Izvestiya Vysshikh Uchebnykh Zavedeniy. Applied Nonlinear Dynamics. 2022;30(6)

Article DOI: 10.18500/0869-6632-2022-30-6-685-699

Dynamics of the Rabinovich–Fabrikant system and its generalized model in the case of negative values of parameters that have the meaning of dissipation coefficients

L. V. Turukina

Saratov Brunch of Kotelnikov Institute
of Radioengineering and Electronics of RAS, Russia
Saratov State University, Russia
E-mail: turukinalv@yandex.ru
Received 20.06.2022, accepted 22.08.2022,
available online 7.11.2022, published 30.11.2022

Abstract. *Purpose* of this work is a numerical study of the Rabinovich–Fabrikant system and its generalized model, which describe the occurrence of chaos during the parametric interaction of three modes in a nonequilibrium medium with cubic nonlinearity, in the case when the parameters that have the meaning of dissipation coefficients take negative values. These models demonstrate a rich dynamics that differs in many respects from what was observed for them, but in the case of positive values of the parameters. *Methods.* The study is based on the numerical solution of the differential equations, and their numerical bifurcation analysis using the MatCont program. *Results.* For investigated models we present a charts of dynamic regimes in the control parameters plane, Lyapunov exponents depending on the parameters, attractors and their basins. On the parameters plane, which have the meaning of dissipation coefficients, bifurcation lines and points are numerically found. They are plotted for equilibrium point and period one limit cycle. For both models we compared dynamics observed in the case when the parameters that have the meaning of dissipation coefficients take negative values, with the one observed in the case when these parameters take positive values. And it is shown that in the first case parameter space has a simpler structure. *Conclusion.* The Rabinovich–Fabrikant system and its generalized model were studied in detail in the case when the parameters which have the meaning of dissipation coefficients take negative values. It is shown that there are a number of differences in comparison with the case of positive values of these parameters. For example, a new type of chaotic attractor appears, multistability that is not related to the symmetry of the system disappears, etc. The obtained results are new, since the Rabinovich–Fabrikant system and its generalized model were studied in detail for the first time in the region of negative values of parameters which have the meaning of dissipation coefficients.

Keywords: Rabinovich–Fabrikant model, generalized Rabinovich–Fabrikant model, chaotic attractors, Lagrange formalism, bifurcation analysis, multistability.

Acknowledgements. Research was carried out under support of the Russian Science Foundation (project no. 21-12-00121), https://rscf.ru/project/21-12-00121/

For citation: Turukina LV. Dynamics of the Rabinovich–Fabrikant system and its generalized model in the case of negative values of parameters that have the meaning of dissipation coefficients. Izvestiya VUZ. Applied Nonlinear Dynamics. 2022;30(6):685–699. DOI: 10.18500/0869-6632-2022-30-6-685-699

Introduction

In 1979, M.,I.Rabinovich and A.,L.Fabrikant investigated the problem of modulation instability in the parametric interaction of modes in a nonequilibrium dissipative medium with cubic nonlinearity and with spectrally narrow amplification [1]. The authors have written down a nonlinear parabolic complex equation, which is a generalization of the well-known Landau model. Further, they assumed the discreteness of the spatial spectrum of solutions in the resonant case for periodic boundary conditions. In addition, only three modes fall into the spectral interval in which the medium is active - the main one and its two satellites, which arise as a result of modulation instability. As a result, they obtained the following system for the problem under consideration:

$$\begin{aligned} \dot{a}_0 &= 2\sigma a_1 a_2 a_0^* \exp(-i\Delta\omega t) + \gamma_0 a_0 + \sigma a_0(|a_0|^2 + 2|a_1|^2 + 2|a_2|^2), \\ \dot{a}_{1,2} &= 2\sigma a_{2,1} a_0^2 \exp(i\Delta\omega t) - \nu_{1,2} a_0 + \sigma a_{1,2}(2|a_0|^2 + |a_{1,2}|^2 + 2|a_{2,1}|^2). \end{aligned} \tag{1}$$

Here a_0 is the complex amplitude of the main mode, $a_{1,2}$ is the complex amplitudes of the satellites; γ_0 is the dissipation coefficient for the main mode, and $\nu_{1,2}$ is the dissipation coefficients for the satellites; $\Delta\omega$ — frequency disorder specified as $\Delta\omega = 2\omega_0 - \omega_1 - \omega_2$, where ω_0 is the frequency of the main mode, and $\omega_{1,2}$ is the frequency of satellites; σ is the parameter characterizing the nonlinearity in the system, t is the dimensionless time. The latter is provided that the main mode dominates the satellites ($a_0 \gg a_1 = a_2$, $\omega_1 = \omega_2 = \omega$ $\nu_1 = \nu_2 = \nu$), has been reduced to a valid three-dimensional model:

$$\begin{aligned} \dot{x} &= y(z - 1 + x^2) + \gamma x, \\ \dot{y} &= x(3z + 1 - x^2) + \gamma y, \\ \dot{z} &= -2z(\nu + xy), \end{aligned} \tag{2}$$

where x, y, z are dynamic variables (x and y are proportional to $\sqrt{a_0}$, and z is — a_1), and γ and ν — parameters that make sense of dissipation coefficients ($\gamma = \gamma_0$).

As studies of recent years have shown [2–11], the system (2) demonstrates rich dynamics: regular and chaotic attractors of different topologies, multistability, when attractors of different types coexist in phase space, etc., So, for example, in [2], the authors investigated a variety of system attractors (2): equilibrium positions, periodic cycles and chaotic attractors, as well as heteroclinic orbits connecting equilibrium positions with stable cycles and chaotic attractors. And in the works [3–5], chaotic attractors of the system (2) were studied for various values of the parameters γ and ν. The authors showed that in the Rabinovich–Fabrikant system (2) typologically different chaotic attractors are observed. At the same time, in the works of [3, 4], the main attention is paid to the so-called «hidden attractors» and «virtual» saddles. In [6], the authors proposed an analog circuit, which is described by the model (2), and conducted a detailed study of it: bifurcation diagrams, phase portraits, «first return» displays, etc., etc. In [7], the authors considered the mechanism of stabilization of a chaotic system in the vicinity of unstable equilibrium points on the example of several systems, including the Rabinovich–Fabrikant system (2).

In the work [8] numerically investigated in detail the Rabinovich–Fabrikant system (2): maps of dynamic modes, dependences of Lyapunov exponents, attractors and pools of their attraction are constructed, lines of the main bifurcations of fixed points and the limit cycle are found. Also in this paper, examples of periodic and chaotic attractors of different topologies are given and it is shown that the system (2) demonstrates multistability when attractors of different types or topologies coexist in the phase space. And finally, in [9], the authors considered and numerically investigated in detail the case of three-mode interaction in the presence of dissipation with cubic nonlinearity of a general form[1]. Using the Lagrange formalism, the authors of [9] wrote down ordinary differential equations of the second order for real variables. Assuming that the system is located in the vicinity of resonance, according to the methodology described in [1], a system was obtained that is a generalization of the Rabinovich model–Fabricant (2) for the case

[1]Recall that in [1] the authors considered a special case of cubic nonlinearity, which corresponded to a complex parabolic equation.

of cubic nonlinearity of the general form:

$$\begin{aligned}
\dot{x} &= [p(x^2 + z) + q(-y^2 + 3z) - 1]y + \gamma x, \\
\dot{y} &= [p(-x^2 + 3z) + q(y^2 + z) + 1]x + \gamma y, \\
\dot{z} &= -2z(\nu + (p + q)xy).
\end{aligned} \tag{3}$$

Here x, y, z — dynamic variables, γ and ν — parameters identical to the parameters Rabinovich–Fabrikant model, a p и q — parameters characterizing nonlinear interaction in the system. At the same time, as shown in [9], the system (3) completely coincides with the system (2), if $p = 1.0$ and $q = 0$.

Note that the models (2) and (3) are universal in a certain sense, since they are applicable to systems of different physical nature in which three-mode interaction takes place in the presence of cubic nonlinearity. For example, in such as the waves Tollmin −Schlichting in hydrodynamic currents [12], wind waves on water [13], waves in chemical media with diffusion [14], radio engineering systems that allow as analog modeling, so is the implementation in a radio engineering device [8], and etc.

In this paper, a detailed numerical study of the systems (2) and (3) is carried out for the case when the parameters γ and ν take negative values. This choice of parameters is due to the fact that in all the above studies, studies were conducted exclusively for positive values of the parameters γ and ν. And since in the equations (1) there are different signs before the terms specifying dissipation, this means that the main mode has negative dissipation, and the satellites have positive. In the case of negative values of the parameters γ and ν, the situation will change to the opposite: the main mode will have a positive dissipation, and the satellites — negative. Thus, the sign of the parameters γ and ν determines at which frequency the energy is pumped, and at which — its selection. At the same time, it was noted in [3,4,6] that although the negative values of the parameters γ and ν do not have physical meaning in relation to the problem formulated in [1], for them the Rabinovich–Fabrikant system (2) it will also demonstrate nontrivial dynamics and a chaotic attractor of a new type.The last one in the work [3] is named as «Gramophon-like chaotic attractor». On the other hand, the systems (2) and (3) can be considered as reference models describing complex dynamics and chaos. And, as a consequence, the parameters present in the equations can take any reasonable values. Thus, it can be expected that the results presented in this paper will complement the results of the above works, primarily [8,9], creating a complete picture of the dynamic behavior of systems (2) and (3).

1. Dynamics of the Rabinovich–Fabrikant system in the case of negative values of parameters γ and ν

First, let's consider the dynamics of the of the Rabinovich–Fabrikant system (2). Let the parameters γ and ν take negative values. Let's build a dynamic mode map for it and its enlarged fragments on the parameter plane (ν, γ) (Fig. 1). Such a map is constructed when scanning the parameter plane, when the type of the observed mode is numerically determined at each of its points, which is indicated by the corresponding color. On maps (see Fig. 1) the following modes are highlighted in color: dark blue corresponds to the equilibrium position, blue — to the limit cycle of the period 1, yellow — to the cycle of the period 2, red — to the cycle of the period 4, etc., etc., black corresponds to the chaotic mode, and the white color indicates the area of «escape trajectories to infinity». The specified cycle periods are determined in a standard way using the Poincare section.

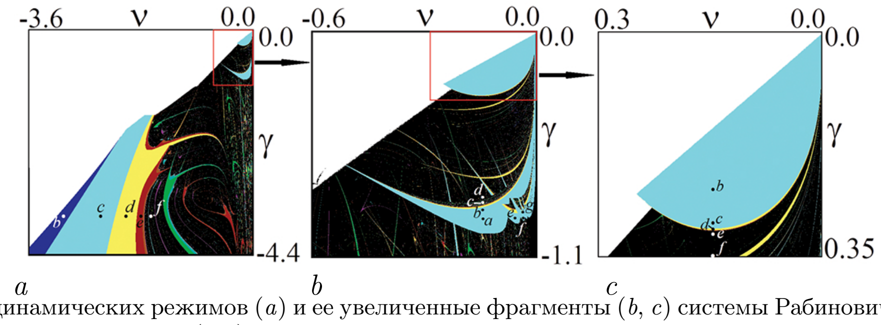

Fig. 1. Карта динамических режимов (a) и ее увеличенные фрагменты (b, c) системы Рабиновича–Фабриканта (2) на плоскости параметров (ν, γ). На фрагменте a буквами обозначены точки, в которых построены аттракторы, представленные на рис. 2. На фрагментах b, c — аттракторы, представленные на рис. 3, 5 (цвет онлайн)

Fig. 1. Chart of dynamical regimes (a) and its enlarged fragments (b, c) of the Rabinovich–Fabrikant model (2) at (ν, γ) parameter plane. On fragment a, the letters indicate the points where the attractors shown in Fig. 2 are plotted. On fragments b, c, the letters indicate the points where the attractors shown in Fig. 3 and Fig. 5 are plotted (color online)

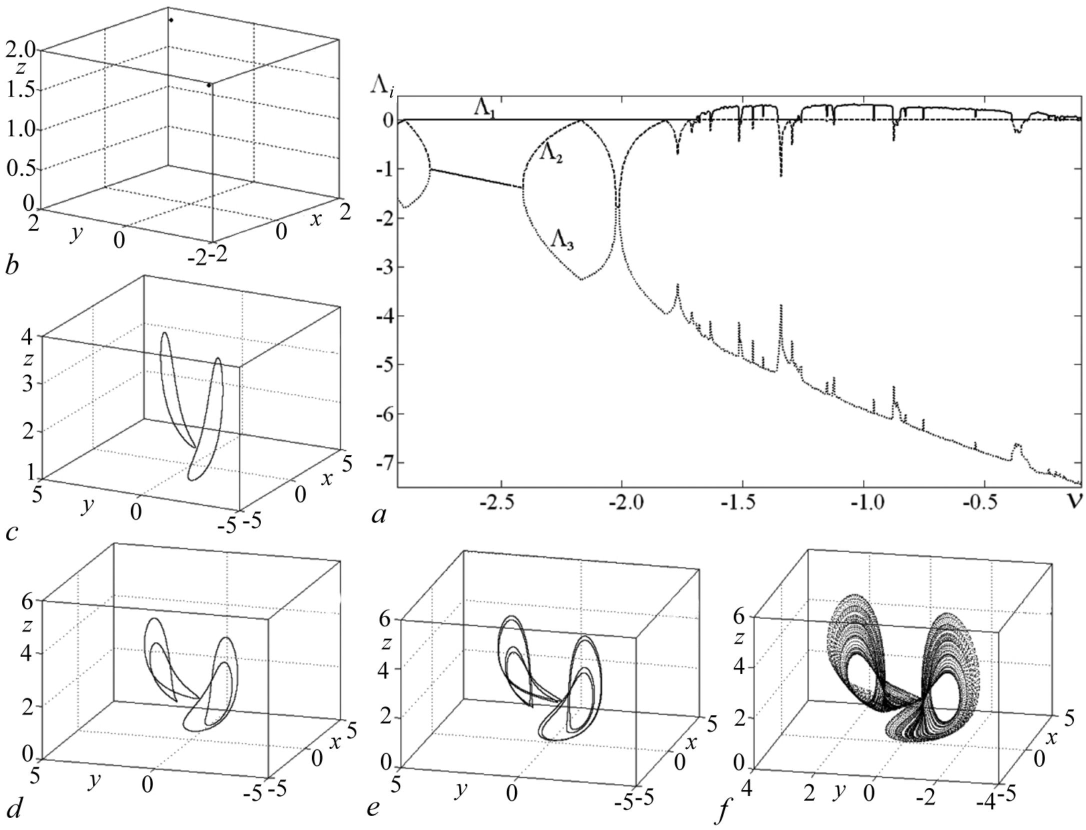

Fig. 2. a — Зависимость показателей Ляпунова системы Рабиновича–Фабриканта (2) от параметра ν для $\gamma = -3.8$. b–f — Аттракторы системы Рабиновича–Фабриканта (2), построенные для следующих значений параметров: $\gamma = -3.8$ и $\nu = -2.9$ (b), $\nu = -2.2$ (c), $\nu = -2.0$ (d), $\nu = -1.8$ (e) и $\nu = -1.5$ (f). Точки, в которых построены аттракторы, отмечены на рис. 1, a соответствующими буквами

Fig. 2. a — Graphs of Lyapunov exponents of the Rabinovich–Fabrikant system (2) on the parameter ν at $\gamma = -3.8$. b–f — Attractors of the Rabinovich–Fabrikant system (2) at $\gamma = -3.8$ and $\nu = -2.9$ (b), $\nu = -2.2$ (c), $\nu = -2.0$ (d), $\nu = -1.8$ (e) and $\nu = -1.5$ (f). In Fig. 1, a the points at which the attractors plotted are marked by the corresponding letters

On the map (Fig. 1, a), two areas can be distinguished that differ in their device. The first one is located at the bottom of the map (parameter $\gamma < -1.5$). Various areas of periodic regimes and chaos are observed here. For a better understanding of the dynamics observed in the specified area for the system Rabinovich–Fabrikant (2) dependences of Lyapunov exponents on the parameter ν and attractors at several points of the parameter space are constructed (рис. 2). It follows from the figure that if you fix the value of the parameter γ, for example $\gamma = -3.8$, and change the value of the parameter ν, moving along the parameter plane from left to right, then first, with large modulo negative values of the parameter ν, all three Lyapunov exponents are negative (Fig. 2, a), and in the phase space of the system under consideration, two symmetrically arranged and transitioning into each other with simultaneous replacement of $x \to -x$ and $y \to -y$ equilibrium positions are observed (Fig. 2, b). Note that the systems (2) and (3) have the property of symmetry with respect to the simultaneous replacement of variables

$$\begin{aligned} x &\to -x, \\ y &\to -y. \end{aligned} \tag{4}$$

This property leads to the fact that in the phase space of the systems under consideration, either a symmetrically located pair of attractors will be observed (when one attractor passes into another when replaced (4)), or one symmetric attractor (which, when replaced (4), passes into itself)[2]. When the parameter ν increases (the modulus of the parameter ν decreases), the senior Lyapunov exponent becomes zero, and the two remaining exponents are still negative. This corresponds to the fact that a symmetric pair of limit cycles of the period 1 is born in the system (Fig. 2, c). If the parameter ν is further increased, a sequence of period doubling bifurcations will be observed (Fig. 2, d, e), until, finally, the senior Lyapunov exponent becomes positive (see Fig. 2, a) and there will be no symmetric pair of chaotic attractors in the system (Fig. 2, f). The points at which the attractors are constructed are marked in Fig. 1, a with the corresponding letters.

Thus, for sufficiently large negative values of the parameter γ the dynamics of the of the Rabinovich–Fabrikant system qualitatively the same as was observed for her in the case when the parameters γ and ν were positive, and the parameter $\gamma > 0.8$ [8]. However, if in the case of positive values of the parameters γ and ν, the width of the regions of periodic and chaotic modes decreased with the growth of the parameter γ, now it increases (see Fig. 1, a).

The second area is located at small, modulo, values of the parameters γ and ν and is presented in more detail in enlarged fragments (Fig. 1, b, c). There are several regions of the limit cycle of the period 1, within which there is a transition to chaos through a sequence of period doubling bifurcations. All these regions, as will be shown below, differ in the topology of the period limit cycle observed inside them 1.

Let's first consider the area shown in Fig. 1, c. In Fig. 3 presents a graph of the dependence of the Lyapunov exponents of the system (2) on the parameter γ and its attractors constructed at some points, which are marked in Fig. 1, c with the corresponding letters. Initially, inside this area, the system (2) demonstrates a symmetric limit cycle of the period 1, which turns into itself when replacing $x \to -x$ or $y \to -y$ (Fig. 3, b). Note that earlier this type of symmetry was not observed either in the Rabinovich–Fabricant system or in its generalized model. When the parameter γ decreases (the modulus of the parameter γ increases), this cycle undergoes a symmetry loss bifurcation and becomes unstable, and instead a symmetric pair of limit cycles of the period 1 is born (Fig. 3, c). These changes are especially noticeable on the projections of these cycles on the plane (x, y), (x, z) and (y, z), which are shown in Fig. 4, a, b. For the convenience of perception, the cycles passing into each other, in Fig. 3, 4 and similar drawings are depicted in red and blue colors. In the future, based on each of the cycles of the period 1,

[2]In the future, when describing the attractors observed in the systems under consideration, we will use the expressions «symmetric pair...» or «symmetrical...» implying the symmetry given by the formula (4).

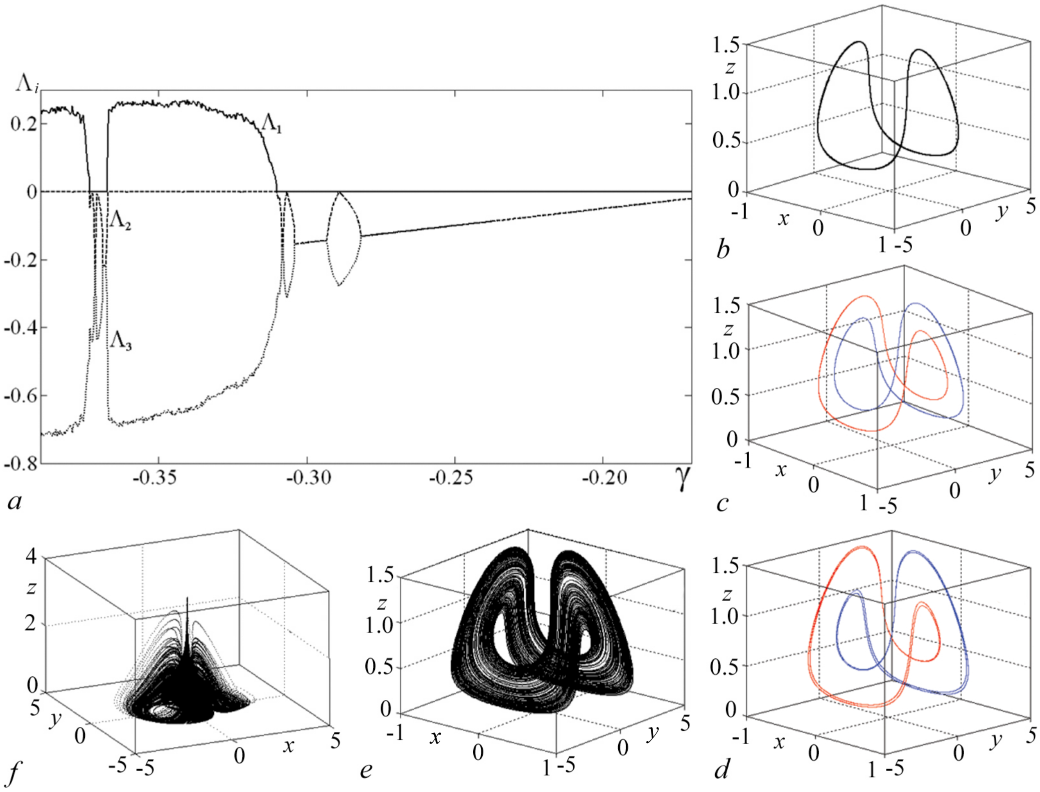

Fig. 3. *a* — Зависимость показателей Ляпунова системы Рабиновича–Фабриканта (2) от параметра γ для ν = −0.15.
b–f — Аттракторы системы Рабиновича–Фабриканта (2), построенные для следующих значений параметров: ν = −0.15 и γ = −0.24 (*b*), γ = −0.29 (*c*), γ = −0.307 (*d*), γ = −0.313 (*e*) и γ = −0.32 (*f*). Точки, в которых построены аттракторы, отмечены на рис. 1, *c* соответствующими буквами (цвет онлайн)

Fig. 3. *a* — Graphs of Lyapunov exponents of the Rabinovich–Fabrikant system (2) on the parameter γ at ν = −0.15.
b–f — Attractors of the Rabinovich–Fabrikant system (2) at ν = −0.15 and γ = −0.24 (*b*), γ = −0.29 (*c*), γ = −0.307 (*d*), γ = −0.313 (*e*) and γ = −0.32 (*f*). In Fig. 1, *c* the points at which the attractors plotted are marked by the corresponding letters (color online)

shown in Fig. 3, *c*, a cascade of period doubling bifurcations will be observed, the corresponding cycles of the period 2 are shown in Fig. 3, *d*, until a chaotic attractor appears in the system (Fig. 3, *e*, *f*). At the same time, at first it will have the same topology as the limit cycle of the period 1 (compare Fig. 3, *c* and 3, *e*, as well as fig. 4, *b* and 4, *c*), and then as a result of the crisis, the topology of the chaotic attractor will change (see Fig. 3, *f*).Note that a chaotic attractor of this type, as shown in Fig. 3, *f*, previously in the Rabinovich–Fabricant system or its generalized model was not observed. In the work [3], the attractors of this configuration were called «Gramophon-like chaotic attractor». The choice of the name is obviously due to the type of projection of the attractor on the plane (x, y) (Fig. 4, *d*).

Now consider the lower part of the figure. 1, *b* of the dynamic mode map fragment of the Rabinovich–Fabrikant system (2). There are several period cycle structures observed here 1. On fig. 5 periodic and chaotic attractors constructed for some of these structures are presented. The points at which they are built are marked in Fig. 1, *b* with the corresponding letters. On fig. 4, *e–j* the projections of some of these attractors on the plane are presented (x, y), (x, z) и (y, z). In the leftmost structure, the system (2) demonstrates a symmetric limit cycle of period 1 (Fig. 5, *a*, fig. 4, *e*). As in the case described above, it first loses symmetry, generating a symmetric pair of limit cycles of period 1 (fig. 5, *b*, fig. 4, *f*). These cycles subsequently undergo a cascade of period doubling bifurcations, as a result of which a chaotic attractor

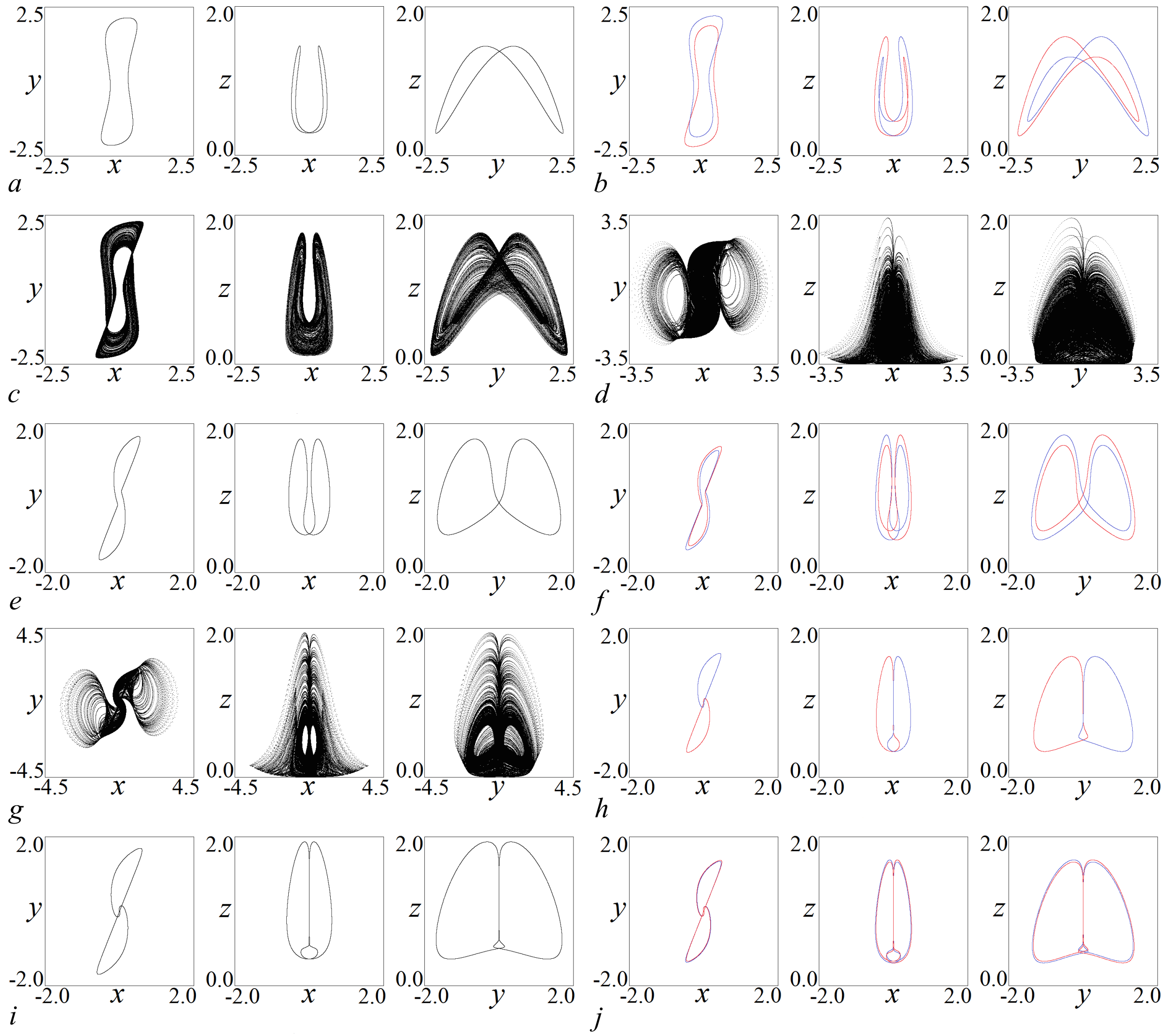

Fig. 4. Проекции аттракторов системы Рабиновича–Фабриканта (2) на плоскости (x, y), (x, z), (y, z), построенные для следующих значений параметров: $\nu = -0.15$, $\gamma = -0.24$ (*a*); $\nu = -0.15$, $\gamma = -0.29$ (*b*); $\nu = -0.15$, $\gamma = -0.313$ (*c*); $\nu = -0.15$, $\gamma = -0.32$ (*d*); $\gamma = -0.9$, $\nu = -0.15$ (*e*); $\gamma = -0.87$, $\nu = -0.15$ (*f*); $\gamma = -0.8$, $\nu = -0.15$ (*g*); $\gamma = -0.9$, $\nu = -0.06$ (*h*); $\gamma = -0.9$, $\nu = -0.04$ (*i*); $\gamma = -0.87$, $\nu = -0.04$ (*j*). Соответствующие трехмерные аттракторы представлены на рис. 3 для фрагментов *a*–*d* и на рис. 5 для фрагментов *e*–*j* (цвет онлайн)

Fig. 4. Attractors projections of the Rabinovich–Fabrikant system (2) at $\nu = -0.15$ and $\gamma = -0.24$ (*a*), $\nu = -0.15$ and $\gamma = -0.29$ (*b*), $\nu = -0.15$ and $\gamma = -0.313$ (*c*), $\nu = -0.15$ and $\gamma = -0.32$ (*d*), $\gamma = -0.9$ and $\nu = -0.15$ (*e*), $\gamma = -0.87$ and $\nu = -0.15$ (*f*), $\gamma = -0.8$ and $\nu = -0.15$ (*g*), $\gamma = -0.9$ and $\nu = -0.06$ (*h*), $\gamma = -0.9$ and $\nu = -0.04$ (*i*), $\gamma = -0.87$ and $\nu = -0.04$ (*j*). Corresponding three-dimensional attractors are shown in fig. 3 for case *a*–*d* and in fig. 5 for case *e*–*j* (color online)

is born, which first has a limit cycle topology (fig. 5, *c*), and then a topology of the type «Gramophone» (fig. 5, *d*, fig. 4, *g*). The other structures will have similar dynamics. The differences will be only in the topology of the limit cycle of the period 1. So, in the second structure from the left for the system (2), a symmetric pair of limit cycles of the period 1 will be immediately observed (Fig. 5, *e* and Fig. 4, *h*). In the third structure from the left, there will again be one symmetric limit cycle of period 1 (Fig. 5, *f* and Fig. 4, *i*), which will lose symmetry before the period doubling bifurcation (Fig. 5, *g* and fig. 4, *j*). Thus, in the area under consideration, there will be an alternation of structures, one of which corresponds to a symmetric limit cycle of period 1, and the second to a symmetric pair of limit cycles of period 1.

In Fig. 6 sections of typical attraction basins of the attractors of the system (2) are represented by the plane z_0=Const. In Fig. 6, *a* pools are shown for the case when the system (2) implements a symmetric pair of limit cycles of period 1, shown in Fig. 2, *b*. It can be seen from the figure that the pools are large, although with an increase in the value of the dynamic variable z_0, their size decreases slightly; there are no areas inside the basins «trajectory escaping to infinity», which was typical in the case of positive values of the parameters γ and ν [8], and their boundary is arranged in a rather complicated way. Note that for all other attractors from this region (the lower part of the dynamic mode map), both periodic and chaotic, the pools of attraction will be the same as in the considered case.

In Fig. 6, *b* the basins for the symmetric limit cycle of period 1 shown in Fig. 3, *b*, and in Fig. 6, *c* — for a symmetric pair of limit cycles of period 1 shown in Fig. 3, *c*. In both cases, the pools have the same size, which does not change with the growth of the value of the dynamic variable z_0, and the shape is in the form of a circle with a complicated border and two vertical stripes. At the same time, in the second case (see Fig.6, *c*) the points starting from which the trajectory comes to one or another cycle fill the pool in a chaotic way. This means that even a small change in the initial values of dynamic variables will lead to the fact that the system (2) will «jump» from one attractor to another. For all other attractors, regardless of their period, from the same region (the region of small values of the parameters γ and ν) the pools of attraction will be the same as in Fig. 6, *b* and fig. 6, *c*.

2. Dynamics of the general model Rabinovich–Fabrikant in the case of negative values of parameters γ and ν

Now let's consider the dynamics of the general model Rabinovich–Fabrikant (3) in the case when the parameters γ and ν take negative values. To begin with, we will build for it maps of dynamic modes on the plane (p, q) for several values of the parameters γ and ν. The corresponding maps are shown in Fig. 7. These and all subsequent maps use the same color palette as in Fig. 1.

On the maps presented above, as well as in the case of positive values of the parameters γ and ν [9], regions of periodic and chaotic modes are observed, which are expanding rays coming out of the origin. The cards themselves have symmetry with respect to the line $p = -q$. However, if, with positive values of the parameters γ and ν, a universal structure of regions of periodic and chaotic modes was observed on the plane (p, q), independent of the parameters γ and ν, then now this is not the case. When the parameters γ and ν vary, the map of the dynamic modes of the generalized model (3) changes, at times quite a lot. Some areas become narrower, others — wider, and some disappear altogether. Another difference is that if, in the case of positive values of the parameters γ and ν, a mode was observed in the generalized model, which is an invariant (stable) set in the form of a circle lying entirely in the plane $z = 0$ (the area of this mode was located on the map of dynamic modes along the line $p = -q$, inside the area of «escape trajectory to infinity» and colored gray), then in the case of negative values of the parameters γ and ν, the specified mode is not observed.

Now let's consider the device of the dynamic mode map of the general model Rabinovich–Fabrikant (3) on the plane (ν, γ). Since in the case under consideration, no universal structure is observed on the parameter plane (p, q), we will build maps of the dynamic modes of the generalized model (3), choosing the parameters p and q so as to bypass the plane (p, q) the origin of the coordinates counterclockwise starting from the point 1, as shown in Fig. 8, *a*. The corresponding maps of dynamic modes are shown in Fig. 8, *b*–*h*.

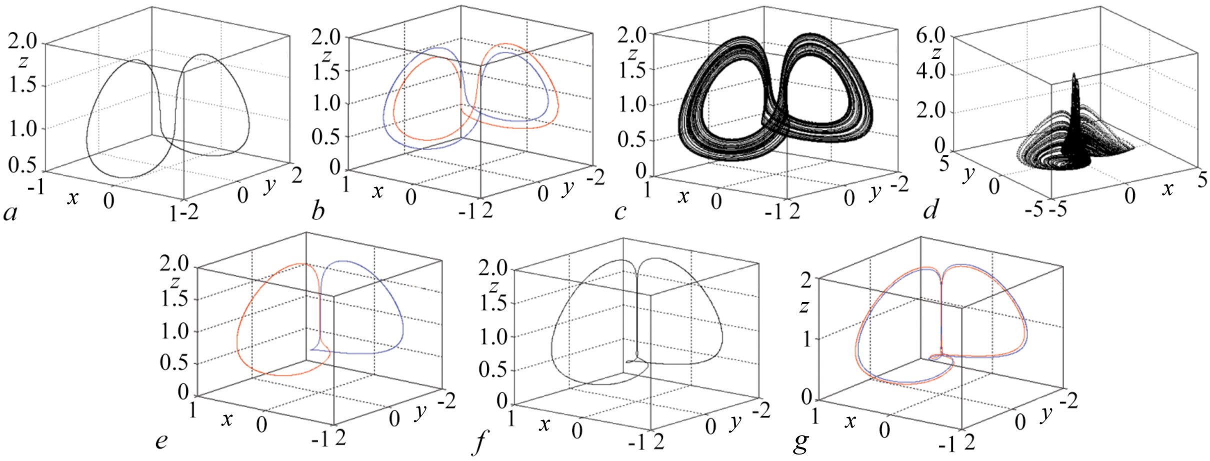

Fig. 5. Аттракторы системы Рабиновича–Фабриканта (2), построенные для следующих значений параметров: γ = −0.9, ν = −0.15 (*a*); γ = −0.87, ν = −0.15 (*b*); γ = −0.83, ν = −0.15 (*c*); γ = −0.8, ν = −0.15 (*d*); γ = −0.9, ν = −0.06 (*e*); γ = −0.88, ν = −0.06 (*f*); γ = −0.86, ν = −0.06 (*g*). Точки, в которых построены аттракторы, отмечены на рис. 1, *b* соответствующими буквами (цвет онлайн)

Fig. 5. Attractors of the Rabinovich–Fabrikant system (2) at γ = −0.9 and ν = −0.15 (*a*), γ = −0.87 and ν = −0.15 (*b*), γ = −0.83 and ν = −0.15 (*c*), γ = −0.8 and ν = −0.15 (*d*), γ = −0.9 and ν = −0.06 (*e*), γ = −0.88 and ν = −0.06 (*f*), γ = −0.86 and ν = −0.06 (*g*). In Fig. 1, *b* the points at which the attractors plotted are marked by the corresponding letters (color online)

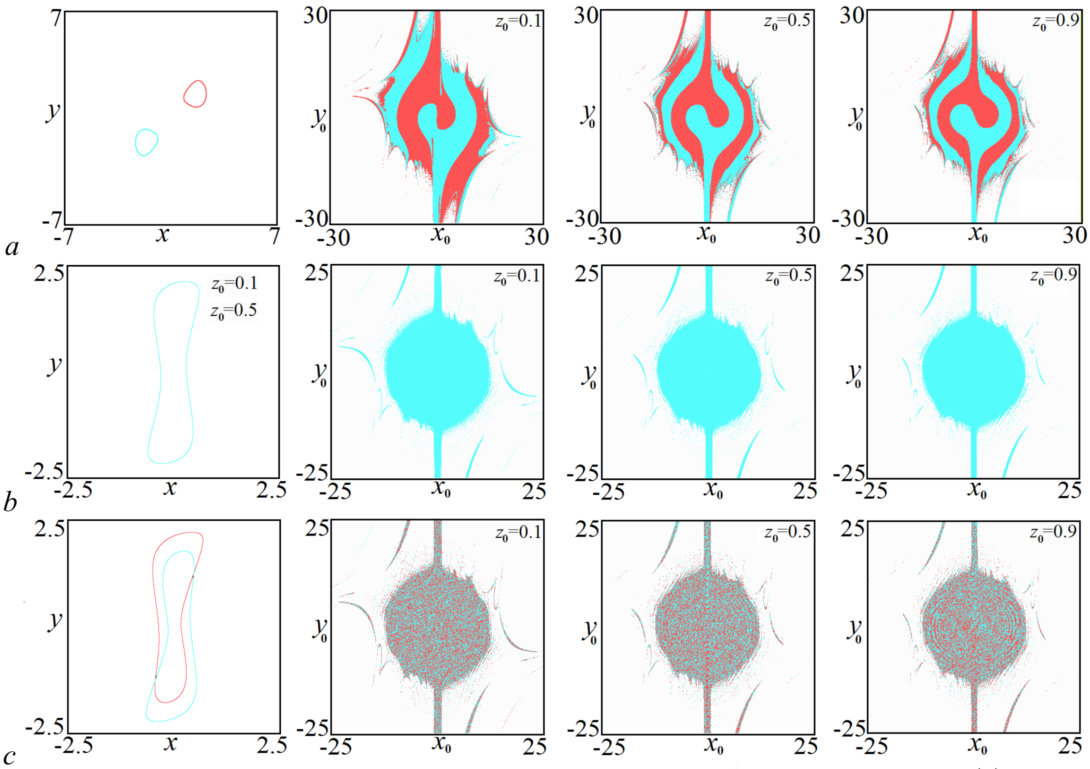

Fig. 6. Сечения бассейнов притяжения аттракторов системы Рабиновича–Фабриканта (2) плоскостью z_0 = Const, построенные для следующих значений параметров γ = −3.8, ν = −2.2 (*a*); γ = −0.24, ν = −0.15 (*b*); γ = −0.29, ν = −0.15 (*c*). Бассейны окрашены в тот же цвет, что и соответствующие аттракторы (цвет онлайн)

Fig. 6. Sections of the basins of the attractors by the z_0 = Const plane plotted for the Rabinovich–Fabrikant system (2), γ = −3.8 and ν = −2.2 (*a*), γ = −0.24 and ν = −0.15 (*b*), γ = −0.29 and ν = −0.15 (*c*). The basin is colored to the same color as the attractors (color online)

At the point 1, the values of the parameters $p = 0.9$ and $q = -0.1$. The map of the dynamic modes of the system (3) for this case is shown in Fig. 8, *b* and represents a strongly distorted picture characteristic of the Rabinovich–Fabrikant (2) (see Fig. 1, *a*). On it, as before, regions of various periodic and chaotic regimes are observed, but the regions themselves have greatly changed their configuration and become smaller, and some areas of periodic regimes have disappeared altogether. First of all, we are talking about the regions observed with small values of the parameters γ and ν. At the point 2 (Fig. 8, *a*), the values of the parameters $p = 1.0$ and $q = 0$. The corresponding map of the dynamic modes of the system (3) is shown in Fig. 8, *c* and will be completely identical to the one observed for the system (2) (compare Fig. 8, *c* and fig. 1, *a*). Which is not surprising, since, as noted earlier (see Introduction), it is at these parameter values that the equations of the system (3) turn into the equations of the system (2).

Now let's go to the point 3. For it, the parameters p and q are equal, respectively, $p = 0.9$ and $q = 0.1$. The map of dynamic modes for this case is shown in Fig. 8, *d*. Let's consider this case in more detail. In Fig. 9 an enlarged fragment of the above-mentioned dynamic mode map is presented. From the comparison of Fig. 1, *a* with fig. 8, *d*, and fig. 1, *b* with fig. 9 it can be seen that if the area of large values of the parameters γ and ν (the lower part of the map) has practically not changed, the only quantitative difference is that the areas of the limit cycles of the period 1 and 2 have become wider, then the area of small values of the

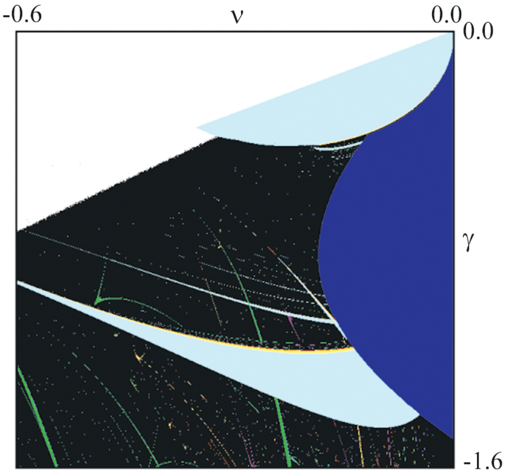

Fig. 9. Enlarged fragment of the chart of dynamical regimes of the general model (3) at (γ, ν) parameter plane. $p = 0.9$

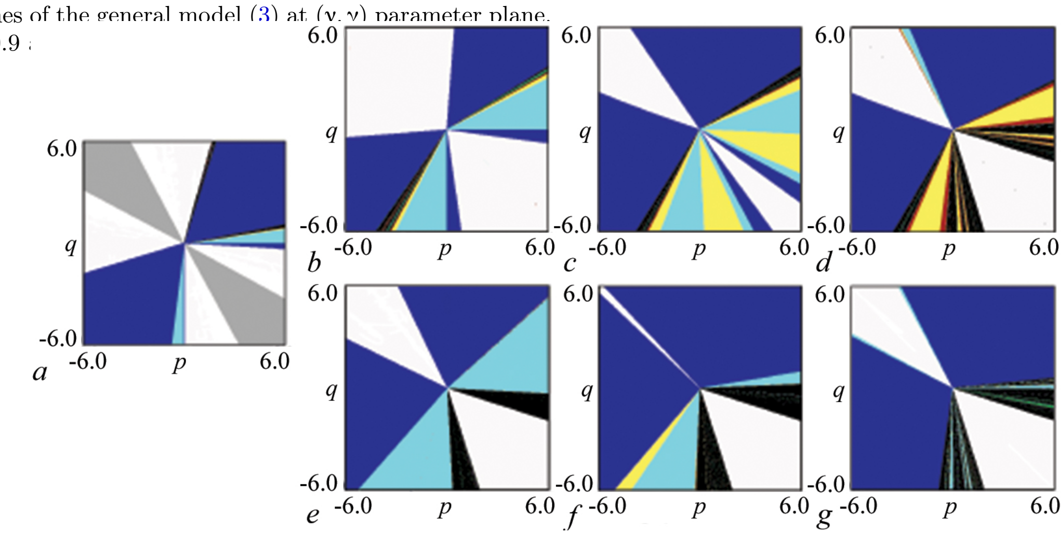

Fig. 7. *a* — Карта динамических режимов системы Рабиновича–Фабриканта (2) на плоскости (p, q). $\gamma = 0.96$ и $\nu = 1.5$. Рисунок взят из работы [9]. *b*–*g* — Карты динамических режимов обобщенной модели Рабиновича–Фабриканта (3) на плоскости параметров (p, q). Параметры γ и ν принимают следующие значения: $\gamma = -3.8$, $\nu = -2.9$ (*b*); $\gamma = -3.8$, $\nu = -2.2$ (*c*); $\gamma = -3.8$, $\nu = -1.5$ (*d*); $\gamma = -0.24$, $\nu = -0.15$ (*e*); $\gamma = -0.35$, $\nu = -0.15$ (*f*); $\gamma = -0.9$, $\nu = -0.15$ (*g*) (цвет онлайн)

Fig. 7. *a* — Chart of dynamical regimes of the Rabinovich–Fabrikant system (2) at (p, q) parameter plane. A figure is taken from the work [9]. *b*–*g* — Charts of dynamical regimes of the general model (3) at (p, q) parameter plane. $\gamma = -3.8$ and $\nu = -2.9$ (*b*), $\gamma = -3.8$ and $\nu = -2.2$ (*c*), $\gamma = -3.8$ and $\nu = -1.5$ (*d*), $\gamma = -0.24$ and $\nu = -0.15$ (*e*), $\gamma = -0.35$ and $\nu = -0.15$ (*f*), $\gamma = -0.9$ and $\nu = -0.15$ (*g*) (color online)

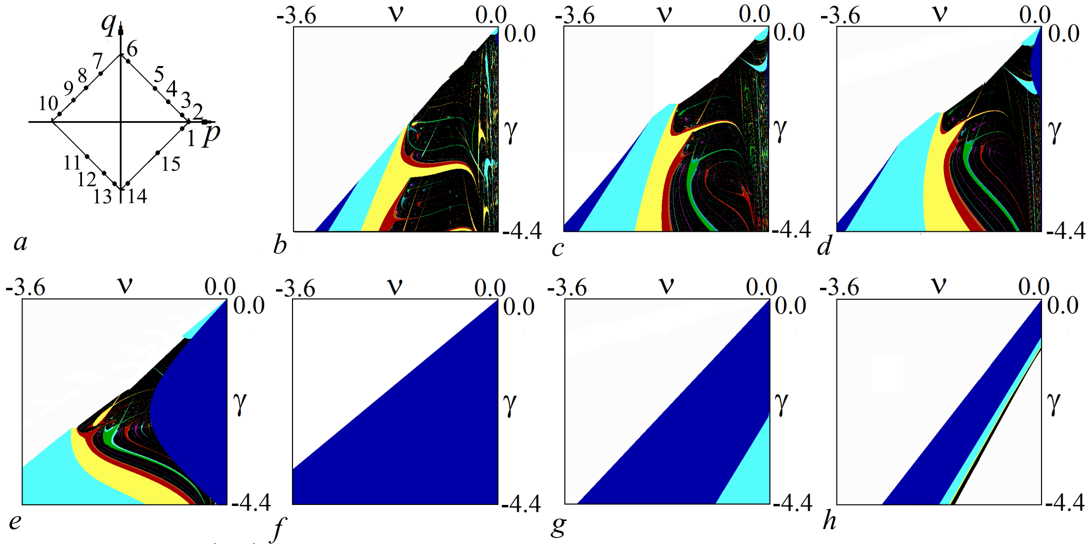

Fig. 8. a — Плоскость (p, q) с указанием точек, в которых строятся карты динамических режимов (b–h) обобщенной модели Рабиновича–Фабриканта (3) на плоскости параметров (ν, γ). Параметры p и q принимают следующие значения: $p = 0.9$, $q = -0.1$ (b); $p = 1$, $q = 0$ (c); $p = 0.9$, $q = 0.1$ (d); $p = 0.7$, $q = 0.3$ (e); $p = 0.5$, $q = 0.5$ (f); $p = 0.1$, $q = 0.9$ (g); $p = -0.3$, $q = 0.7$ (h) (цвет онлайн)

Fig. 8. a — (p, q) parameter plane. The points at which charts of dynamical regimes plotted are shown in this plane. b–h — Charts of dynamical regimes of the general model (3) at (ν, γ) parameter plane. Parameters p, q values: $p = 0.9$ and $q = -0.1$ (b), $p = 1$ and $q = 0$ (c), $p = 0.9$ and $q = 0.1$ (d), $p = 0.7$ and $q = 0.3$ (e), $p = 0.5$ and $q = 0.5$ (f), $p = 0.1$ and $q = 0.9$ (g), $p = -0.3$ and $q = 0.7$ (h) (color online)

parameters γ and ν show noticeable changes. Firstly, in the area of small values of the parameters γ and ν, near the right border, another area corresponding to the new equilibrium position has appeared. As will be shown later, this area does not displace the modes observed with these parameter values, but is superimposed on them. As a consequence, multistability will take place in the considered domain in the generalized model (3): a symmetric pair of new equilibrium positions and either limit cycles or a chaotic attractor will coexist in the phase space. The latter depends on the specific values of the parameters γ and ν. In this case, the topology of the limit cycles and the chaotic attractor will be the same as in the case of the system (2). But the arrangement of attraction pools will change and will not depend on the period of coexisting attractors. For example, in Fig. 10, a a pool is presented for the case when a symmetric pair of equilibrium positions and a chaotic attractor coexist in the phase space. It can be seen from the figure that most of the basin is chaotically filled with points, starting from which the trajectory will come either to one of the equilibrium positions (the corresponding points are colored blue and green), or to a chaotic attractor (the corresponding points are colored blue). In the vicinity of the origin, small «islands» basins of attraction of equilibrium positions will be observed, which is clearly visible on the enlarged fragment of the section of the basin constructed for $z_0 = 0.5$, (see Fig. 10, a). Moreover, with the growth of z_0, these «islands» will decrease until they disappear completely. The second difference between the dynamic mode map constructed for the generalized model (3) and the map constructed for the system (2) is that the regions of all periodic modes are shifted down by the parameter γ (compare Fig. 1, b with fig. 9).

Now let's return to the description of how the maps of the dynamic modes of the generalized model (3) change on the plane (ν, γ) when moving on the plane (p, q) along the route shown in Fig. 8, a. When moving from point 3 to point 4 ($p = 0.7$ and $q = 0.3$), the area of the new equilibrium position will increase in size (see Fig. 8, e), until finally at point 5 ($p = 0.5$ and $q = 0.5$)

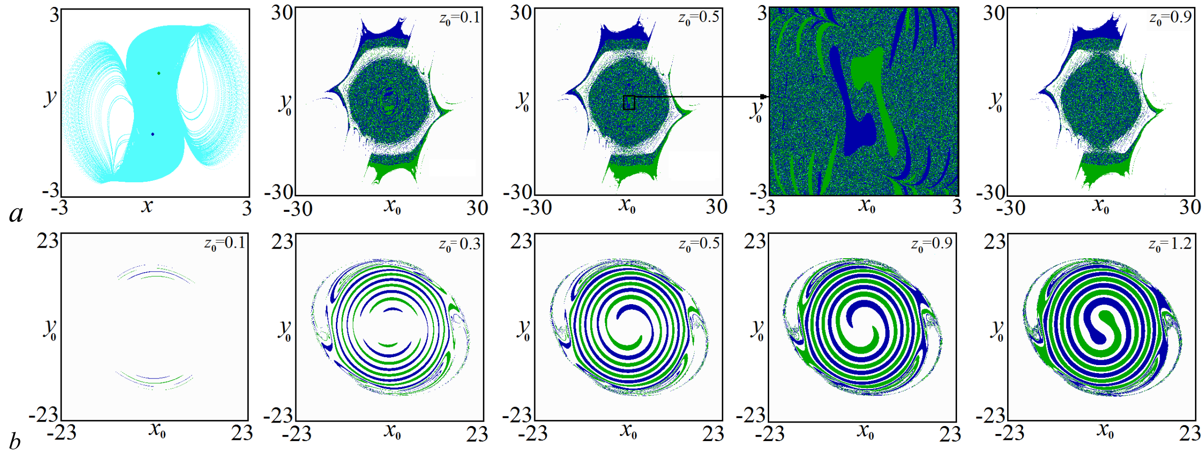

Fig. 10. Сечения бассейнов притяжения аттракторов обобщенной модели (3) плоскостью $z_0 = \text{Const}$, построенные для следующих значений параметров $p = 0.9$, $q = 0.1$, $\gamma = -0.4$, $\nu = -0.1$ (*a*) и $p = -0.3$, $q = 0.7$, $\gamma = -2.0$, $\nu = -0.9$ (*b*). Бассейны окрашены в тот же цвет, что и соответствующие аттракторы (цвет онлайн)

Fig. 10. Sections of the basins of the attractors by the $z_0 = \text{Const}$ plane plotted for the general model (3), $p = 0.9$, $q = 0.1$, $\gamma = -0.4$, $\nu = -0.1$ (*a*) and $p = -0.3$, $q = 0.7$, $\gamma = -2.0$, $\nu = -0.9$ (*b*).The basin is colored to the same color as the attractors (color online)

only the area corresponding to the new equilibrium position will not remain (see Fig. 8, *f*)[3]. Next, at the point 6 ($p = 0.1$ and $q = 0.9$), in addition to the area of the new equilibrium position, an area corresponding to the limit cycle of period 1 will be observed (see Fig. 8, *g*). After that, when moving from point 6 to point 7 ($p = -0.3$ and $q = 0.7$), the regions corresponding to the limit cycles of an increasing and larger period will appear sequentially. Until, finally, at the point 7, an area corresponding to the chaotic mode appears (see Fig. 8, *h*). At the same time, all areas observed on this map have the appearance of narrow bands, especially in the case of limit cycles and chaos. Note that for all observed modes, the pools of attraction will be the same and will have the form shown in Fig. 10, *b*. This figure shows the basins for the case when a symmetric pair of equilibrium positions is observed in the phase space. It can be seen that for small z_0 pools are narrow arcs that connect with the growth of z_0, forming a spiral. With the growth of z_0, the number of turns of the spiral will decrease, and their thickness will increase (see Fig. 10, *b*). As it turned out, with further movement along the selected route shown in Fig. 8, *h* the stripes will get narrower and narrower until they completely disappear, and at the point 8 ($p = -0.5$ and $q = 0.5$) in the general system Rabinovich–Fabrikant (3), only «escape trajectory to infinity» will be observed.

If we continue moving along the plane of parameters (p, q) according to the above route, then the transformation of dynamic mode maps will go in reverse order. Namely, at the point 9, the picture shown in Fig. 8, *h*, at point 10 — in Fig. 8, *g*, at point 11 — in Fig. 8, *f*, etc., etc. At point 15, as well as at point 8, only «escape of the trajectory to infinity» will be observed.

3. Bifurcation analysis of the system Rabinovich–Fabricant and its general model

In conclusion, a bifurcation analysis was performed for the Rabinovich–Fabrikant system (2) and its generalized model (3) using the program *MatCont*. The lines of the main bifurcations for the equilibrium positions and limit cycles of the period were numerically found and constructed 1.

[3]Note that the area of «escape trajectory to infinity» will be present on all maps of dynamic modes shown in Fig. 8. Therefore, when describing the changes observed on the maps, it will not be mentioned.

The corresponding bifurcation diagrams are shown in Fig. 11. They are in good agreement with the dynamic mode maps constructed for the system (2) and the system (3) for the same parameter values and presented in Fig. 1, fig. 8, *d* and fig. 9.

First, consider the bifurcation diagram for the Rabinovich–Fabrikant system (2). It is shown in Fig. 11, *a*. It follows from the diagram that in the region of large values of the parameter γ ($\gamma < -2$), when moving along the plane from left to right, first on the line SN, stable and unstable equilibrium positions are born as a result of saddle-node bifurcation. Further, the stable equilibrium position becomes unstable on the H line, as a result of the Andronov–Hopf bifurcation. At the same time, a stable limit cycle is born in the system, the period of which doubles on the bifurcation line of the doubling of the PD period (Fig. 11, *a*, left fragment). As follows from the view of the dynamic mode map (Fig. 1, *a*), then there will be a cascade of period doubling bifurcations, as a result of which a chaotic attractor appears in the system (2).

In the region of small values of the parameters γ and ν, several regions of the existence

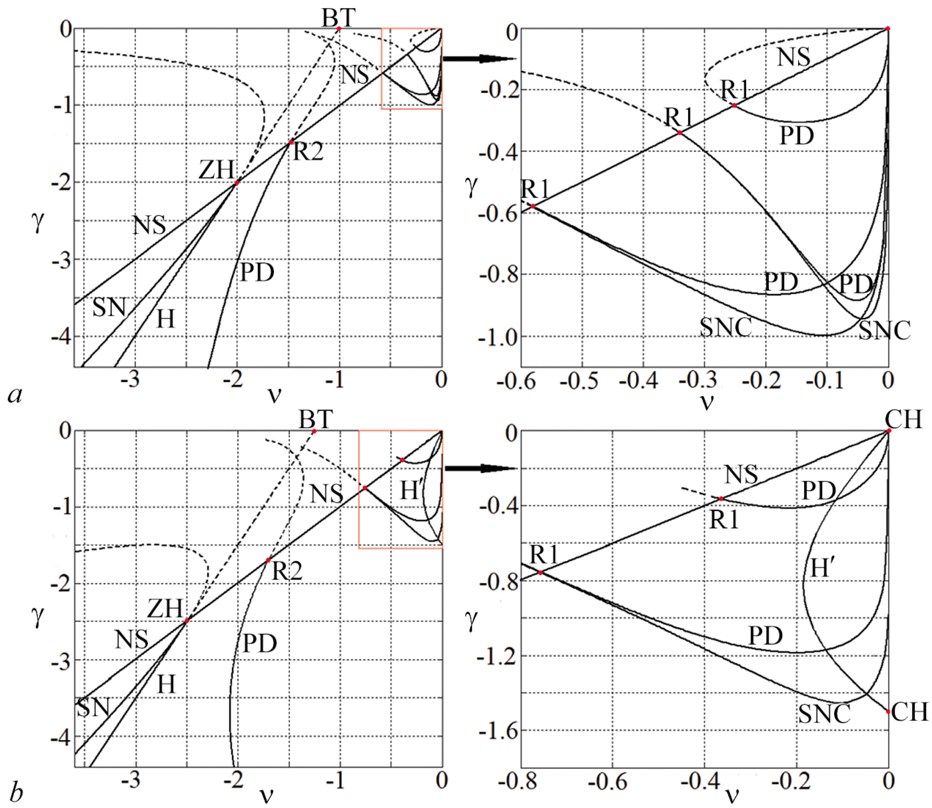

Fig. 11. Бифуркационные линии и точки системы Рабиновича–Фабриканта (2) (*a*) и ее обобщенной модели (3) для случая $p = 0.9$ и $q = 0.1$ (*b*) на плоскости (ν, γ). SN — седло-узловая бифуркация неподвижной точки, SNC — седло-узловая бифуркация предельных циклов, H — прямая бифуркация Андронова–Хопфа, H′ — обратная бифуркация Андронова–Хопфа, PD — бифуркация удвоения периода предельных циклов, NS — обратная бифуркация Неймарка–Сакера, ZH — точка пересечения линии касательной бифуркации неподвижной точки и линии бифуркации Андронова–Хопфа (точка бифуркации Zero–Hopf), $R1$ — резонанс 1 : 1, $R2$ — резонанс 1 : 2, BT — точка бифуркации Богданова–Такенса. Сплошными линиями показаны бифуркации устойчивых режимов, а пунктирными — неустойчивых

Fig. 11. Bifurcations lines and points of the Rabinovich–Fabrikant system (2) and its general model (3) at $p = 0.9$ and $q = 0.1$ on the (ν, γ) parameter plane. SN is saddle-node bifurcation of the stable point, SNC is saddle-node bifurcation of the limit cycle, H is direct Hopf bifurcation, H′ is inverse Hopf bifurcation, PD is period doubling bifurcation of the limit cycle, NS is inverse Neimark–Saker bifurcation, ZH is Zero–Hopf bifurcation point, $R1$ is resonance 1:1, $R2$ is resonance 1:2, BT is Bogdanov–Takens bifurcation point. The bifurcation lines corresponded unstable regimes are indicated by dotted lines

of the limit cycle of period 1 are observed (see Fig. 11, *a*, right fragment). The upper region is bounded by the lines of the doubling bifurcation of the PD period, which is a closed curve, and the inverse Neumark bifurcation−Sakera NS. And the lower regions (in the right fragment of Fig. 11, *a* two such regions are shown) are bounded by the lines of the saddle-node bifurcation of the limit cycles SNC and the doubling bifurcation of the PD period. Note that the regions of periodic and chaotic behavior are bounded by the Neumark inverse bifurcation line−A Saker NS on which there are bifurcation points of codimension two, such as ZH — bifurcation point Zero−Hopf, $R1$ — resonance $1:1$, $R2$— resonance $1:2$.

Now consider the bifurcation diagram for the general model of the Rabinovich−Fabrikant system (3). It is shown in Fig. 11, *b*. From the comparison of Fig. 11, *a* and fig. 11, *b* it follows that, in general, the bifurcation pattern of the generalized model (3) is identical to that observed in the Rabinovich−Fabrikant system (2) except for two differences, taking place with small values of the parameters γ and ν. The first difference is that the bifurcation line of the doubling of the PD period, bounding the upper region of the limit cycle of the period 1, is not a closed curve in the generalized model (see Fig. 11, *b*, right fragment). The second is that in the generalized model (3), in the region of small values of the parameters γ and ν, there is a bifurcation line, which was not in the Rabinovich−Fabrikant system. This is the H′ line, which corresponds to the inverse Andronov−Hopf bifurcation and on which a stable equilibrium position and an unstable limit cycle are born (see Fig. 11, *b*, right fragment).

Conclusion

In this paper, the Rabinovich−Fabricant system and its generalized model were numerically investigated using methods of dynamic chaos theory in the case when the parameters γ and ν, which have the meaning of dissipation coefficients, take negative values. For both systems, maps of dynamic modes, attractors and their projections were constructed, typical for these systems attraction pools of attractors. The main bifurcations of equilibrium positions and limit cycles of period one are found. The comparison of the Rabinovich−Fabricant system with its generalized model, as well as with the results obtained earlier for the case of positive values of the parameters γ and ν.

The study showed that in both models under study, regions corresponding to stable equilibrium positions, limit cycles of various periods and chaotic attractors are observed in the parameter space. In this case, just as in the case of positive values of the parameters γ and ν, all attractors, regardless of type and period, arise either in pairs, passing into each other with simultaneous replacement of $x \to -x$ and $y \to -y$, or if the attractor is one, then it has symmetry with respect to the same substitution of variables. That is, there is a multistability in the system associated with the internal symmetry of the systems. But multistability, when attractors of different periods and topologies coexist in the phase space, characteristic of the case when the parameters γ and ν take positive values, in the case of negative values of these parameters in the Rabinovich−Fabricant system is completely absent. And in its generalized model, it is observed only in the region of small values of parameters that make sense of dissipation coefficients.

The study also showed that in the case of negative values of the parameters γ and ν in the Rabinovich−Fabricant system and in its generalized model, a chaotic attractor of the type «Gramophone» is observed, which for positive values of the specified no parameters are observed. In addition, in the case of negative values of the parameters γ and ν, the pools of attraction of all attractors have almost the same device: they are significantly larger than in the case of positive values of these parameters, and do not contain areas corresponding to «escape trajectory to infinity».

Another difference. If in the case of positive values of the parameters γ and ν in the generalized model, the plane of parameters characterizing the nonlinearity of the system had a universal device, then in the case of negative values of these parameters, this is not the case. Although the qualitative structure of the plane of parameters characterizing the nonlinearity of the system, for negative values of the parameters γ and ν is the same as in the case of positive values, but it changes quite a lot when the parameters vary.

Thus, it can be noted that this work complements the work of [8, 9] forming a complete picture of the dynamics of the Rabinovich−Fabricant system and its generalized model for any values of the parameters included in them.

References

1. Rabinovich MI, Fabrikant AL. Stochastic self-modulation of waves in nonequilibrium media. Journal of Experimental and Theoretical Physics. 1979;77(2):617–629 (in Russian).
2. Danca MF, Feckan M, Kuznetsov N, Chen G. Looking more closely to the Rabinovich-Fabrikant system. International Journal of Bifurcation and Chaos. 2016;26(2):1650038. DOI: 10.1142/S0218127416500383.
3. Danca MF. Hidden transient chaotic attractors of Rabinovich–Fabrikant system. Nonlinear Dynamics. 2016;86(2):1263–1270. DOI: 10.1007/s11071-016-2962-3.
4. Danca MF, Kuznetsov N, Chen G. Unusual dynamics and hidden attractors of the Rabinovich–Fabrikant system. Nonlinear Dynamics. 2017;88(1):791–805. DOI: 10.1007/s11071-016-3276-1.
5. Luo X, Small M, Danca MF, Chen G. On a dynamical system with multiple chaotic attractors. International Journal of Bifurcation and Chaos. 2007;17(9):3235–3251. DOI: 10.1142/S0218127407018993.
6. Danca MF, Chen G. Bifurcation and chaos in a complex model of dissipative medium. International Journal of Bifurcation and Chaos. 2004;14(10):3409–3447. DOI: 10.1142/S0218127404011430.
7. Srivastava M, Agrawal SK, Vishal K, Das S. Chaos control of fractional order Rabinovich–Fabrikant system and synchronization between chaotic and chaos controlled fractional order Rabinovich–Fabrikant system. Applied Mathematical Modelling. 2014;38(13):3361–3372. DOI: 10.1016/j.apm.2013.11.054.
8. Kuznetsov AP, Kuznetsov SP, Turukina LV. Complex dynamics and chaos in the Rabinovich–Fabrikant model. Izvestiya of Saratov University. Physics. 2019;19(1):4–18 (in Russian). DOI: 10.18500/1817-3020-2019-19-1-4-18.
9. Kuznetsov SP, Turukina LV. Generalized Rabinovich–Fabrikant system: equations and its dynamics. Izvestiya VUZ. Applied Nonlinear Dynamics. 2022;30(1):7–29 (in Russian). DOI: 10.18500/0869-6632-2022-30-1-7-29.
10. Liu Y, Yang Q, Pang G. A hyperchaotic system from the Rabinovich system. Journal of Computational and Applied Mathematics. 2010;234(1):101–113. DOI: 10.1016/j.cam.2009.12.008.
11. Agrawal SK, Srivastava M, Das S. Synchronization between fractional-order Ravinovich–Fabrikant and Lotka–Volterra systems. Nonlinear Dynamics. 2012;69(4):2277–2288. DOI: 10.1007/s11071-012-0426-y.
12. Hocking LM, Stewartson K. On the nonlinear response of a marginally unstable plane parallel flow to a two-dimensional disturbance. Proc. R. Soc. Lond. A. 1972;326(1566):289–313. DOI: 10.1098/rspa.1972.0010.
13. Andronov AA, Fabrikant AL. Landau damping, wind waves, and whistler. In: Gaponov-Grekhov AV, editor. Nonlinear Waves. Moscow: Nauka; 1979. P. 68–104 (in Russian).
14. Kuramoto Y, Yamada T. Turbulent state in chemical reactions. Progress of Theoretical Physics. 1976;56(2):679–681. DOI: 10.1143/PTP.56.679.