin Q_h w \cdot Q_h v + f(Q_h w, Q_h v),$$

where $f(Q_h w, Q_h v) = K\gamma\,(\cos Q_h w(\cos Q_h v - 1) - \sin Q_h w(\sin Q_h v - Q_h v))$.

The decomposition of the nonlinear function $f(Q_h v, Q_h w)$ into a series by powers of v begins с v^2 и $f(Q_h w, 0) = 0$.

The problem (1)–(4) with respect to v will take the form

$$\frac{\partial v}{\partial t} + v = \mu \triangle v - K\gamma \sin Q_h w \cdot Q_h v + f(Q_h v, Q_h w), \quad (r, \theta) \in S, \quad t \geqslant 0, \quad (5)$$

with conditions of the second kind on the boundary

$$\frac{\partial v(r_1, \theta, t)}{\partial r} = g_1(\theta, t), \quad \frac{\partial v(r_2, \theta, t)}{\partial r} = g_2(\theta, t), \quad g_i(\theta, t) \in H_{2\pi}^2, \quad i = 1, 2, \quad (6)$$

the initial condition

$$v\,(r,\theta,0) = v_0(r,\theta), \tag{7}$$

periodicity condition

$$v\,(r,\theta+2\pi,t) = v\,(r,\theta,t)\,. \tag{8}$$

Keeping in decomposition $f(Q_h v, Q_h w)$ a finite number of terms of the series, we obtain a series of model equations.

In [31], a detailed analysis of the partial solutions of $u(r,\theta,t) = w$ of the equation (1): stationary, equal to a constant, $u = w = \text{const}$; stationary, depending only on r, $u = w(r)$; stationary, depending only on θ, $u = w(\theta)$; stationary, depending on r and θ, $u = w(r,\theta)$; unsteady, depending only on t, $u = w(t)$; unsteady, depending on t и θ, $u = w(\theta,t)$.

The equation (5), depending on the specifics of the problem statement, is convenient to represent in three operator forms

$$\frac{\partial v}{\partial t} = A_j v + B_j, \quad j = 1,2,3,$$

$$A_1 v = \mu\triangle v, \quad B_1 = -v - K\gamma\sin Q_h w\cdot Q_h v + f(Q_h v, Q_h w);$$
$$A_2 v = \mu\triangle v - v, \quad B_2 = -K\gamma\sin Q_h w\cdot Q_h v + f(Q_h v, Q_h w);$$
$$A_3 v = \mu\triangle v - v - K\gamma\sin Q_h w\cdot Q_h v, \quad B_3 = f(Q_h v, Q_h w).$$

Lemma 1. *The operators A_j, $j = 1,2,3$ have a full in $L_2(\Omega)$,$\Omega = \{(r,\theta)\,|\,r_1 \leqslant r \leqslant r_2, 0 \leqslant \theta \leqslant 2\pi\}$ an orthonormal system of eigenfunctions*

$$\psi_{n,m}(r,\theta) = R_{n,m}(\lambda_{n,m} r)\exp[in\theta], \quad n = 0,\pm1,\pm2,\ldots; \quad m = 1,2,\ldots,$$

where

$$R_{n,m}(r) = R_{n,m}(\lambda_{n,m} r) = J_n\left(\lambda_{n,m} r\right)\cdot Y_n'\left(\lambda_{n,m} r_1\right) - Y_n\left(\lambda_{n,m} r\right)\cdot J_n'\left(\lambda_{n,m} r_1\right) \tag{9}$$

are defined through the Bessel functions J_n,Y_n [32] of the first and second kind, respectively, of the order n :

$$J_n(x) = \left(\frac{x}{2}\right)^n\cdot\widetilde{\psi}_n(x), \text{ где } \quad \widetilde{\psi}_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\cdot\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{2k},$$

$$Y_n(x) = \lim_{\alpha\to n}\left(\operatorname{ctg}\pi\alpha\cdot J_\alpha(x) - \frac{1}{\sin\pi\alpha}\cdot J_{-\alpha}(x)\right),$$

$\lambda_{n,m} = \widetilde{\lambda}$ *— a sequence of numbered in ascending order of the roots of the equation*

$$J_n'\left(\widetilde{\lambda} r_1\right)\cdot Y_n'\left(\widetilde{\lambda} r_2\right) - J_n'\left(\widetilde{\lambda} r_2\right)\cdot Y_n'\left(\widetilde{\lambda} r_1\right) = 0. \tag{10}$$

The functions $R(r) = R_{n,m}\left(\lambda_{m,n}\right)$ are solutions of the boundary value problem for the Bessel equation

$$r^2 R''(r) + rR'(r) + \left(\widetilde{\lambda}^2 r^2 - n^2\right)R(r) = 0, \quad R'(r_1) = 0, \quad R'(r_2) = 0, \quad n = 0,\pm1,\pm2,\ldots \tag{11}$$

Eigenvalues: $\lambda = -\mu\lambda_{n,m}^2$ (for the operator A_1); $\lambda = -1 - \mu\lambda_{n,m}^2$ (for A_2); $\lambda = -1 - \mu\lambda_{n,m}^2 + \Lambda\exp[inh]$ (for A_3), $n = 0,\pm1,\pm2,\ldots$,$\Lambda = -K\gamma\sin Q_h w = -K\gamma\sin w$, for $w = \text{const}$.

Proof. Eigenfunctions $\psi_{n,m}(r,\theta)$ we get as a result of applying the method of separation of variables for equations $\frac{\partial v}{\partial t} = A_j v$, $j = 1,2,3$. For example, for an equation with the operator A_3, representing $v(r,\theta,t) = X(r,\theta)\cdot T(t) \equiv R(r)\cdot\Phi(\theta)\cdot T(t)$, we come to the problem of Storming – Liouville for $X(r,\theta)$:

$$\begin{gathered}\mu\triangle X(r,\theta) - X(r,\theta) + \Lambda Q_h X(r,\theta) = \lambda X(r,\theta),\\ \frac{\partial X(r_1,\theta)}{\partial r} = 0, \quad \frac{\partial X(r_2,\theta)}{\partial r} = 0, \quad X(r,\theta+2\pi) = X(r,\theta)\end{gathered} \tag{12}$$

and the equation for the function $T(t)$: $T'(t) - \lambda T(t) = 0$.

Separating the variables in (12), for $R(r)$ we come to the problem (11), and for $\Phi(\theta)$, unlike the problem with operators $A_{1,2}$ $(\Lambda \equiv 0)$, the Storming problem– of Liouville has the form:

$$\Phi''(\theta) + \nu\Phi(\theta) = 0, \quad Q_h\Phi(\theta) + \alpha\Phi(\theta) = 0, \quad \Phi(\theta+2\pi) = \Phi(\theta), \quad 0 \leqslant \theta \leqslant 2\pi,$$

$$\nu = \nu_n = n^2, \quad \Phi = \Phi_n(\theta) = \frac{1}{\sqrt{2\pi}}\exp[\pm in\theta], \quad \alpha_n = -\frac{Q_h\Phi_n(\theta)}{\Phi_n(\theta)}, \quad n = 0, \pm1, \pm2, \ldots$$

Let's define the eigenvalues of the λ operator A_3 (12).

Laying out in a row $X(r,\theta) = \sum\limits_{n=-\infty}^{+\infty}\sum\limits_{m=1}^{+\infty} v_{n,m}\psi_{n,m}(r,\theta)$ by functions $\psi_{n,m}(r,\theta)$, we get for the coefficients of the decomposition $v_{n,m}$

$$\left(-1 - \mu\lambda_{n,m}^2 + \Lambda\exp[inh]\right) v_{n,m} = \lambda v_{n,m}, \quad n = 0, \pm1, \pm2, \ldots, \quad m = 1, 2, \ldots$$

Whence it follows that for A_3

$$\lambda = -1 - \mu\lambda_{n,m}^2 + \Lambda\exp[inh], \quad n = 0, \pm1, \pm2, \ldots, \quad m = 1, 2, \ldots$$

With $\Lambda = 0$, we get eigenvalues for the operator A_2 and, obviously, $\lambda = -\mu\lambda_{n,m}^2$ for the operator A_1.

The solutions of the boundary value problem (11) are the functions $R(r) = C_1 J_n(r) + C_2 Y_n(r)$. Given the boundary conditions of the problem (11), we get:

$$\begin{gathered}C_1 J_n'(\widetilde{\lambda} r_1) + C_2 Y_n'(\widetilde{\lambda} r_1) = 0,\\ C_1 J_n'(\widetilde{\lambda} r_2) + C_2 Y_n'(\widetilde{\lambda} r_2) = 0.\end{gathered} \tag{13}$$

The system (13) with respect to C_1 and C_2 has a nontrivial solution if $\widetilde{\lambda}$ is a solution of the equation (10). It is known that the equation (10) has a countable number of positive roots [32] $\lambda_{n,m}$, $n = 0, \pm1, \pm2, \ldots$, $m = 1, 2, \ldots$ Then C_1 and C_2 are determined from any equation of the system (13) (that is, the solution is determined (9)):

$$C_1 = -C_2\frac{Y_n'(\lambda_{n,m} r_1)}{J_n'(\lambda_{n,m} r_1)} \quad \text{или} \quad C_2 = -C_1\frac{J_n'(\lambda_{n,m} r_1)}{Y_n'(\lambda_{n,m} r_1)}. \qquad \square$$

Based on the calculations of the method of separating variables of Lemma 1, we write down the Green function for the operator A_1:

$$\begin{gathered}G_1(r,\rho,\theta,\varphi,t,\tau) = \frac{\rho}{2\pi}\sum_{n=-\infty}^{+\infty}\sum_{m=1}^{\infty}\frac{\exp[-in(\theta-\varphi)] R_{n,m}(r) R_{n,m}(\rho)\exp[-\mu\lambda_{n,m}^2(t-\tau)]}{d_{n,m}^2},\\ d_{n,m}^2 = \frac{2}{\pi^2\lambda_{n,m}^2 r_1^2}\left[\frac{\pi^2 r_1^2}{4}\left(\lambda_{n,m}^2 r_2^2 - n^2\right)\left(R_{n,m}(r_2)\right)^2 - \left(\lambda_{n,m}^2 r_1^2 - n^2\right)\right],\\ n = 0, \pm1, \pm2, \ldots \quad m = 1, 2, \ldots\end{gathered} \tag{14}$$

For a inhomogeneous linear equation with operator A_1 with conditions reduced to homogeneous boundary and initial conditions

$$v_t = A_1 v + v_0(r,\theta)\delta(t) + g_2(\theta,t)\delta(r-r_2) - g_1(\theta,t)\delta(r-r_1) + f(r,\theta,t) \equiv A_1 v + f_1,$$

the solution can be represented through the Green function

$$v(r,\theta,t) = \int\limits_0^t \int\limits_{r_1}^{r_2} \int\limits_0^{2\pi} G_1(r,\rho,\theta,\varphi,t,\tau) f_1(\rho,\varphi,\tau)\, d\varphi \rho d\rho d\tau.$$

Green's function G_2 for operator A_2 will differ from G_1 by the multiplier $\exp[-(1+\mu\lambda_{n,m}^2)t]$ instead of $\exp[-\mu\lambda_{n,m}^2 t]$, and G_3 for the operator A_3 is a multiplier $\exp[(-1-\mu\lambda_{n,m}^2 + \Lambda\exp[inh])t]$.

In the case of nonlinear equations with operators A_j with zero boundary and non-zero initial conditions, the use of Green's functions G_j leads to nonlinear equations of the following form

$$v_j(r,\theta,t) = \int\limits_0^t \int\limits_{r_1}^{r_2} \int\limits_0^{2\pi} G_j(r,\rho,\theta,\varphi,t,\tau)[B_j(v(\rho,\varphi,\tau)) + v_0(\rho,\varphi)\delta(\tau)]\, d\varphi \rho d\rho d\tau, \quad j=1,2. \tag{15}$$

The equations (15) are convenient for approximate calculations and estimates.

3. Solution bifurcation, asymptotic representation

Further in this section we consider spatially inhomogeneous stationary solutions bifurcating from spatially homogeneous stationary solution $u(r,\theta,t) = w = \text{const}$, which is defined by the equality

$$w = K(1+\gamma\cos w). \tag{16}$$

Known [33], that with an increase in K, the number of roots of the equation (16) grows and their composition changes. Let's fix a smooth branch corresponding to one of the solutions (16)

$$w = w(K,\gamma), \quad 1 + K\gamma\sin w(K,\gamma) \neq 0.$$

We linearize the equation (1) on the selected stationary spatially homogeneous solution $w(K,\gamma)$, we replace $u = v + w$ and, having selected the linear part, we obtain the equation:

$$\frac{\partial v}{\partial t} + v = \mu\triangle v - K\gamma\sin w \cdot Q_h v + f(Q_h v, w), \ (r,\theta) \in S, \ t \geqslant 0.$$

To detect solutions that have been observed in experiments (for example, [34]), assume that the operator Q_h is involutive: $Q_h^p = I$. Let's choose $h = 2\pi/p$, $p \in \mathbb{N}$. $p \geqslant 3$ is of interest (the case of $p=2$ was investigated earlier [24]).

Operator eigenvalues A_3

$$\lambda = -1 - \mu\lambda_{n,m}^2 - K\gamma\sin w \exp\left[i\frac{2\pi n}{p}\right], \quad n = 0, \pm1, \pm2, \ldots, \quad m = 1, 2, \ldots. \tag{17}$$

Lemma 2. *The stability of the solution v is determined by the sign of the real part of the expression (17): $\lambda = \alpha_{n,m} + i\beta_{n,m}$, где $\operatorname{Re}\lambda = \alpha_{n,m} = -1 - \mu\lambda_{n,m}^2 - K\gamma\sin w\cos(2\pi n/p)$, $\operatorname{Im}\lambda = \beta_{n,m} = -K\gamma\sin w\sin(2\pi n/p)$. If $\operatorname{Re}\lambda < 0$, then the solution v is stable, if $\operatorname{Re}\lambda > 0$, then the solution v is unstable [35, page 29].*

The real part $\alpha_{n,m} = -1-\mu\lambda^2_{n,m}+\Lambda\cos(2\pi n/p)$ contains the parameters K, γ, w, μ, of which in the general case, K and $\mu aresignificant$. Let's choose the diffusion coefficient μ as a bifurcation parameter, fixing at this Λ.

In general, for $Q_h\Phi(\theta) = \Phi(\theta + 2\pi/p)$, the following statement is true.

Statement 1. *The solution of the linearized problem corresponding to* (5)–(8) *can be represented as*

$$\sum_{n=-\infty}^{+\infty}\sum_{m=0}^{+\infty} C_{n,m}R_{n,m}(r)\exp\left[-in\theta\right]\exp\left[\left(-1-\mu\lambda^2_{n,m}+\Lambda\exp\left[i\frac{2\pi n}{p}\right]\right)t\right].$$

Depending on the values of the real and imaginary parts of λ, various types of solutions can be obtained, in particular, when $\operatorname{Re}\lambda \neq 0, \operatorname{Im}\lambda = 0$, we obtain stationary solutions (5)–(8), for $\operatorname{Re}\lambda = 0, \operatorname{Im}\lambda \neq 0$ we get purely periodic solutions (5)–(8)

$$A_{n,m}(r)\exp\left[-i\left(n\theta + \Lambda\sin\frac{2\pi n}{p}t\right)\right].$$

3.1. The method of central manifolds. To study bifurcation phenomena, we use an accepted methodology based on the construction of a hierarchy of simplified models in the vicinity of bifurcation points [20, 36].

Next, consider the case of $\operatorname{Re}\lambda \neq 0$. Using the linearization of the equation (5) on a dedicated stationary spatially homogeneous solution $w(K,\gamma)$, we will replace $u = v + w$ and consider one of the model problems of the problem (5)–(8):

$$\frac{\partial v}{\partial t} + v = \mu\triangle v - K\gamma\sin w\cdot Q_h v - \frac{K\gamma\cos w}{2!}\cdot Q_h v^2 + \frac{K\gamma\sin w}{3!}\cdot Q_h v^3,$$
$$0 < r_1 \leqslant r \leqslant r_2,\quad 0 \leqslant \theta \leqslant 2\pi,\quad t \geqslant 0, \tag{18}$$

with conditions of the second kind on the boundary

$$\frac{\partial v(r_1,\theta,t)}{\partial r} = 0,\quad \frac{\partial v(r_2,\theta,t)}{\partial r} = 0, \tag{19}$$

the initial condition

$$v\left(r,\theta,0\right) = 0 \tag{20}$$

and the periodicity condition

$$v\left(r,\theta+2\pi,t\right) = v\left(r,\theta,t\right). \tag{21}$$

Given that $\Lambda = -K\gamma\sin w$, and, denoting $\Omega = -K\gamma(\operatorname{ctg} w/2)$, (18) we write as

$$\frac{\partial v}{\partial t} = -v + \mu\triangle v + \Lambda Q_h v + \Omega Q_h v^2 - \frac{\Lambda}{6}Q_h v^3,$$
$$0 < r_1 \leqslant r \leqslant r_2,\quad 0 \leqslant \theta \leqslant 2\pi,\quad t \geqslant 0. \tag{22}$$

The equation (22), linearized in the neighborhood of the zero solution, is represented in the form

$$\frac{\partial v}{\partial t} = A_3 v, \tag{23}$$

where $A_3 v = -v + \mu\triangle v + \Lambda Q_h v$.

Next, we assume that $h = \pi/3$ (other cases can be considered similarly).

The linear operator A_3 with domain of definition H^2, considered as an unbounded operator in space H, is a self-adjoint operator. Based on Lemma 1, it is established

Lemma 3. *For the case of rotation,* $h = \pi/3$, $Re\lambda_n \neq 0$, $\operatorname{Im}\lambda_n = \beta_{n,m} = K\gamma \sin w \sin(\pi n/3) = 0$, *the ratorus* A_3 *corresponds to a series expansion by eigenfunctions* $\psi_{3s,m}(r,\theta) = R_{3s,m}\cos 3s\theta, s = 1, 2, \ldots$ *with eigenvalues*

$$\lambda_{3s} = -\mu\lambda_{3s,m}^2 - 1 + (-1)^s\Lambda, \tag{24}$$

where $\lambda_{3s,m}$ *— m-the root of the equation* (10).

The proof follows from the general case (Lemma 1) of the decomposition of the linear operator A_3, considered in the Hilbert space H with domain of definition H^2 by a complete orthonormal system of eigenfunctions $\psi_{n,m}(r,\theta)$.

The following theorem holds.

Theorem 1. *For* $h = \pi/3$, $\Lambda < -1$, *there exists* $\delta > 0$, *such that for a fixed value of* $m = 1$ *and for any values of the parameter* μ *satisfying the inequality* $\mu_1 - \delta < \mu < \mu_1$, *где* $\mu_s = (-1-(-1)^s\Lambda)/\lambda_{3s,m}^2$, $s = 1, 2, \ldots$, *there is a continuous branch of stationary points* $z(\mu) > 0$ *of the equation*

$$\dot{z} = \lambda_3(\mu)z + \frac{1}{2d_{3,1}^2}\left(\frac{\Lambda\gamma_1}{4} - \frac{\Omega^2\gamma_2^2}{(2\lambda_3 - \lambda_6)\,d_{6,1}^2}\right)z^3 + \ldots, \tag{25}$$

which corresponds to the stationary solution $v = \varphi(r,\theta,\mu)$ *equations (22), defined by equality*

$$\varphi(r,\theta,\mu) = zR_{3,1}(r)\cos 3\theta + z^2P_6(r,\mu)\cos 6\theta + z^3P_9(r,\mu)\cos 9\theta + \xi(z,r,\theta,\mu)\,|_{z=z(\mu)}, \tag{26}$$

$$P_6(r,\mu) = \frac{\Omega\gamma_2}{2\,(2\lambda_3 - \lambda_6)\,d_{6,1}^2}\cdot R_{6,1}(r), \tag{27}$$

$$P_9(r,\mu) = \frac{1}{2\,(3\lambda_3 - \lambda_9)\,d_{9,1}^2}\left[-\frac{\Omega^2\gamma_2\gamma_3}{(2\lambda_3 - \lambda_6)\,d_{6,1}^2} + \frac{\Lambda\gamma_4}{12}\right]\cdot R_{9,1}(r), \tag{28}$$

where $\xi(z,r,\theta,\mu) = O(|z|^4)$, $R_{3s,1}$ *and* $d_{3s,1}^2$, $s = 1, 2, 3$ *are defined by the equalities* (9) *and* (14), *respectively,*

$$\gamma_1 = \int_{r_1}^{r_2} rR_{3,1}^4(r)dr, \quad \gamma_2 = \int_{r_1}^{r_2} rR_{3,1}^2(r)R_{6,1}(r)dr,$$
$$\gamma_3 = \int_{r_1}^{r_2} rR_{3,1}(r)R_{6,1}(r)R_{9,1}(r)dr, \quad \gamma_4 = \int_{r_1}^{r_2} rR_{3,1}^3(r)R_{9,1}(r)dr. \tag{29}$$

The solution $\varphi(r,\theta,\mu)$ *— is orbitally stable.*

Proof. According to Theorem 5.1.1 of [36] and Lemmas 1, 2, if $\Lambda > 1$, then the null solution (23) is unstable for any $\mu > 0$. If $-1 < \Lambda < 1$, then the null solution (23) is asymptotically stable for any $\mu > 0$. The case of $\Lambda < -1 is of interest$. Now let's choose K so that the condition is met: $\Lambda = \Lambda\,(K) < -1$.

Here $\mu_s = (-1-(-1)^s\Lambda)/\lambda_{3s,m}^2$, $s = 1, 2, \ldots$. If $\mu > \mu_1$, then according to the Lemma 2 the null solution of the problem (23) is stable. When the parameter μ decreases and it passes through the value μ_1, one eigenvalue λ_3 passes through the imaginary axis.

If $\mu_2 < \mu < \mu_1$, then the instability index of the null solution is 1. The instability index of the null solution increases by one when μ decreases and it passes through μ_s, $s = 2, 3, \ldots$.

In neighborhood $v = 0$ for μ satisfying the inequality $\mu_1 - \delta < \mu < \mu_1$, there exists a central manifold [36], which can be represented as

$$\varphi(r,\theta,\mu) = zR_{3,1}(r)\cos 3\theta + z^2P_6(r,\mu)\cos 6\theta + z^3P_9(r,\mu)\cos 9\theta + \xi(z,r,\theta,\mu)\,|_{z=z(\mu)}, \tag{30}$$

where $P_6(r,\mu)$, $P_9(r,\mu),\ldots$ functions from space $L_2^r[r_1,r_2]$. On the manifold (30), the equation (22) takes the form

$$\dot{z} = \lambda_3(\mu)z + C_2z^2 + C_3z^3. \tag{31}$$

Find the coefficients of the expansions (30) and (31). For this, we substitute (30) and (31) into equation (22):

$$\begin{aligned}
&\Big(\lambda_3(\mu)z + C_2z^2 + C_3z^3\Big)\Big(zR_{3,1}(r)\cos 3\theta + 2zP_6(r,\mu)\cos 6\theta + 3z^2P_9(r,\mu)\cos 9\theta\Big) = \\
&= \mu\Big(zR''_{3,1}(r)\cos 3\theta + z^2P''_6(r,\mu)\cos 6\theta + z^3P''_9(r,\mu)\cos 9\theta + \\
&+\frac{zR'_{3,1}(r)\cos 3\theta}{r} + \frac{z^2P'_6(r,\mu)\cos 6\theta}{r} + \frac{z^3P'_9(r,\mu)\cos 9\theta}{r} - \\
&-\frac{9zR''_{3,1}(r)\cos 3\theta}{r^2} - \frac{36z^2P''_6(r,\mu)\cos 6\theta}{r^2} - \frac{81z^3P''_9(r,\mu)\cos 9\theta}{r^2}\Big) + \\
&+\Lambda\Big(-zR_{3,1}(r)\cos 3\theta + z^2P_6(r,\mu)\cos 6\theta - z^3P_9(r,\mu)\cos 9\theta\Big) + \\
&+\Omega\Big(-zR_{3,1}(r)\cos 3\theta + z^2P_6(r,\mu)\cos 6\theta - z^3P_9(r,\mu)\cos 9\theta\Big)^2 - \\
&-\frac{\Lambda}{6}\Big(-zR_{3,1}(r)\cos 3\theta + z^2P_6(r,\mu)\cos 6\theta - z^3P_9(r,\mu)\cos 9\theta\Big)^3.
\end{aligned} \tag{32}$$

Due to the orthogonality of the system of eigenfunctions $\cos 3s\theta$, $s = 1,2,3$, we equate the coefficients for these functions in the left and right parts of equality (32). R $\cos 3\theta$ we get

$$\begin{aligned}
&R_{3,1}(r)\Big[z\Big(\lambda_3 + 1 + \Lambda\Big) + C_2z^2(t) + C_3z^3\Big] = \\
&= \mu z\Big(R''_{3,1}(r) + \frac{R'_{3,1}(r)}{r} - \frac{9R_{3,1}(r)}{r^2}\Big) - \Omega z^3R_{3,1}(r)P_6(r,\mu) + \frac{\Lambda z^3R^3_{3,1}(r)}{8}.
\end{aligned}$$

Since $R_{3,1}(r)$ — solution of the boundary value problem of the Bessel equation (11), then

$$\begin{aligned}
&R_{3,1}(r)\Big[z\Big(\lambda_3 + 1 + \Lambda + \mu\lambda^2_{3,1}\Big) + C_2z^2 + C_3z^3\Big] = \\
&= -\Omega z^3R_{3,1}(r)P_6(r,\mu) + \frac{\Lambda z^3R^3_{3,1}(r)}{8}.
\end{aligned}$$

Since $\lambda_3 = -1 - \mu\lambda^2_{3,1} - \Lambda$, then $C_2 = 0$, $C_3R_{3,1}(r) = -\Omega R_{3,1}(r)P_6(r,\mu) + \dfrac{\Lambda R^3_{3,1}(r)}{8}$. Therefore,

$$C_3 = \frac{1}{d^2_{3,1}}\left(-\Omega\int\limits_{r_1}^{r_2} rR^2_{3,1}(r)P_6(r,\mu)dr + \frac{\Lambda\gamma_1}{8}\right), \tag{33}$$

where $\gamma_1 = \int\limits_{r_1}^{r_2} rR^4_{3,1}(r)dr$, $P_6(r,\mu) = \dfrac{\gamma_2\Omega R_{6,1}(r)}{2d^2_{6,1}(2\lambda_3 - \lambda_6)}$.

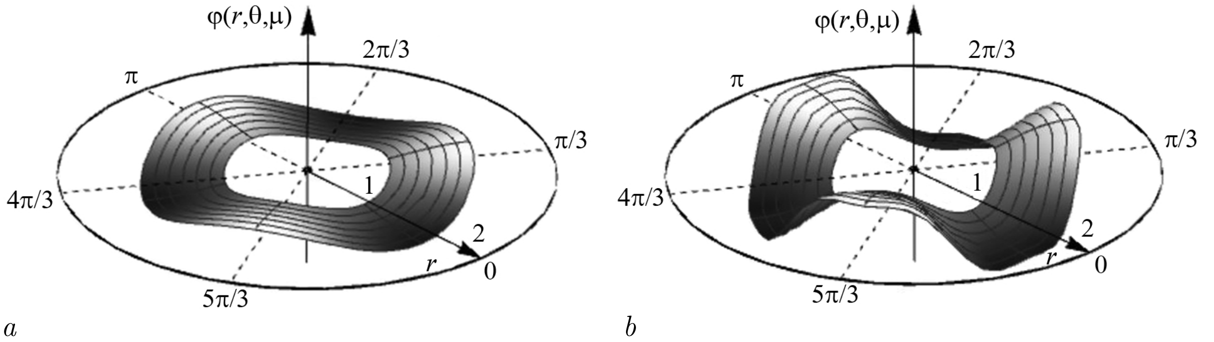


Fig. 1. Приближённое стационарное решение (30) для $\Lambda = -3/2$, $h = \pi/3$ в цилиндрической системе координат при $\mu = 0.1$ (a) и $\mu = 0.01$ (b)

Fig. 1. Approximate stationary solution of (30) for $\Lambda = -3/2$, $h = \pi/3$ in a cylindrical coordinate system for $\mu = 0.1$ (a) and $\mu = 0.01$ (b)

Based on the condition $\Lambda < -1$ and the equality (24), it is obvious that $C_3 < 0$, then there is a supercritical bifurcation of the "fork"type (see [36], ch. 6. 3) and from the trivial singular the points of the equation (25) branch off two stable stationary points.

Carrying out similar (cumbersome) calculations for $s = 2, 3$, we obtain the statements of the Theorem.

Thus, in some neighborhood μ_1 there exists a stationary solution $v = \varphi(r, \theta, \mu)$ the equations (22), defined by the equalities (26)–(28). The solution $\varphi(r, \theta, \mu)$ — is orbitally stable. □

The theorem is local in nature.

When using the package «Wolfram Mathematica 11.3» for $\Lambda = -3/2$, $h = \pi/3$ approximate solutions of (30) obtained in the Theorem are constructed for various values of the bifurcation parameter μ (fig. 1).

Corollary 1. The results obtained are consistent with the one-dimensional case when a narrow ring can be replaced by a circle. For $h = \pi/3$, $\Lambda < -1$, there exists $\delta > 0$ and $\mu_1 = (-1 - \Lambda)/9$, such that for any values of the parameter μ satisfying the inequality $\mu_1 - \delta < \mu < \mu_1$,there is a continuous branch of stationary points $z(\mu) > 0$ of the equation

$$\dot{z} = \lambda_3(\mu)z + \frac{1}{2}\left(\frac{\Lambda}{4} - \frac{\Omega^2}{(2\lambda_3 - \lambda_6)}\right) z^3 + \ldots,$$

which corresponds to the stationary solution $v{=}\varphi_1(\theta, \mu)$ equations (22) on circle $S\,(r_1{=}r_2)$ with condition 2π-periodicity, defined by equality

$$\varphi_1(\theta, \mu) = z\cos 3\theta + z^2 \frac{\Omega}{2\,(2\lambda_3 - \lambda_6)} \cos 6\theta + \\ + z^3 \frac{1}{2\,(3\lambda_3 - \lambda_9)} \left[-\frac{\Omega^2}{(2\lambda_3 - \lambda_6)} + \frac{\Lambda}{12}\right] \cos 9\theta + \xi(z, \theta, \mu)\,|_{z=z(\mu)},$$

where $\xi(z, \theta, \mu) = O(|z|^4)$, $\lambda_{3s} = -1 + (-1)^s\Lambda - (3s)^2\mu$, $s = 1, 2, 3$.

The solution $\varphi_1(\theta, \mu)$ — is orbitally stable.

3.2. The Galerkin method. In order to investigate the asymptotics of stationary solutions of the problem (18)–(21) with a decrease in the bifurcation parameter μ and its departure from the critical value of μ_1, we will use the Galerkin method, according to which we will present approximate solutions in the following form

$$\varphi^*(r,\theta) = \sum_{k=1}^{N} \left(z_k \exp[ik\theta] + \overline{z}_k \exp[-ik\theta]\right) R_{k,1}(r), \tag{34}$$

here $z_k, \overline{z}_k$ are complex conjugate expressions.

Requiring that the function $\varphi^*(r,\theta)$, defined by the equality (34), satisfies the equation (18), we obtain a system of ordinary differential equations

$$\begin{aligned} \dot{z}_k &= \lambda_k z_k + \sigma_k(z,\overline{z}), \\ \dot{\overline{z}}_k &= \overline{\lambda}_k \overline{z}_k + \overline{\sigma}_k(z,\overline{z}), \end{aligned} \tag{35}$$

where $\lambda_k(\mu) = -1 - \mu\lambda_{k,1}^2 + \exp[ikh]\Lambda$, $\overline{\lambda}_k(\mu) = -1 - \mu\lambda_{k,1}^2 + \exp[-ikh]\Lambda$, $\sigma_k(z,\overline{z})$, $\overline{\sigma}_k(z,\overline{z})$ — forms of the third degree from $z_k, \overline{z}_k$, $k = 1, 2, \ldots, N$.

One of the solutions of the (35) system is a null solution whose stability is determined by the spectrum $\{\lambda_k(\mu),\ \overline{\lambda}_k(\mu)\}$ of the corresponding stability matrix. As above, we assume that the condition is met $\Lambda < -1$.

Let $h = \pi/3$, then the first critical value of the bifurcation parameter at which the zero stationary solution of the system (35) loses stability, $\mu_1 = (-\Lambda - 1)/\lambda_{3,1}^2$. As a result, a bifurcation of the type «fork» occurs and at $\mu < \mu_1$ a pair of stable stationary points $\pm z^*(\mu) = \{0, 0, \pm z_3^*, 0, 0, \pm z_6^*, \ldots\}$, being solutions of an algebraic system of equations

$$\lambda_k z_k + \varepsilon_k(z_l) = 0, \quad k, l = 1, 2, \ldots, N, \tag{36}$$

where $\varepsilon_k(z_l)$ is a third degree polynomial containing the second and third degrees z_l.

Based on this, taking into account (34), the spatially inhomogeneous stationary solution of the problem (18)–(21) is determined by the asymptotic equality

$$\varphi^*(r,\theta,\mu) = \sum_{k=1}^{[N/3]} z_{3k}(\mu) \cos[3k\theta] R_{3k,1}(r). \tag{37}$$

For example, for $N = 3$, the solution is $z^*(\mu)$ is determined by the system

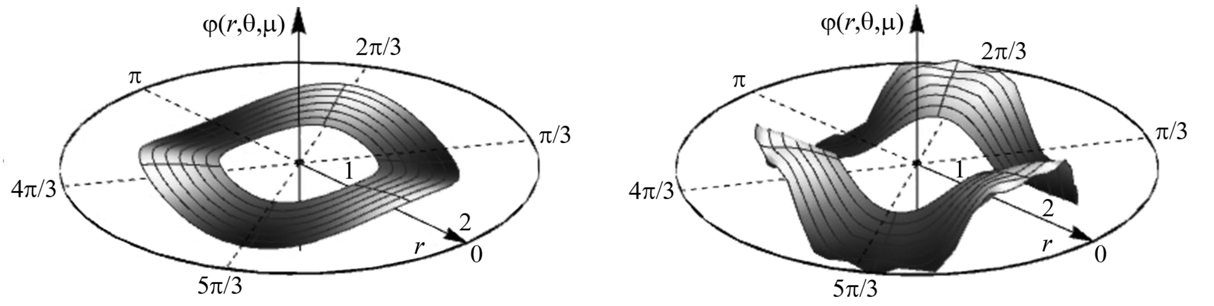
$$\lambda_3 z_1 + \frac{1}{8d_3^2}\left[\Lambda\left(\beta_3 z_1^3 + 2\delta_{36} z_2^2 z_1 + 2\delta_{39} z_3^2 z_1 + \zeta_{39} z_1^2 z_3 + \xi_{639} z_2^2 z_3\right) - 8\Omega z_2\left(\delta_{369} z_3 + \xi_{36} z_1\right)\right] = 0,$$

$$\lambda_6 z_2 + \frac{1}{8d_6^2}\left[-\Lambda z_2\left(\beta_6 z_2^2 + 2\delta_{36} z_1^2 + 2\delta_{69} z_3^2 + 2\xi_{639} z_1 z_3\right) + \Omega z_1\left(8\delta_{369} z_3 + 4\xi_{36} z_1\right)\right] = 0,$$

$$\lambda_9 z_3 + \frac{1}{24d_9^2}\left[3\Lambda\left(\beta_9 z_3^3 + 2\delta_{39} z_1^2 z_3 + 2\delta_{69} z_2^2 z_3 + \zeta_{39}\Lambda z_1^3 + \xi_{639} z_2^2 z_1\right) - 24\delta_{369}\Omega z_2 z_1\right] = 0,$$

where

$$\begin{gathered} \beta_k = \int_{r_1}^{r_2} r R_{k,1}^4(r)dr, k = 3, 6, 9; \quad \delta_{kl} = \int_{r_1}^{r_2} r R_{k,1}^2(r) R_{l,1}^2(r)dr, \quad k, l = 3, 6, 9 (k < l); \\ \zeta_{39} = \int_{r_1}^{r_2} r R_{3,1}^3(r) R_{9,1}(r)dr, \quad \xi_{36} = \int_{r_1}^{r_2} r R_{3,1}^2(r) R_{6,1}(r)dr, \\ \delta_{369} = \int_{r_1}^{r_2} r R_{3,1}^2(r) R_{6,1}^2(r) R_{9,1}^2(r)dr, \quad \xi_{639} = \int_{r_1}^{r_2} r R_{6,1}^2(r) R_{3,1}(r) R_{9,1}(r)dr. \end{gathered} \tag{38}$$



a *b*

Fig. 2. Приближённое решение (30), полученное с применением метода Галёркина, для $\Lambda = -3/2$, $h = \pi/3$ в цилиндрической системе координат при $\mu = 0.1$ (a) и $\mu = 0.01$ (b)

Fig. 2. Approximate solution (30) obtained using the Galerkin method for $\Lambda = -3/2$, $h = \pi/3$ in a cylindrical coordinate system for $\mu = 0.1$ (a) и $\mu = 0.01$ (b)

Numerical analysis for $N = 5$ was carried out at fixed values of the parameters $\Lambda = -3/2$, $Omega = 0.129264$, which corresponds to $K = 2, \gamma = 0.761058$, $w = 1.74147$. The following results were obtained.

1. Critical value of the bifurcation parameter $\mu^* \approx 0.113315$.
2. For $\mu > \mu^*$, the null solution of the system (36) is stable.
3. When the parameter μ decreases and the critical value μ^* passes, one of its own values of the spectrum of the stability matrix of the null solution λ_3 passes through zero and becomes positive. As a result, a bifurcation of the type «fork» occurs and a pair of stable stationary solutions branches off from the zero solution that is losing stability. In particular, for $\mu = 0.11331$, the solution of the system is (36) $z^*(\mu) = = \{0, 0, \pm 0.0481462.0, 0, \pm 0.0000198428, 0, \ldots\}$.
4. When the parameter μ is further reduced, the eigenvalue of λ_3 remains positive.
5. The spectrum of the stability matrix of the solution $z^*(\mu)$ lies on the negative semi-axis.

In the package «Wolfram Mathematica 11.3» for various values of the bifurcation parameter μ, approximate solutions of $\varphi(r, \theta, \mu) obtainedusingtheGalerkinmethodareconstructed$, determined by the equality (37) (Fig. 2).

Approximate solutions of the problem (18)–(21), constructed using the method of central manifolds and the Galerkin method, practically coincide.

3.3. Running wave. Note that, unlike the results obtained above, the presence of rotation of spatial coordinates can simulate a situation where a spatially homogeneous solution loses stability in an oscillatory manner when the parameters in the problem (μ, K) change. In this case, a traveling wave occurs.

Next, using the representation of the solution (34) of the problem (18)–(21) in the Galerkin method, we construct a two-mode approximation of its periodic solution of the type «traveling wave», which is born as a result of the Andronov – Hopf's bifurcation hspace1pt at the highest critical value of the parameter $\mu = \mu^* : \{\operatorname{Re}\lambda(\mu^*) = 0\}$ (24) from the zero solution losing vibrationally stability systems (35).

We are looking for the specified solution in the form

$$\begin{aligned} &z_1(t) = \rho_1 \exp[i\theta_1], \quad z_2(t) = 0, \quad z_3(t) = \rho_3 \exp[i(3\theta_1 + \alpha_3)]; \\ &\overline{z}_1(t) = \rho_1 \exp[-i\theta_1], \quad \overline{z}_2(t) = 0, \quad \overline{z}_3(t) = \rho_3 \exp[-i(3\theta_1 + \alpha_3)], \end{aligned} \tag{39}$$

where $\rho_k = \rho_k(t, \mu) > 0$, $\theta_k = \theta_k(t, \mu)$, $k = 1, 3$.

Counting $\theta_1(t, \mu) = \omega(\mu)t$. Substitute (39) into (35), we get a system for determining ρ_k, α_3 :

$$
\begin{aligned}
&\rho_1(\lambda_1^* d_1^2 - i\omega) - \frac{1}{2}\Lambda \exp[ih]\left[\xi_{13}\exp[i\alpha_3]\rho_3\rho_1^2 - \beta_1\rho_1^3 - 2\delta_{13}\rho_3^2\rho_1\right] = 0,\\
&\rho_1(\hat\lambda_1^* d_1^2 + i\omega) - \frac{1}{2}\Lambda \exp[-ih]\left[\xi_{13}\exp[-i\alpha_3]\rho_3\rho_1^2 - \beta_1\rho_1^3 - 2\delta_{13}\rho_3^2\rho_1\right] = 0,\\
&\rho_3(\lambda_3^* d_3^2 - 3i\omega) - \frac{1}{6}\Lambda \exp[3ih]\left[\xi_{13}\exp[-i\alpha_3]\rho_1^3 - 3\beta_3\rho_3^3 - 6\delta_{13}\rho_3\rho_1^2\right] = 0,\\
&\rho_3(\hat\lambda_3^* d_3^2 + 3i\omega) - \frac{1}{6}\Lambda \exp[-3ih]\left[\xi_{13}\exp[i\alpha_3]\rho_1^3 - 3\beta_3\rho_3^3 - 6\delta_{13}\rho_3\rho_1^2\right] = 0.
\end{aligned}
\tag{40}
$$

Here $d_k^2(k=1,3)$ is defined by the equality (14), $\beta_k(k=1,3)$, δ_{13} are defined by the equalities (38), $\xi_{13} = \int\limits_{r_1}^{r_2} rR_{1,1}^3(r)R_{3,1}(r)dr$.

With $\mu > \mu^*$, the system has only zero ($\rho_1 = 0, \rho_3 = 0$) sustainable solution. When the parameter μ decreases and the critical value μ^* passes, the null solution loses stability and at the same time a solution of the form is born:

$$
\begin{aligned}
&\rho_1^2(\mu) = \frac{6\Psi_3^c}{\Lambda\left(\Psi_1^c\left(3\gamma_1(\Psi_3^c)^2 - 4\delta_{13}\Psi_3^c - \gamma_3\Psi_1^c\right)\right)}\Bigg[\omega\left(3\omega\sin 4h + d_3^2\Upsilon_{13}^c\right) + \\
&\qquad + \left(d_1^2\left(3\omega\Upsilon_{31}^c + d_3^2\left(\Lambda\sin 3h\left(1+\mu\lambda_{3,1}^2\right) - \Upsilon^s\left(1+\mu\lambda_{1,1}^2\right)\right)\right)\right)\Bigg],\\
&\sin^2\alpha_3(\mu) = -\frac{12\Psi_1^c\Psi_3^c}{\Lambda^2\xi_{13}^2\rho_1(\mu)^4},\\
&\cos\alpha_3(\mu) = \frac{\sin\alpha_3(\mu)\left(\gamma_1\Lambda\rho_1^2(t) - 2d_1^2\Lambda\right)\Psi^s}{2\Psi_1^c} + \frac{4\delta_{13}\Psi_1^c}{\Lambda\xi_{13}^2\sin\alpha_3(\mu)\rho_1^2(t)},\\
&\rho_3(\mu) = \frac{-2\Psi_1^c}{\Lambda\xi_{13}\sin\alpha_3(\mu)\rho_1(\mu)},
\end{aligned}
\tag{41}
$$

where $\Psi_k^c = k\omega\cos(kh) - d_k^2\sin(kh)(\mu\lambda_{k,1}^2 + 1)$, $k = 1,3$, $\Psi^s = \omega\sin h + d_1^2\cos h(\mu\lambda_{1,1}^2 + 1)$, $\Upsilon_{jk}^c =$ $= -\Lambda\cos jh + \cos 4h(\mu\lambda_{k,1}^2 + 1)$, $(j,k = 1,3, j\neq k)$, $\Upsilon^s = -\Lambda\sin h + \sin 4h\left(\mu\lambda_{3,1}^2 + 1\right)$, the value of ω is determined from the basic trigonometric identity for $\alpha_3(\mu)$, the sign $\sin\alpha_3(\mu)$ is chosen opposite to the sign Ψ_1^c.

Therefore, the system (35) at $\mu < \mu^*$ has the solution

$$
\begin{aligned}
&z_1(t,\mu) = \rho_1(\mu)\exp[i\theta_1(\mu)t], \quad z_2(t) = 0, \quad z_3(t,\mu) = \rho_3(\mu)\exp[i(3\theta_1(\mu)t + \alpha_3(\mu))];\\
&\overline{z}_1(t,\mu) = \rho_1(\mu)\exp[-i\theta_1(\mu)], \quad \overline{z}_2(t) = 0,\ \overline{z}_3(t,\mu) = \rho_3(\mu)\exp[-i(3\theta_1(\mu)t + \alpha_3(\mu))].
\end{aligned}
\tag{42}
$$

Substituting (42) into (34), we get the periodic solution $\varphi^*(r,\theta,t,\mu)$ tasks (18)–(21). The specified solution is born stable.

Numerical simulation was performed for $h = 2\pi/3$ with fixed values of $\Lambda = -3/2$, $\Omega =$ 0.129264, which correspond to the parameters $K = 2$, $\gamma = 0.761058$, $w = 1.74147$ of the original problem. In the package «Wolfram Mathematica 11.3», Galerkin approximations of periodic solutions of $\varphi^*(r,\theta,t,\mu)$ were constructed for various values of the bifurcation parameter μ at $N = 5$ (Fig. 3).

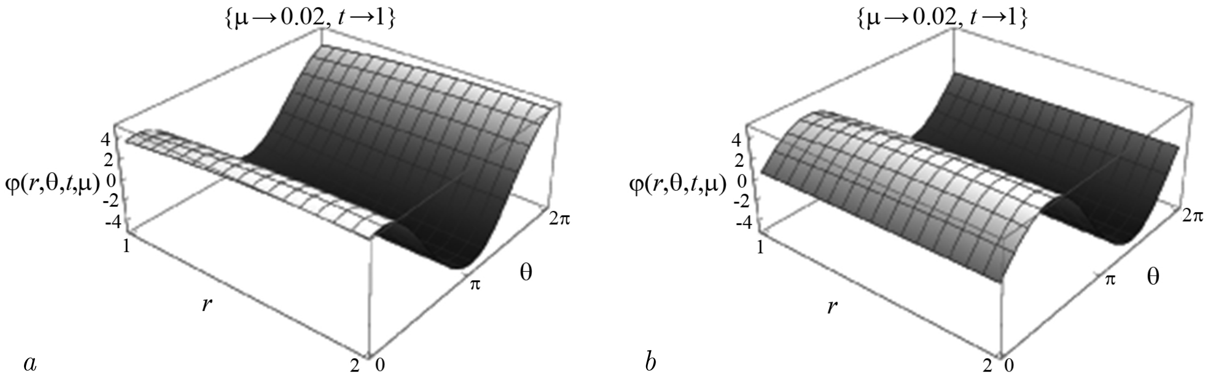


Fig. 3. Приближённое решение (30) типа «бегущая волна», полученное с применением метода Галёркина, для $\Lambda = -3/2$, $h = 2\pi/3$ при $\mu = 0.1$ (a) и $\mu = 0.01$ (b)

Fig. 3. An approximate solution (30) of the "traveling wave" type obtained using the Galerkin method for $\Lambda = -3/2$, $h = 2\pi/3$ for $\mu = 0.1$ (a) and $\mu = 0.01$ (b)

Conclusion

The paper considers an initial boundary value problem for a parabolic functional differential equation in a ring domain, which describes the dynamics of phase modulation of a light wave that has passed through a thin layer of a Kerr-type nonlinear medium in an optical system with a feedback loop, with an involution operator and by Neumann conditions on the boundary in the class of periodic functions. Using the Green's function, an integral representation of the equation under consideration is obtained, which makes it easier to find the coefficients of asymptotic expansions, prove the existence and uniqueness theorems (similar to [30]), and also use a different number of coefficients of expansion of the nonlinear component in the right side of the original equation in the neighborhood of the selected solution (for example, stationary).

Using the method of central manifolds, a theorem is proved on the existence in the vicinity of the bifurcation value of the parameter μ (diffusion coefficient) of a spatially inhomogeneous solution that branches off from a spatially homogeneous solution. Using the Galerkin method, numerical modeling of bifurcating spatially inhomogeneous stationary solutions and traveling waves at fixed parameter values was carried out.

The considered mathematical model corresponds to an optical scheme in which the phase of the light wave is visualized due to the Kerr nonlinearity. The phase distribution corresponds to the intensity distribution in the cross section. Visualization of the numerical solution confirms the theoretical calculations and shows the possibility of forming complex phase structures.

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