Bifurcation in Dynamical Systems. Deterministic Chaos. Quantum Chaos

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Representation of exact trajectory solutions for chaotic one-dimensional maps in Schröder form

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Abstract. *Purpose* of the article is to illustrate the genesis, meaning and significance of the functional Schröder equation, introduced in the theory of iterations of rational functions, for the theory of deterministic chaos by analytical calculations of exact trajectory solutions, invariant densities and Lyapunov exponents of one-dimensional chaotic maps. We demonstrate the *method* for solving the functional Schröder equation for various chaotic maps by passing to a topologically conjugate mappings, for which finding the exact trajectory solution is a simpler mathematical procedure. *Results* of the analytical solution of the Schröder equation for 12 chaotic mappings of various types and the calculation of the corresponding expressions for exact trajectory solutions, invariant densities and Lyapunov exponents are presented. *Conclusion* is made about the methodological expediency of formulating and solving the Schröder equations by the study of the dynamics of one-dimensional chaotic mappings.

Keywords: iteration theory, deterministic chaos, one-dimensional maps, Schröder equation, exact solutions.

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Introduction

In 1870–1871 the German mathematician Friedrich Wilhelm Karl Ernst Schröder (1841–1902) published in the journal «Mathematische Annalen» two interrelated pioneer articles containing studies of iterations of rational functions on the complex plane in application to finding the roots of nonlinear equations [1, 2].

In the article «Über unendlich viele Algorithmen zur Auflösung der Gleichungen» [1], he demonstrated the application of Newton's algorithm to solving the *quadratic* equation $f(z) = z^2 - 1 = 0$, that is, he considered a difference iterative scheme

$$z_{n+1} = g(z_n) = z_n - \frac{z_n^2 - 1}{2z_n} = \frac{z_n^2 + 1}{2z_n}, \quad n = 0, 1, 2, ...$$

E. Schröder established the convergence regions to the two available solutions of the equation under consideration: at the initial value of z_0 with a positive real part ($\operatorname{Re} z_0 > 0$), the iterations of z_n converge to the positive root of $z_1^* = 1$, at the initial value of z_0 with a negative real part ($\operatorname{Re} z_0 < 0$) the attracting point is the negative value of the root $z_2^* = -1$. In the case of a purely imaginary initial value ($\operatorname{Re} z_0 = 0$), there is no convergence. Many years later, this Schröder's result was condescendingly called «simple» [3, p. 55].

Note that a few years after the publication of Schröder's article, in 1879 , the Englishman Arthur Cayley (1821–1895) considered in the complex plane the problem of convergence of iterations to the values of the roots of the *cubic* equation $z^3 - 1 = 0$ (one real root and two complex conjugate), also using Newton 's algorithm:

$$z_{n+1} = z_n - \frac{z_n^3 - 1}{3z_n^2} = \frac{2z_n^3 + 1}{3z_n^2}, \quad n = 0, 1, 2, ...$$

Against the background of the results of solving this problem (Cayley's problem), Schröder's result for a second-order equation looked «simple». The areas of attraction of all three roots of the cubic equation are closely and intricately intertwined. Their boundaries are fractals [4–6]. The Cayley problem played the role of a «start» for Pierre Joseph Louis Fatou (1878-1929) and Gaston Maurice Julia (1893-1978) in constructing the theory of iterated holomorphic maps, the theory of sets of Fatou and Julia [7, 8]. A E. Schröder took the first place in the chronological list of creators of holomorphic dynamics.

In the following work «Über iterirte Funktionen» [2] E. Schröder proposed a special methodological toolkit for investigating the convergence problem of iterative procedures based on the solution of the ***functional equation*** constructed by him. Now it bears his name and contains some unknown function, the finding of which allows us to further explicitly investigate the asymptotic behavior of the iterative process [9, 10].

Initially, mathematicians focused on analytical solutions of the Schröder equation on the complex plane. Works in this direction were devoted to proving the existence of Schröder functions, their analytical representation, establishing general conditions (theorems) for the convergence of iterative procedures, establishing connections of the Schröder equation with other functional equations and investigating the features of the latter's application in the analysis of iterative processes, studying the spectral properties of the so-called composite operator distinguished in the Schröder equation and other related problems. As a result, the Schröder equation gained the status of one of the most important equations of functional analysis [10, 11].

For the theory of deterministic chaos, the main topic of research is not the problems of convergence of iterations in complex or real domains to certain limits, but, «divergence», the implementation of chaotic regimes in dynamical systems. The Schröder equation is also applicable to the analysis of similar problems. Moreover, *exact finite* solutions of Schröder equations for this class of «chaotic» problems can be given much more than exact finite (not representable by infinite series) solutions for describing converging processes! From a historical point of view, it is also interesting that in the article [2] (it was written in 1869), Schröder gave an exact trajectory solution of his equation for the logistic mapping, written in complex variables as

$$z_{n+1} = 4z_n(1 - z_n), \quad |z_n| < 1, \quad n = 0, 1, 2, ...$$

This publication illustrates the applied significance of the functional Schröder equation in establishing exact expressions for the points of orbits of one-dimensional chaotic maps defined on a real numerical line as functions of the initial value x_0 and the number n of the iteration step $x_n = x_n(x_0, n)$. Knowledge of exact solutions allows us to analytically calculate the invariant density of the mapping, the corresponding Lyapunov exponent and the autocorrelation function of its trajectories. The review is conducted on the examples of test chaotic maps and new synthesized chaotic maps with invariant density in the form of classical probability distributions. Trajectory solutions are obtained in the context of constructing an additional chaotic mapping, topologically conjugate with the one under consideration, but more convenient when calculating trajectory and probabilistic characteristics. A similar «convenience» occurs if the conjugating function has specific (periodic) properties.

1. Schröder equation: formulation

Consider a mapping given by a real function of a real argument on some interval of the numeric axis (a, b):

$$x_n = g(x_{n-1}), \quad x_n \in (a, b), \quad n = 0, 1, 2, ... \tag{1}$$

The Schröder functional equation is understood as the equation

$$\omega(g(x)) = \lambda\omega(x), \tag{2}$$

where $\omega(x)$ and λ are the real function and number to be found. From function $\omega(x)$ requires one-to-one reversibility and differentiability of the inverse function:

$$u = \omega(x), \quad x = \omega^{-1}(u) = \Omega(u), \tag{3}$$

where $\omega^{-1} = \Omega(u)$ denotes the inverse function for $\omega(x)$:

$$\Omega\big(\omega(x)\big) \equiv x, \quad \omega\big(\Omega(u)\big) \equiv u.$$

Using the reversibility property (3) of the function $\omega(x)$, we obtain from the equation (2)

$$g(x) = \Omega\big(\lambda\omega(x)\big). \tag{4}$$

Using (4), it is possible to express all the members of the sequence x_n generated by (1) in terms of the initial value x_0 and the number of iterations n. Based on the starting point x_0, we get the trajectory «continuation» x_1:

$$x_1 = g(x_0) = \Omega\big(\lambda\omega(x_0)\big). \tag{5}$$

If we substitute the value x_1b in the right part of (4), then based on (5) we get the following representation for x_2:

$$x_2 = g(x_1) = \Omega\big(\lambda\omega(x_1)\big) = \Omega\left(\lambda\omega\big(\Omega(\lambda\omega(x_0))\big)\right) = \Omega\left(\lambda^2\omega(x_0)\right). \tag{6}$$

Let's assume that for the nth iteration step, the relation is valid

$$x_n = \Omega\big(\lambda^n\omega(x_0)\big). \tag{7}$$

Then, using the actions applied at the first step of iterations, we get

$$x_{n+1} = g(x_n) = \Omega\left(\lambda^{n+1}\omega(x_0)\right). \tag{8}$$

Taken together, relations (5)–(8) constitute a simple proof of exact expression (7) for the n iteration by mathematical induction. In addition to the initial value x_0 and the number of iterations n, the expression (7) contains a numeric parameter λ, the value of which is determined for a specific mapping as a result of solving the functional equation (2), that is, in the process of finding the functions $\omega(x)$ and $\Omega(x)$. The relation (7) is hereinafter referred to as a trajectory solution in the Schröder form.

2. Schröder equation: Genesis

Schröder formulated equation (2) in the context of generalizing the results he obtained for the trajectories of various iterative processes. Consider described by Newton 's formula,

$$x_{n-1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ..., \tag{9}$$

convergent process to the value of the root of the equation given by the function of the real argument,

$$f(x) = x^2 - a = 0, \quad a > 0,$$

starting from some value x_0, that is, we write down a numerical algorithm for extracting the square root from some positive number a. The ratio (9) will take the form

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 0, 1, 2, ... \tag{10}$$

This iterative procedure for extracting the square root was known to mathematicians of the Ancient World. One of its names is the iterative formula of Heron of Alexandria (he cited it in his work «Metrica», about the 60s A.D.). There are also references to the use of this algorithm by the Babylonians.

Let's denote the iterated function in (10) as

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) = \frac{\sqrt{a}}{2}\left(\frac{x}{\sqrt{a}} + \frac{\sqrt{a}}{x}\right), \quad a > 0. \tag{11}$$

Let's introduce into (11) a monotone (reversible) replacement of variables using *hyperbolic* functions

$$\frac{x}{\sqrt{a}} = \operatorname{cth}(u), \quad u = \operatorname{cth}^{-1}\frac{x}{\sqrt{a}}, \tag{12}$$

where $\operatorname{cth}(u)$ is a hyperbolic cotangent, $\operatorname{ctg}^{-1}(t)$ is a hyperbolic areacotangent (inverse function). Substituting (12) into (11) leads to a beautiful result:

$$g(u) = \frac{\sqrt{a}}{2}\left(\operatorname{cth}(u) + \frac{1}{\operatorname{cth}(u)}\right) = \sqrt{a}\operatorname{cth}(2u), \tag{13}$$

since the relation is valid for the hyperbolic cotangent of a double argument (for example, [12, c. 18]):

$$\operatorname{cth}(2u) = \frac{1}{2}\left(\operatorname{cth}(u) + \frac{1}{\operatorname{cth}(u)}\right) = \frac{\operatorname{cth}^2(u) + 1}{2\operatorname{cth}(u)}.$$

Accordingly, a single iteration for (11) based on (13) will be written as

$$x_1 = g(x_0) = \sqrt{a}\operatorname{cth}\left(2\operatorname{cth}^{-1}\frac{x_0}{\sqrt{a}}\right). \tag{14}$$

To clarify the structure of the solution, we write an expression for the second iteration using (14):

$$x_2 = g(x_1) = g^2(x_0) = \sqrt{a}\,\mathrm{cth}\left(2\sqrt{a}\,\mathrm{cth}^{-1}\frac{x_1}{\sqrt{a}}\right) =$$
$$= \sqrt{a}\,\mathrm{cth}\left(2\sqrt{a}\,\mathrm{cth}^{-1}\frac{\sqrt{a}\,\mathrm{cth}(2\,\mathrm{cth}^{-1}(x_0/\sqrt{a})}{\sqrt{a}}\right) = \sqrt{a}\,\mathrm{cth}\left(2^2\,\mathrm{cth}^{-1}\frac{x_0}{\sqrt{a}}\right). \tag{15}$$

When writing (15), the composition property of the forward and inverse functions is taken into account: $\mathrm{cth}(\mathrm{cth}^{-1}(t)) \equiv t$. From the form (14) and (15), we can assume that for the n iteration of the function (11), $x_n = g^n(x_0)$, the relation is valid

$$x_n = g^n(x_0) = \sqrt{a}\,\mathrm{cth}\left(2^n\,\mathrm{cth}^{-1}\frac{x_0}{\sqrt{a}}\right). \tag{16}$$

To complete the proof of this assumption by mathematical induction, it should be shown that the structure (16) is also valid for *(n+1)* iteration of the function *g(x)*. Using (16) and the result for a single iteration of (14), we will arrive at the desired result:

$$x_{n+1} = g(x_n) = g^{n+1}(x_0) = \sqrt{a}\,\mathrm{cth}\left(2\sqrt{a}\,\mathrm{cth}^{-1}\frac{x_n}{\sqrt{a}}\right) =$$
$$= \sqrt{a}\,\mathrm{cth}\left(2\sqrt{a}\,\mathrm{cth}^{-1}\frac{\sqrt{a}\,\mathrm{cth}(2^n\,\mathrm{cth}^{-1}(x_0/\sqrt{a})}{\sqrt{a}}\right) = \sqrt{a}\,\mathrm{cth}\left(2^{n+1}\,\mathrm{cth}^{-1}\frac{x_0}{\sqrt{a}}\right). \tag{17}$$

The resulting exact expression for the n iteration of (17) of the function in question (11) allows:

a). directly establish the convergence of the iterative process (16) to the value of $\sqrt{a}$;
b). set the correspondence of the representation (16) with the form of the solution of the Schröder equation.

Calculating the limit of the function (16) taking into account the limitation of the hyperbolic tangent and the monotonic tendency of its value to 1 with the growth of the argument, we see:

$$\lim_{n\to\infty} x_n = \sqrt{a}\lim_{n\to\infty}\mathrm{cth}\left(2^n\,\mathrm{cth}^{-1}\frac{x_0}{\sqrt{a}}\right) = \sqrt{a}.$$

If you enter the Schröder notation,

$$u = \omega(\tilde{x}) = \mathrm{cth}^{-1}(\tilde{x}), \quad \tilde{x} = \Omega(u) = \mathrm{cth}(u), \quad \lambda = 2, \quad \tilde{x} = x/\sqrt{a},$$

then the solution (16) will take the form of a solution to the Schröder equation (7) for $\lambda = 2 > 1$. In the theory of the functional Schröder equation, the value of $\lambda < 1$ is correlated with the convergence of the iterative process [10, 11]. The result obtained with $\lambda = 2$ is a kind of «counterexample», due to the asymptotic properties of the conjugating hyperbolic function with the simultaneous existence of an argument doubling formula for it.

3. Schröder equation: solution method

The Schröder method in solving the functional equation (2),which leads to finding unknown characteristics in (7), is based on the search (construction) of a transformation topologically conjugate to the map in question. Let T and $\tilde{T}$ be transformations on the real axis. The transformations $T\colon X \to X$ and $\tilde{T}\colon \tilde{X} \to \tilde{X}$ are called conjugate (topologically equivalent, isomorphic) [13, 14] if there exists a mapping $h\colon X \to \tilde{X}$, having the following properties::

a). h is a one-to-one (monotone, reversible) transformation (there is a single inverse differentiable map $h^{-1}\colon \tilde{X} \to X$);

b). for specific points, the compositional relations $h \circ Tx = \tilde{T} \circ h(x)$ or $Tx = h^{-1} \circ \tilde{T} \circ h(x)$ are executed for all $x \in X$.

The first condition means preserving the structure of conjugate numeric spaces. The second condition requires that h uniquely translates the elements of T in $\tilde{T}$ *regardless of the «path» of the transition.*

The transformation conjugate to the mapping $\tilde{T}\colon X \to \tilde{X}$ will therefore have the form:

$$\tilde{T}\tilde{x} = h \circ T \circ h^{-1}(\tilde{x}).$$

The meaning of the replacements of variables in the original transformation and the resulting relation (7) is to find such a conjugating function $h(x)$ (in Schröder notation $\omega(x)$), which would provide an analytical expression representing the formula of dependence x_n on the initial value of x_0 and the number of iterations of n. The solution of the functional Schröder equation (2) is focused on finding specific functions $\omega(x)$ and $\Omega(u) = \omega^{-1}(u)$, which should be interpreted as functions specifying a suitable replacement of variables (conjugation of discrete dynamical systems) in order to fulfill the relation (7).

In the case of chaotic conjugate maps with a known invariant density of one of them (we denote it by $\rho_1(x)$) the invariant density of the second mapping $\rho_2(y)$ when trajectories of maps are connected via the function $y = h(x)$ is calculated (for example, [14]) as

$$\begin{aligned}\rho_2(y) &= \int\limits_{-\infty}^{\infty} \rho_1(x)\delta(y - h(x))dx = \\ &= \int\limits_{-\infty}^{\infty} \rho_1(h^{-1}(u))\delta(y-u)dh^{-1}(u) = \rho_1(h^{-1}(y))|dh^{-1}(y)/dy|. \end{aligned} \tag{18}$$

The expression (18) is significantly simplified if $\rho_1(x)$ describes the uniform distribution. For conjugate maps, the Lyapunov exponents and the eigenvalues of the Perron–Frobenius operators associated with the maps are numerical invariants [14].

An important problem of mathematical analysis in the book by S. Ulam «Unsolved mathematical Problems» is to find out the possibility of conjugating arbitrary functions (in particular, polynomials) that map a segment into itself with *piecewise linear* maps. As noted in [15, p. 84], «a positive answer to this question would reduce the study of iterations to a purely combinatorial study of the properties of piecewise linear functions».

Test example. Schröder's article [2] is one of the first articles in the mathematical literature, where a classical example of a mapping with chaotic properties is given, namely, the logistic mapping with simultaneous recording of an exact trajectory solution. In 1947. J. von Neumann and S. Ulam first applied this mapping as a pseudorandom number sensor [16, 17]. This was the reason to call it (in the context of the problem of machine generation of pseudorandom number sequences) the Ulam-von Neumann map [14].

Starting from the map

$$x_{n+1} = 4x_n(1 - x_n), \quad x_n \in (0, 1), \quad n = 0, 1, 2, ..., \tag{19}$$

by replacing the variables

$$x = h(t) = \sin^2 \frac{\pi t}{2}, \quad t = h^{-1}(x) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad t \in (0,1),$$

we come to the relation:

$$x_{n+1} = \sin^2 \frac{\pi t_n}{2} = 4 \sin^2 \frac{\pi t_n}{2} \left(1 - \sin^2 \frac{\pi t_n}{2}\right) = \sin^2 \pi t_n = \sin^2 \left(2 \arcsin \sqrt{x_n}\right).$$

On the way to the formula of the new mapping, the double angle formula for the sine plays a key role. This makes it possible to obtain in parallel, in a compact form, an exact representation for the coordinates of the points of the trajectory of the original mapping through the initial value x_0 and the number of iterations is n:

$$x_n = \sin^2 \left(2^n \arcsin \sqrt{x_n}\right). \tag{20}$$

The conjugate for the logistic mapping (19) is the pyramidal mapping (tent map)

$$t_{n+1} = 1 - |2t_n - 1|, \quad n = 0, 1, 2, ...,$$

having an exact trajectory solution [14]

$$t_n = 1 - 2\left\{2^{n-1} t_0\right\}$$

(curly brackets denote the operation of allocating the fractional part of a number).

Invariant density. The presence of an exact trajectory solution (formula(20)) allows us to calculate analytically the invariant density of the chaotic mapping, which is the fixed point of the associated Perron-Frobenius operator. The invariant density on the unit interval is defined as follows [18, p. 36]:

$$\rho(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta\left(x - g^k(x_0)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta(x - x_k). \tag{21}$$

Here $g^k(x_0) = x_k$ is the k-th composition of the map. For the logistic mapping (19), we use the expression for the exact trajectory solution (20). From (21) we get

$$\rho(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta(x - x_k) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta\left(x - \sin^2\left(2^k \arcsin \sqrt{x_0}\right)\right). \tag{22}$$

Let's convert the argument of the trigonometric function in (22):

$$\sin^2\left(2^k \arcsin \sqrt{x_0}\right) = \sin^2\left(\pi \cdot 2^k \cdot \frac{1}{\pi} \arcsin \sqrt{x_0}\right) = \sin^2(\pi\alpha_k),$$

where

$$\alpha_k = 2^k \frac{1}{\pi} \arcsin \sqrt{x_0} = \left[2^k \frac{1}{\pi} \arcsin \sqrt{x_0}\right] + \left\{2^k \frac{1}{\pi} \arcsin \sqrt{x_0}\right\} = [\alpha_k] + \{\alpha_k\}$$

(square and curly brackets denote the operations of selecting the integer part and the fractional part of the number α_k). Then (22) will take the form:

$$\rho(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta\left(x - \sin^2(\pi\alpha_k)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta\left(x - \sin^2(\pi\{\alpha_k\})\right). \qquad (23)$$

The irrational value $\alpha_0 = \frac{1}{\pi} \arcsin\sqrt{x_0}$ takes values from the interval $(0, 1)$. According to the criterion of H. Weyl [19,20] on the uniform distribution of fractional fractions of real numbers in the unit interval sequence

$$\{\alpha_k\} = \left\{2^k \frac{1}{\pi} \arcsin\sqrt{x_0}\right\}, \quad k = 1, 2, \ldots$$

will be evenly distributed in the area of $(0, 1)$. This means that (23) can be represented by a Riemann integral on a unit interval with an integral function $\delta(x - \sin^2(\pi\alpha))$

$$\rho(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta\left(x - \sin^2\left(\pi\{\alpha_k\}\right)\right) = \int_0^1 \delta\left(x - \sin^2(\pi\alpha)\right) d\alpha =$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \delta\left(x - \sin^2(\alpha)\right) d\alpha = \frac{2}{\pi} \int_0^1 \delta(x - \xi) \frac{d\xi}{2\sqrt{\xi(1-\xi)}} = \frac{1}{\pi\sqrt{x(1-x)}}, \quad x \in (0, 1).$$

Thus, the calculation of the density of the invariant Ulam–von Neumann distribution based on the exact solution for iterations leads to the result

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad x \in (0, 1),$$

The correctness of the calculation can be confirmed by differentiating the function $h^{-1}(x) = \frac{2}{\pi} \arcsin\sqrt{x}$.

Lyapunov exponent. Lyapunov exponent $\Lambda(x_0)$ for logistic mapping as a characteristic of the degree of sensitivity to initial conditions when iterating the function (19) can be calculated based on the exact solution $x_n = x_n(x_0, n)$:

$$\Lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln\left|\frac{dg^n(x_0)}{dx_0}\right|.$$

We get according to (16):

$$\Lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln \frac{d}{dx_0} \left(\sin^2\left(2^n \arcsin\sqrt{x_0}\right)\right) = \ln 2.$$

Autocorrelation function. For a known invariant density , the expression for the autocorrelation function of the trajectories of the chaotic map is represented by an integral [14,18]:

$$R(m) = \int_0^1 x g^m(x)\rho(x)dx - \left(\int_0^1 x\rho(x)dx\right)^2$$

(averaging is carried out according to the invariant density of the mapping). In the case of the Ulam-von Neumann logistic mapping we have [14]:

$$R(m) = \begin{cases} 1/8, & m = 0, \\ 0, & m \geqslant 0. \end{cases}$$

The lack of correlation between the cross sections of the process correlates with the concept of *discrete white noise*. This circumstance fuels a special interest in the Ulam–von Neumann mapping.

4. Schröder equation: Chaos in Newton's scheme

Let's change the sign in the algorithm for extracting the square root according to the Newton algorithm (10), counting $a > 0$, that is, we present the difference equation as:

$$x_{n+1} = \frac{1}{2}\left(x_n - \frac{a}{x_n}\right) = \frac{\sqrt{a}}{2}\left(\frac{x_n}{\sqrt{a}} - \frac{\sqrt{a}}{x_n}\right), \quad n = 0, 1, 2, ...; \quad x_n \in (-\infty, +\infty). \tag{24}$$

The iterative scheme (24) demonstrates chaotic behavior, i.e. trajectory is wandering along the entire (!) numeric axis. To find the exact expression for the trajectories of the mapping (24) we introduce a continuous monotonic replacement of variables based on *trigonometric* functions:

$$x_n = \sqrt{a} \cdot \operatorname{ctg}(\pi\nu_n), \quad \nu_n = \frac{1}{\pi} \operatorname{arcctg} \frac{x_n}{\sqrt{a}}, \quad \nu_n \in (0, 1). \tag{25}$$

When setting (25) to (24), a situation arises related to the application of the double angle cotangent formula

$$\operatorname{ctg}(2u) = \frac{\operatorname{ctg}^2 u - 1}{2 \operatorname{ctg} u} = \frac{1}{2}\left(\operatorname{ctg} u - \frac{1}{\operatorname{ctg} u}\right).$$

This allows us to obtain from (24), taking into account the periodicity of the cotangent, the equation

$$\operatorname{ctg}(\pi\nu_{n+1}) = \operatorname{ctg}(\pi \cdot 2\nu_n) = \operatorname{ctg}(\pi[2\nu_n] + \pi\{2\nu_n\}) \equiv \operatorname{ctg}(\pi\{2\nu_n\}).$$

Hence follows the difference equation for the variable ν_n

$$\nu_{n+1} = \{2\nu_n\} = 2\nu_n \bmod 1, \quad n = 0, 1, ..., \quad \nu_n \in (0, 1) \tag{26}$$

(curly brackets denote the operation of allocating the fractional part of a number). The transformation (26) represents a chaotic piecewise linear mapping called the Bernoulli shift. The trajectory solution for it has the form

$$\nu_n = 2^n \nu_0 \bmod 1, \quad n = 0, 1, ...$$

When using induction, the exact solution for the mapping trajectories (24) can be represented as

$$x_n = \sqrt{a} \operatorname{ctg}\left(2^n \operatorname{arcctg} \frac{x_0}{\sqrt{a}}\right). \tag{27}$$

Due to the topological conjugation with chaotic Bernoulli shift, the map (24) also has chaotic properties. The Lyapunov exponent, being an invariant for conjugate maps, is positive in this case and is equal to $\Lambda = \ln 2$. Establishing the conjugacy of the maps (24) and (26) allows us to determine the type of invariant density of the mapping (24) by simply differentiating the inverse function in (25):

$$\rho(x) = \left|\frac{d}{dx}\frac{1}{\pi}\left(\operatorname{arcctg} \frac{x}{\sqrt{a}}\right)\right| = \frac{1}{\pi} \cdot \frac{\sqrt{a}}{a + x^2}, \quad x \in (-\infty, +\infty). \tag{28}$$

The law (28) defines the Cauchy distribution on the entire real numerical axis. It belongs to the number of «pathological» distributions for which the mathematical expectation and initial moments are not defined. The appearance of chaos in the (24) scheme can be interpreted as the result of the formal application of Newton's difference scheme to calculate the square root of a negative number in the framework of arithmetic of real numbers.

5. Representation of chaotic trajectory equations in the Schröder form

Examples of chaotic mappings whose trajectory solutions are reduced to the Schröder form are presented in the Table. The difference equations for 12 mappings demonstrating chaotic behavior are written in the first column of the Table. They are obtained by topological conjugation with piecewise linear chaotic maps. The type of conjugating functions «is read» in expressions for invariant densities (second column) and exact trajectory solutions (third column) corresponding to the mappings from the first column.

Chaotic mappings in the Table are divided into four groups.

The first group contains the maps on the unit interval. The logistic mapping «heads» this group. Two other examples show inventive possibilities in the synthesis of new mappings based on trigonometric functions.

The second group of chaotic transformations is maps in the form of Chebyshev polynomials of the first kind on the interval $(-1, 1)$. All Chebyshev polynomials of the first kind can serve as chaos generators (but with the same invariant distribution!), since they are conjugated with chaotic piecewise linear maps. In the first mapping from this group «veiled» the formula of the sine of a double angle, in the second — the formula of the sine of a triple angle, in the third — the formula for calculating the sine of a fivefold increased angle. The values of the Lyapunov exponent for these maps are respectively $\ln 2$, $\ln 3$, $\ln 5$.

The third block of the Table presents chaotic mappings, the scope of which extends to infinite intervals (subintervals). The conjugating functions are chosen in a special way so that the invariant densities coincide with the known probabilistic distribution laws (Cauchy, F-distributions, Z-distributions), which are widely used in various problems of physics, biophysics, reliability theory [21]. Obtaining a mapping with an invariant distribution in the form of Cauchy's law was discussed in detail above. Other mappings from this block are based on the formula (18).

In the final block of the Table, mappings are presented that include dependence on the parameter and demonstrate chaotic behavior for the *area of its continuous change*. These mappings are constructed on the basis of elliptic Jacobi functions [22].

Recall that the elliptic Jacobi sine $\mathrm{sn}(u,k)$ is defined as the inversion of an elliptic integral of the first kind [23, chapter 22]:

$$u = \int\limits_0^{\mathrm{sn}(u,k)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad \mathrm{sn}^{-1}\left(\sqrt{x}, k\right) = \int\limits_0^{\sqrt{x}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$(0 < k < 1)$. The elliptic Jacobi cosine $\mathrm{cn}(u,k)$ is the inverse of the integral

$$u = \int\limits_{\mathrm{cn}(u,k)}^{1} \frac{dt}{\sqrt{(1-t^2)(k'^2+t^2)}}, \quad \mathrm{cn}^{-1}(x,k) = \int\limits_x^0 \frac{dt}{\sqrt{(1-t^2)(k'^2+t^2)}}$$

$(k'^2 = 1-k^2)$. Elliptic function $\mathrm{dn}(u,k)$ is inversion of elliptic integral

$$u = \int\limits_{\mathrm{dn}(u,k)}^{1} \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}, \quad \mathrm{dn}^{-1}(x,k') = \int\limits_x^1 \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}.$$

A complete elliptic integral of the first kind:

$$K = \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Table. Characteristics of chaotic mappings with an exact trajectory solution in the Schröder form

Display	**Invariant density**	**Exact trajectory solution**
Displays on a unit interval		
$x_{n+1} = 4x_n(1 - x_n)$, $x_n \in (0, 1)$, $n = 0, 1, 2, \ldots$	$\rho(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$	$x_n = \sin^2(2^n \arcsin\sqrt{x_0})$
$x_{n+1} = 16\left(1 - \sqrt{x_n}\right)^2$, $x_n \in (0, 1)$	$\rho(x) = \dfrac{1}{2\pi x^{3/4}\sqrt{1-\sqrt{x}}}$	$x_n = \sin^4(2^n \arcsin\sqrt[4]{x_0})$
$x_{n+1} = \sqrt{2}(1 - x_n^4)^{1/4}$, $x_n \in (0, 1)$	$\rho(x) = \dfrac{4x}{\pi\sqrt{1-x^4}}$	$x_n = \sqrt{\lvert\sin(2^n \arcsin x_0^2\rvert}$
Chaotic maps based on Chebyshev polynomials		
$x_{n+1} = 2x_n^2 - 1$, $x_n \in (-1, 1)$	$\rho(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$	$x_n = \cos(2^n \arccos x_0)$
$x_{n+1} = 4x_n^3 - 3x_n$, $x_n \in (-1, 1)$	$\rho(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$	$x_n = -\sin(3^n \arcsin x_0)$
$x_{n+1} = 16x_n^5 - 20x_n^3 + 5x_n$, $x_n \in (-1, 1)$	$\rho(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$	$x_n = \sin(5^n \arcsin x_0)$
Chaotic mappings on infinite (semi-infinite) intervals		
$x_{n+1} = \dfrac{1}{2}\left(x_n - \dfrac{\lvert a\rvert}{x_n}\right)$, $x \in (-\infty, +\infty)$	Cauchy distribution $\rho(x) = \dfrac{1}{\pi} \cdot \dfrac{\sqrt{a}}{a^2 + x^2}$	$x_n = \sqrt{a}\,\mathrm{ctg}\left(2^n \,\mathrm{arcctg}\,\dfrac{x_0}{\sqrt{a}}\right)$
$x_{n+1} = \dfrac{4x_n}{(1 - x_n)^2}$, $x_n \in (0, +\infty)$	F-distribution $\rho(x) = \dfrac{1}{\pi\sqrt{x}(1+x)}$	$x_n = \mathrm{tg}^2\left(2^n \,\mathrm{arcctg}\sqrt{x_0}\right)$
$x_{n+1} = -\ln\lvert\sinh x_n\rvert$, $x_n \in (-\infty, +\infty)$	Z-distribution $\rho(x) = \dfrac{1}{\pi} \cdot \dfrac{1}{\cosh(x)}$	$x_n = -\ln\lvert\mathrm{ctg}\,(2^n \,\mathrm{arcctg}(\exp(-x_0)))\rvert$
Mappings that generates chaos in the area of continuous parameter change		
$x_{n+1} = 4x_n(1 - x_n)\dfrac{1 - k^2 x_n}{(1 - k^2 x_n^2)^2}$, $0 < x_n < 1$	$\rho(x) = \dfrac{1}{2K\sqrt{x(1-x)(1-k^2x)}}$	$x_n = \mathrm{sn}^2\left(2^n \mathrm{sn}^{-1}\left(\sqrt{x_0}, k\right), k\right)$
$x_{n+1} = 1 - \dfrac{2x_n^2}{(1 - k^2 x_n^2)^2}$, $-1 < x_n < 1$	$\rho(x) = \dfrac{1}{2K\sqrt{(1-x^2)(k'^2+k^2x^2)}}$	$x_n = -\mathrm{cn}^2\left(2^n \mathrm{cn}^{-1}\left(x_0, k\right), k\right)$
$x_{n+1} = \dfrac{k'^2 - 2k'^2 x_n^2 + x_n^4}{k'^2 + 2x_n^2 - x_n^4}$, $k' \leqslant x_n \leqslant 1$	$\rho(x) = \dfrac{1}{K\sqrt{(1-x^2)(x^2-k'^2)}}$	$x_n = \mathrm{dn}\left(2^n \mathrm{dn}^{-1}\left(x_0, k'\right), k'\right)$, $k' \leqslant x_0 \leqslant 1$

The periodicity of Jacobi functions makes it possible to reduce expressions for trajectory solutions of chaotic maps to the Schröder form.

Conclusion

The representation of expressions for the trajectories of iterative processes in Schröder form allows *analytically* to assess the presence or absence of the reducibility of a computational procedure to the solution of a given equation. The solution of the Schröder equation is connected with the construction of a mapping topologically conjugate to the original one. The article shows that the method of constructing topologically conjugate maps is very productive. Theoretical and applied interest in the idea of isomorphic transformations of basic (piecewise linear) endomorphisms, for which the features of their chaotic behavior (ergodicity, mixing, accuracy) are revealed, is stimulated by the following reasons:

a). the prospect of constructing new nonlinear chaotic generators for various applications with various statistical characteristics (setting areas, given exact invariant densities, smooth or discontinuous iterative functions, Lyapunov exponents, etc.);

b). the possibility of using invariant properties and characteristics of chaotic mappings, the results of trajectory, multiple and spectral analysis of known mappings in the study of new mappings;

c). development of analytical methods for solving difference equations «generating» chaos, direct and inverse problems for the Frobenius-Perron integral equation with a singular kernel or the corresponding functional equation linking invariant densities and iterative functions.

Convergence is as important for mappings exhibiting chaotic behavior as it is for regular processes of computational mathematics. But this convergence has a thermodynamic connotation: we are talking about convergence to an equilibrium state determined by the establishment of an invariant distribution in a dynamical system. And in this case, we can talk about the convergence of the process to «point». But this point is the fixed point of the linear non-self-adjoint Perron-Frobenius operator associated with the chaotic mapping [24].

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