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Longitudinal waves in the walls of an annular channel filled with liquid and made of a material with fractional nonlinearity

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Abstract. *Purpose* of this paper is to study the evolution of longitudinal strain waves in the walls of an annular channel filled with a viscous incompressible fluid. The walls of the channel were represented as coaxial shells with fractional physical nonlinearity. The viscosity of the fluid and its influence on the wave process was taken into account within the study. *Metods.* The system of two evolutionary equations, which are generalized Schamel equations, was obtained by the two-scale asymptotic expansion method. The fractional nonlinearity of the channel wall material leads to the necessity to use a computational experiment to study the wave dynamics in them. The computational experiment was conducted based on obtaining new difference schemes for the governing equations. These schemes are analogous to the Crank–Nicholson scheme for modeling heat propagation. *Results.* Numerical simulation showed that over time, the velocity and amplitude of the deformation waves remain unchanged, and the wave propagation direction concurs with the positive direction of the longitudinal axis. The latter specifies that the velocity of the waves is supersonic. For a particular case, the coincidence of the computational experiment with the exact solution is shown. This substantiates the adequacy of the proposed difference scheme for the generalized Schamel equations. In addition, it was shown that solitary deformation waves in the channel walls are solitons.

Keywords: wave dynamics, annular channel, viscous fluid, fractional nonlinearity, computational experiment.

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Introduction

The wave technologies of nondestructive testing are increasingly being used in different industries, for example, for pipeline diagnostics. Papers [1–5] are devoted to various aspects of wave propagation in rods, plates and shells. In the above papers, the effect on the wave process in the shell of a viscous liquid located inside it was not studied. The shell–fluid interaction was studied in [6] outside the consideration of wave processes. On the other hand, in [7] the wave process in an annular channel is studied considering the fluid inertia forces and cubic nonlinearity of shells forming the channel. In the cases of shells containing a viscous fluid, the use of qualitative analysis methods for the analytical study of nonlinear models for deformation waves causes significant difficulties [7–9]. Consequently, for these cases it is necessary to carry out computational experiments [10]. In our study, the system of governing equations for wave processes in two cylindrical shells forming an annular channel are obtained by the perturbations' method with respect to a small parameter of the problem, as well as considering the fractional physical nonlinearity of the shell material. This system is two generalized Schamel equations and allows us to estimate the wiggle of dissipative properties of the fluid on longitudinal strain waves in the channel walls. For a particular case, an exact solution of this system is found, for the general case, difference schemes are developed, and its numerical solution is performed.

1. Mathematical statement of the problem

Let us consider an annular channel formed by two cylindrical shells. Further in the paper we denote by the index $i = 1$ the parameters for the outer shell and $i = 2$ the parameters for the inner shell. We assume that the Cartesian coordinate system xyz associated with the channel symmetry axis and the corresponding cylindrical coordinate system $r\theta x$ are introduced. The x-axis coincides with the symmetry axis and the z-axis is directed along the normal to the unperturbed middle surface of the shells. For the shells material, relationship between the stress and strain tensors, as well as deformation intensity in the frames of the plasticity theory of A. A. Ilyushin [11, 12] is

$$\begin{gathered}
\sigma_x^{(i)} = E\left(\mu_0\varepsilon_\theta^{(i)} + \varepsilon_x^{(i)}\right)\left[1 + \varepsilon_u^{(i)\frac{1}{2}}\frac{m}{E}\right] / \left(1 - \mu_0^2\right), \\
\sigma_\theta^{(i)} = E\left(\varepsilon_\theta^{(i)} + \mu_0\varepsilon_x^{(i)}\right)\left[1 + \varepsilon_u^{(i)\frac{1}{2}}\frac{m}{E}\right] / \left(1 - \mu_0^2\right), \\
\varepsilon_u^{(i)} = 2\left(\mu_1\left(\varepsilon_\theta^{(i)2} + \varepsilon_x^{(i)2}\right) - \mu_2\varepsilon_\theta^{(i)}\varepsilon_x^{(i)}\right)^{\frac{1}{2}} / \sqrt{3}, \\
\mu_1 = \frac{1}{3}\left[1 + \frac{\mu_0}{(1-\mu_0)^2}\right], \quad \mu_2 = \frac{1}{3}\left[1 - \frac{2\mu_0}{(1-\mu_0)^2}\right],
\end{gathered} \tag{1}$$

here μ_0 is the shell material Poisson's ratio, E is the shell material Young's modulus, m is the constant determined from tensile-compression experiments [13], σ_x, σ_θ are the normal stresses along the x and θ axes; ε_x, ε_θ are the tensile-compression strains along the x and θ axes; ε_u is the strain intensity. Note that the relation of stresses σ_x, σ_θ with strains ε_x, ε_θ and strain intensity ε_u on the basis of the physical law with nonlinearity in the form of power function with a fractional value of the exponent for the case of incompressible material, i.e. when $\mu_0 = 1/2$, is considered in [5, 14].

The relationship between the components of the strain tensor and i-th shell displacements has the form

$$\begin{gathered}\varepsilon_x^{(i)}=\frac{\partial U^{(i)}}{\partial x}-z\frac{\partial^2 W^{(i)}}{\partial x^2},\\ \varepsilon_\theta^{(i)}=-\frac{W^{(i)}}{R^{(i)}}-z\frac{W^{(i)}}{R^{(i)2}}\quad\text{at}\quad-\frac{h_0^{(i)}}{2}\leqslant z\leqslant\frac{h_0^{(i)}}{2}.\end{gathered}\tag{2}$$

Here z is the local coordinate along the axis normal to the shell middle surface ($z=0$ corresponds to the shell middle surface), x is coordinate along the longitudinal axis of the shell median surface, $W^{(i)}$ is the deflection of the i-th shell, the positive direction of which is taken to the shell curvature center, $U^{(i)}$ is the longitudinal displacement of i-th shell, $R^{(i)}$ is the median surface radius of the i-th shell, $h_0^{(i)}$ is the i-th shell thickness.

The asymptotic analysis carried out in [7] showed that the intensity of deformations in (1), (2) can be considered on the middle surface ($z=0$) for longitudinal waves. Consequently, forces acting on the element of the shell middle surface are determined by the formulas

$$\begin{gathered}N_x^{(i)}=\int\limits_{-h_0^{(i)}/2}^{h_0^{(i)}/2}\sigma_x^{(i)}dz=\frac{Eh_0^{(i)}}{1-\mu_0^2}\left(\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}\right)\times\\ \times\left(1+\frac{m}{E}\left(\frac{2}{\sqrt{3}}\right)^{\frac12}\left[\mu_1\left(\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right)+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{\frac14}\right),\\ N_\theta^{(i)}=\int\limits_{-h_0^{(i)}/2}^{h_0^{(i)}/2}\sigma_\theta^{(i)}dz=\frac{Eh_0^{(i)}}{1-\mu_0^2}\left(\mu_0\frac{\partial U^{(i)}}{\partial x}-\frac{W^{(i)}}{R^{(i)}}\right)\times\\ \times\left(1+\frac{m}{E}\left(\frac{2}{\sqrt{3}}\right)^{\frac12}\left[\mu_1\left(\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right)+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{\frac14}\right),\end{gathered}\tag{3}$$

and the moment is defined as

$$M_x^{(i)}=\int\limits_{-h_0^{(i)}/2}^{h_0^{(i)}/2}\sigma_x^{(i)}zdz=-\frac{Eh_0^{(i)3}}{12\left(1-\mu_0^2\right)}\left(\frac{\partial^2 W^{(i)}}{\partial x^2}+\mu_0\frac{W^{(i)}}{R^{(i)2}}\right),\tag{4}$$

We write the dynamics equations for the i-th shell

$$\begin{gathered}\frac{\partial N_x^{(i)}}{\partial x}=\rho_0 h_0^{(i)}\frac{\partial^2 U^{(i)}}{\partial t^2}-q_x^{(i)}\Big|_{R^{(i)}},\\ \frac{\partial^2 M_x^{(i)}}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial W^{(i)}}{\partial x}N_x^{(i)}\right)+\frac{1}{R^{(i)}}N_\theta^{(i)}=\rho_0 h_0^{(i)}\frac{\partial^2 W^{(i)}}{\partial t^2}-(-1)^{i-1}q_n\Big|_{R^{(i)}}.\end{gathered}\tag{5}$$

Here $\rho_0^{(i)}$ is the i-th shell material density, $q_x^{(i)}$, q_n are the shear and normal fluid stresses; r, x are the cylindrical coordinates, t is the time.

Substituting (3), (4) into the shell dynamics equations, we obtain the equations in displacements

$$\frac{Eh_0^{(i)}}{1-\mu_0^2}\frac{\partial}{\partial x}\left\langle\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}+\frac{m}{E}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}}\left\{[\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}]\left[\mu_1\left[\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right]+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{\frac{1}{4}}\right\}\right\rangle=\rho_0h_0^{(i)}\frac{\partial^2U^{(i)}}{\partial t^2}-\left[q_x^{(i)}\right]_{R^{(i)}},$$

$$\frac{Eh_0^{(i)}}{12\left(1-\mu_0^2\right)}\frac{\partial^2}{\partial x^2}\left\langle-\frac{h_0^{(i)2}}{12}\left(\frac{\partial^2W^{(i)}}{\partial x^2}+\mu_0\frac{W^{(i)}}{R^{(i)2}}\right)\right\rangle+$$

$$+\frac{Eh_0}{1-\mu_0^2}\frac{\partial}{\partial x}\left\langle\frac{\partial W^{(i)}}{\partial x}\left[\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}+\frac{m}{E}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}}\left\{[\frac{\partial U^{(i)}}{\partial x}-\mu_0\frac{W^{(i)}}{R^{(i)}}]\left[\mu_1\left[\left(\frac{\partial U^{(i)}}{\partial x}\right)^2+\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.+\left(\frac{W^{(i)}}{R^{(i)}}\right)^2\right]+\mu_2\frac{\partial U^{(i)}}{\partial x}\frac{W^{(i)}}{R^{(i)}}\right]^{\frac{1}{4}}\right\}\right]\right\rangle+\frac{Eh_0^{(i)}}{1-\mu_0^2}\frac{1}{R^{(i)}}\left\langle\mu_0\frac{\partial U^{(i)}}{\partial x}-\frac{W^{(i)}}{R^{(i)}}+\right.$$

$$\left.+\frac{m}{E}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}}\left\{\left[\mu_0\frac{\partial U^{(i)}}{\partial x}-\frac{W^{(i)}}{R^{(i)}}\right]\left[\mu_1\left(\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{W}{R}\right)^2\right)+\mu_2\frac{\partial U}{\partial x}\frac{W}{R}\right]^{\frac{1}{4}}\right\}\right\rangle=$$

$$=\rho_0h_0^{(i)}\frac{\partial^2W}{\partial t^2}-\left[(-1)^{i-1}q_n\right]_{R^{(i)}}. \quad (6)$$

2. Analysis of equations for shells containing fluid by perturbation method

To analyze the wave process in the channel walls, we consider a dimensionless axisymmetric problem and choose small parameters of the problem. Namely, we assume

$$W^{(i)}=w_mu_3^{(i)},\quad U^{(i)}=u_mu_1^{(i)},\quad x^*=\frac{x}{l},$$
$$t^*=\frac{c_0}{l}t,\quad r^*=\frac{r}{R^{(i)}},\quad w_m=h_0,\ u_m=\frac{h_0l}{R^{(i)}}, \quad (7)$$

here $c_0=\sqrt{E/(\rho_0(1-\mu_0^2))}$ is the sound velocity in the shell, l is the wavelength, u_m, w_m are the characteristic values of elastic shells displacements. Let us make the following assumptions

$$\frac{h_0^{(i)}}{R^{(i)}}=\varepsilon\ll1,\quad \frac{R^{(i)^2}}{l^2}=O\left(\varepsilon^{\frac{1}{2}}\right),\quad \frac{w_m}{h_0^{(i)}}=O(1),$$
$$\frac{u_m}{l}\frac{R^{(i)}}{h_0^{(i)}}=O(1),\quad \frac{m}{E}=O(1), \quad (8)$$

here ε is the small parameter.

The method of two-scale expansions is applied and the dependent variables as an asymptotic expansion are represented

$$u_1^{(i)} = u_{10}^{(i)} + \varepsilon^{\frac{1}{2}} u_{11}^{(i)} + \ldots, u_3^{(i)} = u_{30}^{(i)} + \varepsilon^{\frac{1}{2}} u_{31}^{(i)} + \ldots. \quad (9)$$

Independent variables are introduced in the form of

$$\xi = x^* - \sqrt{1-\mu_0^2}\, t^*, \quad \tau = \varepsilon^{\frac{1}{2}} t^*. \quad (10)$$

Here τ is the slow time.

Substituting (7)–(10) into (5) in the zeroth approximation by ε gives

$$u_{30}^{(i)} = \mu_0 \frac{\partial u_{10}^{(i)}}{\partial \xi}, \quad \frac{\partial^2 u_{10}^{(i)}}{\partial \xi^2} - \mu_0 \frac{\partial u_{30}}{\partial \xi} = \left(1-\mu_0^2\right) \frac{\partial^2 u_{10}^{(i)}}{\partial \xi^2}. \quad (11)$$

Thus $u_{10}^{(i)}$ is an arbitrary function. Then the system of equations in the next approximation with account (11) is we obtain in the form of

$$\frac{\partial^2 u_{10}^{(i)}}{\partial \xi \partial \tau} + \frac{m}{E}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}} \frac{3}{4}\sqrt{1-\mu_0^2}\left(\mu_1+\mu_2\mu_0+\mu_1\mu_0^2\right)^{\frac{1}{4}}\left(\frac{\partial u_{10}^{(i)}}{\partial \xi}\right)^{\frac{1}{2}} \frac{\partial^2 u_{10}^{(i)}}{\partial \xi^2} + \frac{\mu_0^2\sqrt{1-\mu_0^2}}{2}\frac{\partial^4 u_{10}^{(i)}}{\partial \xi^4} =$$

$$= -\frac{1}{2\sqrt{1-\mu_0^2}} \frac{l}{\varepsilon^{\frac{3}{2}} \rho_0 h_0^{(i)} c_0^2}\left[\left(q_x^{(i)}\right) - \mu_0 \varepsilon^{\frac{1}{4}} \frac{\partial\left((-1)^{i-1} q_n\right)}{\partial \xi}\right]_{R^{(i)}}. \quad (12)$$

The obtained Eqs. (12) are the generalized Schamel equations for $\frac{\partial u_{10}^{(i)}}{\partial \xi}$. If the fluid is excluded, we will have two homogeneous uncoupled Schamel equations. To determine fluid stresses in Eqs. (12) it is required to study the fluid motion in the channel.

3. Determination of stresses acting on the shell from the liquid

Two coaxial infinitely long shells forming an annular channel with a viscous fluid whose density is constant are considered. The width of the slot occupied by the liquid $\delta = R_1 - R_2$, where R_1 is the inner surface radius of the outer shell and R_2 is the outer surface radius of the inner shell. The fluid motion equations of creeping flow for the problem under consideration have the following form [15]

$$\begin{aligned} &\frac{1}{\rho}\frac{\partial p}{\partial r} = \nu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial x^2} - \frac{V_r}{r^2}\right), \\ &\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r}\frac{\partial V_x}{\partial r} + \frac{\partial^2 V_x}{\partial x^2}\right), \\ &\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{\partial V_x}{\partial x} = 0. \end{aligned} \quad (13)$$

At the boundary of the shells and the liquid, the no-slip conditions of the liquid are satisfied [15]

$$V_x = \frac{\partial U^{(i)}}{\partial t}, \quad V_r = -\frac{\partial W^{(i)}}{\partial t} \quad \text{at} \quad r = R_i - W^{(i)}, \tag{14}$$

here r, x is the cylindrical coordinates, V_x, V_r are fluid velocity projections on the coordinate axis, p is the fluid pressure; ρ is the fluid density, ν is the kinematic coefficient of fluid viscosity.

The fluid stresses $q_x^{(i)}$ and q_n are determined at $r = R^{(i)}$

$$q_n = -p + 2\rho\nu\frac{\partial V_r}{\partial r}, \quad q_x^{(i)} = -\rho\nu\left(\frac{\partial V_x}{\partial r} + \frac{\partial V_r}{\partial x}\right). \tag{15}$$

Dimensionless variables and parameters are introduced

$$\begin{gathered} V_r = h_0^{(i)}\frac{c_0}{l}v_r, \quad V_x = h_0^{(i)}\frac{c_0}{\delta}\mathrm{v}_x, \quad r^* = \frac{r - R^{(2)}}{\delta}, \quad t^* = \frac{c_0}{l}t, \quad x^* = \frac{x}{l}, \quad p = \frac{\rho\nu c_0 l h_0^{(i)}}{\delta^3}P, \\ \psi = \frac{\delta}{R^{(2)}} = \varepsilon^{\frac{1}{2}}, \quad \lambda = \frac{h_0^{(i)}}{\delta} = \varepsilon^{\frac{1}{2}}, \quad \frac{h_0^{(i)}}{R^{(i)}} = \varepsilon, \quad \frac{h_0^{(i)}}{l} = \varepsilon^{\frac{5}{4}}, \quad \frac{\delta}{l} = \varepsilon^{\frac{3}{4}}, \quad \mathrm{Re} = \frac{\delta}{l}\frac{\delta c_0}{\nu} = \varepsilon. \end{gathered} \tag{16}$$

The variables (16) are substituted into Eqs. (13), (14), (15), and the following decompositions

$$P = P^0 + \varepsilon^{\frac{1}{2}}P^1 + \ldots \quad v_r = v_r^0 + \varepsilon^{\frac{1}{2}}v_r^1 + \ldots, \quad v_x = v_x^0 + \varepsilon^{\frac{1}{2}}v_x^1 + \ldots \tag{17}$$

are taken into account. After that, for the first terms of expansion (17) are obtained dynamics equations of the thin fluid layer for creeping flow [7, 15]

$$\frac{\partial P^0}{\partial r^*} = 0, \quad \frac{\partial P^0}{\partial x^*} = \frac{\partial^2 v_x^0}{\partial r^{*2}}, \quad \frac{\partial v_x^0}{\partial x^*} + \frac{\partial v_r^0}{\partial r^*} = 0 \tag{18}$$

with boundary conditions

$$v_r^0 = -\frac{\partial u_3^{(1)}}{\partial t^*}, \quad v_x^0 = 0 \text{ at } r^* = 1, \quad v_r^0 = -\frac{\partial u_3^{(2)}}{\partial t^*}, \quad v_x^0 = 0 \text{ at } r^* = 0 \tag{19}$$

as well as with the accuracy to ψ, $\varepsilon^{\frac{1}{2}}$ from (15) are obtained, too

$$q_x^{(1)} \approx -\rho\nu\frac{h_0^{(1)}c_0}{\delta^2}\frac{\partial v_x^{0*}}{\partial r^*} \quad \text{at} \quad r^* = 1, \quad q_x^{(2)} \approx -\rho\nu\frac{h_0^{(2)}c_0}{\delta^2}\frac{\partial v^{0*}}{\partial r^*} \quad \text{at} \quad r^* = 0, \quad q_n \approx -\frac{\rho\nu c_0 l h_0^{(i)}}{\delta^3}P^0. \tag{20}$$

Solving the problem (18), (19) we obtain

$$P^0 = 12\int\left[\int\left(\frac{\partial u_3^{(2)}}{\partial t^*} - \frac{\partial u_3^{(1)}}{\partial t^*}\right)dx^*\right]dx^*, \quad \frac{\partial v_x^0}{\partial t^*} = 6\left(r^{*2} - r^*\right)\int\left(\frac{\partial^2 u_3^{(2)}}{\partial t^{*2}} - \frac{\partial^2 u_3^{(1)}}{\partial t^{*2}}\right)dx^*. \tag{21}$$

Bearing in mind (9), the new variables ξ, τ (10) and with the accuracy to $\varepsilon^{\frac{1}{2}}$ we write (21) as

$$P^0 = 12\sqrt{1 - \mu_0^2}\int\left(u_{30}^{(1)} - u_{30}^{(2)}\right)d\xi, \quad \frac{\partial v_x^0}{\partial r^*} = 6\sqrt{1 - \mu_0^2}\,(2r^* - 1)\left(u_{30}^{(1)} - u_{30}^{(2)}\right), \tag{22}$$

and then

$$\frac{\partial P^0}{\partial \xi} = 12\sqrt{1-\mu_0^2}\left(u_{30}^{(1)} - u_{30}^{(2)}\right),$$

$$\left.\frac{\partial v_x^0}{\partial r^*}\right|_{r^*=1} = 6\sqrt{1-\mu_0^2}\left(u_{30}^{(1)} - u_{30}^{(2)}\right), \quad \left.\frac{\partial v_x^0}{\partial r^*}\right|_{r=0}^{*} = -\left.\frac{\partial v_x^0}{\partial r^*}\right|_{r^*=1}. \tag{23}$$

Given that according (11) $u_{30}^{(i)} = \mu_0 \partial u_{10}^{(i)}/\partial\xi$ submiting (22), (23) into (20) and assuming $R^{(1)} = R^{(2)} = R$, $h_0^{(1)} = h_0^{(2)} = h_0$ due to smallness ψ, λ the right-hand side of the first equation $(i = 1)$ of system (12) was obtained as

$$-6\mu_0^2 \frac{\rho l}{\rho_0 h_0}\frac{\nu}{R c_0 \varepsilon^{\frac{1}{2}}}\left(\frac{R}{\delta}\right)^3\left(\frac{\partial u_{10}^{(1)}}{\partial \xi} - \frac{\partial u_{10}^{(2)}}{\partial \xi}\right), \tag{24}$$

and the right-hand side of the second equation $(i = 2)$ of system (12) was obtained in the form of

$$-6\mu_0^2 \frac{\rho l}{\rho_0 h_0}\frac{\nu}{R c_0 \varepsilon^{\frac{1}{2}}}\left(\frac{R}{\delta}\right)^3\left(\frac{\partial u_{10}^{(2)}}{\partial \xi} - \frac{\partial u_{10}^{(1)}}{\partial \xi}\right). \tag{25}$$

4. Governing equations of deformation waves

Taking into account the found right-hand parts of Eqs. (24) and (25), the system of Eqs. (12) is rewritten as

$$\frac{\partial^2 u_{10}^{(1)}}{\partial\xi\partial\tau} + \frac{m}{E}\frac{3}{4}\sqrt{1-\mu_0^2}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}}\left(\mu_1 + \mu_2\mu_0 + \mu_1\mu_0^2\right)^{\frac{1}{4}}\left(\frac{\partial u_{10}^{(1)}}{\partial \xi}\right)^{\frac{1}{2}}\frac{\partial^2 u_{10}^{(1)}}{\partial \xi^2} + \frac{\mu_0^2\sqrt{1-\mu_0^2}}{2}\frac{\partial^4 u_{10}^{(1)}}{\partial \xi^4} =$$
$$= -6\mu_0^2 \frac{\rho l}{\rho_0 h_0}\frac{\nu}{R c_0 \varepsilon}\left(\frac{R}{\delta}\right)^3\left(\frac{\partial u_{10}^{(1)}}{\partial \xi} - \frac{\partial u_{10}^{(2)}}{\partial \xi}\right),$$
$$\frac{\partial^2 u_{10}^{(2)}}{\partial\xi\partial\tau} + \frac{m}{E}\frac{3}{4}\sqrt{1-\mu_0^2}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}}\left(\mu_1 + \mu_2\mu_0 + \mu_1\mu_0^2\right)^{\frac{1}{4}}\left(\frac{\partial u_{10}^{(2)}}{\partial \xi}\right)^{\frac{1}{2}}\frac{\partial^2 u_{10}^{(2)}}{\partial \xi^2} + \frac{\mu_0^2\sqrt{1-\mu_0^2}}{2}\frac{\partial^4 u_{10}^{(2)}}{\partial \xi^4} =$$
$$= -6\mu_0^2 \frac{\rho l}{\rho_0 h_0}\frac{\nu}{R c_0 \varepsilon}\left(\frac{R}{\delta}\right)^3\left(\frac{\partial u_{10}^{(2)}}{\partial \xi} - \frac{\partial u_{10}^{(1)}}{\partial \xi}\right). \tag{26}$$

The following notations are introduced

$$\partial u_{10}^{(1)}/\partial\xi = c_3\phi^{(1)}, \quad \partial u_{10}^{(2)}/\partial\xi = c_3\phi^{(2)}, \quad \eta = c_1\xi, \quad t = c_2\tau,$$
$$c_2 = 6\mu_0^2\frac{\rho}{\rho_0\varepsilon^2}\frac{\nu}{\delta c_0 \varepsilon^{\frac{3}{4}}}, \quad c_1 = \left(2c_2/\left(\mu_0^2\sqrt{1-\mu_0^2}\right)\right)^{\frac{1}{3}},$$
$$c_3 = \left[6\frac{c_2}{c_1}\frac{E}{m}\frac{4}{3\sqrt{1-\mu_0^2}(2/\sqrt{3})^{\frac{1}{2}}\left(\mu_1 + \mu_2\mu_0 + \mu_1\mu_0^2\right)^{\frac{1}{4}}}\right]^2. \tag{27}$$

Considering (27) in (26) the system of governing equations for the study of longitudinal deformation waves in the walls of the annular channel is obtained

$$\begin{aligned} &\frac{\partial \phi^{(1)}}{\partial t} + 6{\phi^{(1)}}^{\frac{1}{2}}\frac{\partial \phi^{(1)}}{\partial \eta} + \frac{\partial^3 \phi^{(1)}}{\partial \eta^3} + \phi^{(1)} - \phi^{(2)} = 0, \\ &\frac{\partial \phi^{(2)}}{\partial t} + 6{\phi^{(2)}}^{\frac{1}{2}}\frac{\partial \phi^{(2)}}{\partial \eta} + \frac{\partial^3 \phi^{(2)}}{\partial \eta^3} + \phi^{(2)} - \phi^{(1)} = 0. \end{aligned} \tag{28}$$

System of Eqs. (28) has exact partial solution

$$\varphi^{(1)} = \varphi^{(2)} = \frac{25}{4}k^4\left(1 + \operatorname{ch} k\left(\eta - 4k^2 t\right)\right)^{-2}, \tag{29}$$

but the general case requires a numerical solution to this system. By implementing a numerical solution to equations (28), the initial conditions at $t = 0$ in the form of solutions (29) can be used

$$\varphi^{(1)}(0,\eta) = \varphi^{(2)}(0,\eta) = \frac{25}{4}k^4(1 + \operatorname{ch} k\eta)^{-2} \tag{30}$$

or

$$\varphi^{(1)}(0,\eta) = \frac{25}{4}k^4(1 + \operatorname{ch} k\eta)^{-2}, \quad \varphi^{(2)}(0,\eta) = 0. \tag{31}$$

5. Computational experiment results

The computational experiment was carried out similarly to [7], but taking into account the fractional nonlinearity. The desired difference scheme for the numerical solution of the system of equations (28) was obtained using the Gröbner basis technique in the Maple computer algebra system. The resulting difference scheme is similar to the Crank–Nicholson scheme for the heat equation [16] and has the form

$$\begin{aligned}
&\frac{{u^{(1)}}_j^{n+1} - {u^{(1)}}_j^{n}}{\tau} + 4\frac{\left({u^{(1)}}^{3/2}{}_{j+1}^{n+1} - {u^{(1)}}^{3/2}{}_{j-1}^{n+1}\right) + \left({u^{(1)}}^{3/2}{}_{j+1}^{n} - {u^{(1)}}^{3/2}{}_{j-1}^{n}\right)}{4h} + \\
&+\frac{\left({u^{(1)}}_{j+2}^{n+1} - 2{u^{(1)}}_{j+1}^{n+1} + 2{u^{(1)}}_{j-1}^{n+1} - {u^{(1)}}_{j-2}^{n+1}\right)}{4h^3} + \frac{\left({u^{(1)}}_{j+2}^{n} - 2{u^{(1)}}_{j+1}^{n} + 2{u^{(1)}}_{j-1}^{n} - {u^{(1)}}_{j-2}^{n}\right)}{4h^3} + \\
&\qquad + \frac{{u^{(1)}}_j^{n+1} + {u^{(1)}}_j^{n}}{2} - \frac{{u^{(2)}}_j^{n+1} + {u^{(2)}}_j^{n}}{2} = 0, \\
&\frac{{u^{(2)}}_j^{n+1} - {u^{(2)}}_j^{n}}{\tau} + 4\frac{\left({u^{(2)}}^{3/2}{}_{j+1}^{n+1} - {u^{(2)}}^{3/2}{}_{j-1}^{n+1}\right) + \left({u^{(2)}}^{3/2}{}_{j+1}^{n} - {u^{(2)}}^{3/2}{}_{j-1}^{n}\right)}{4h} + \\
&+\frac{\left({u^{(2)}}_{j+2}^{n+1} - 2{u^{(2)}}_{j+1}^{n+1} + 2{u^{(2)}}_{j-1}^{n+1} - {u^{(2)}}_{j-2}^{n+1}\right)}{4h^3} + \frac{\left({u^{(2)}}_{j+2}^{n} - 2{u^{(2)}}_{j+1}^{n} + 2{u^{(2)}}_{j-1}^{n} - {u^{(2)}}_{j-2}^{n}\right)}{4h^3} + \\
&\qquad + \frac{{u^{(2)}}_j^{n+1} + {u^{(2)}}_j^{n}}{2} - \frac{{u^{(1)}}_j^{n+1} + {u^{(1)}}_j^{n}}{2} = 0.
\end{aligned} \tag{32}$$

Within this scheme, terms with fractional nonlinearity for the next time step are linearized as

$$\nu_{k+1}^{3/2} = \nu_{k+1}^{3/2} - \nu_k^{3/2} + \nu_k^{3/2} = \left(\nu_{k+1}^{1/2} - \nu_k^{1/2}\right)\left(\nu_{k+1} + v_{k+1}^{1/2}\nu_k^{1/2} + \nu_k\right) + \nu_k^{3/2} =$$

$$= \left(\nu_{k+1}^{1/2} - \nu_k^{1/2}\right)\frac{\nu_{k+1}^{1/2} + \nu_k^{1/2}}{\nu_{k+1}^{1/2} + \nu_k^{1/2}}\left(\nu_{k+1} + \nu_{k+1}^{1/2}\nu_k^{1/2} + \nu_k\right) + \nu_k^{3/2} \approx (\nu_{k+1} - \nu_k)\,\frac{3}{2}\nu_k^{1/2} - 2\nu_k^{3/2} =$$

$$= \frac{3}{2}\nu_k^{1/2}\nu_{k+1} - \frac{1}{2}\nu_k^{3/2}. \quad (33)$$

Models (28), (30) and (28), (31) were numerically studied by using this difference scheme. We consider the initial condition (30) with $k = 0.2$, the numerical simulation results of the wave process are shown in Fig. 1.

According to Fig. 1, it can be seen that the waves propagate to the right without changing the speed and amplitude (supersonic speed). The numerical solution coincides with the analytic solution (29). Then the initial conditions (31) are consider with $k = 0.2$ and the calculation results are shown in Fig. 2.

According to Fig. 2, it can be seen that in the presence of a disturbance in the outer shell and its absence in the inner shell at the initial moment of time, the wave amplitude in the outer shell decreases with time, while in the opposing shell it increases. The wave amplitudes are equalized, which indicates the transfer of energy through the liquid layer between the shells.

Let us consider the case when at the initial moment a perturbation is given in the form of two waves (30) with different amplitudes and speeds assuming $k = 0.225$ for the first wave and $k = 0.2$ for the second wave. The numerical simulation results are presented in Fig. 3.

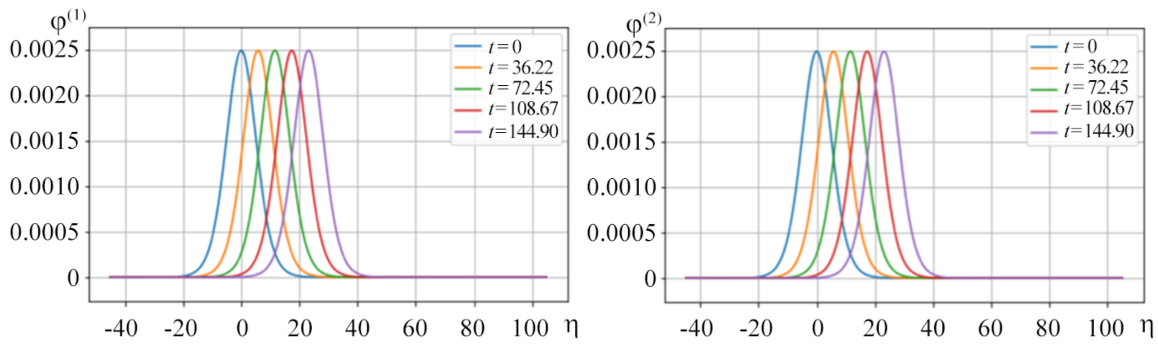

Fig. 1. Results of numerical solution of equations (28) with initial conditions (30) (color online)

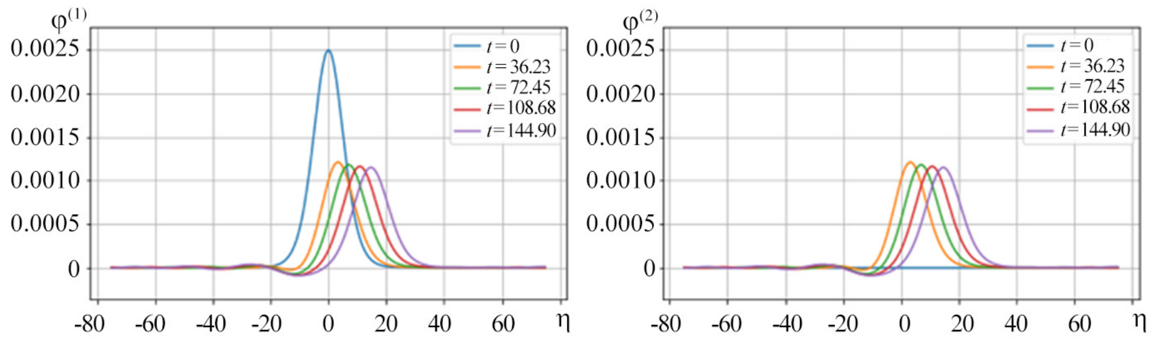

Fig. 2. Results of numerical solution of equations (28) with initial conditions (31) (color online)

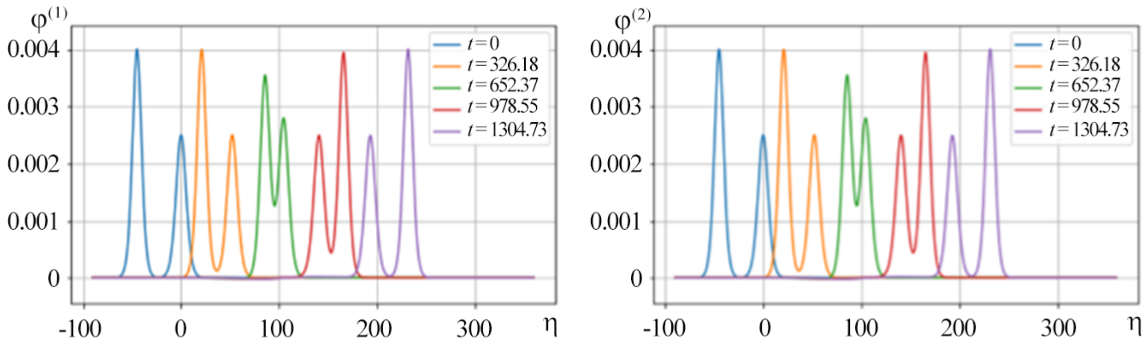

Fig. 3. Results of numerical solution of equations (28) with two initial conditions (30) for k=0.225 and k=0.2 (color online)

It follows from Fig. 3 that there is an elastic interaction of waves as particles. This means that deformation waves are solitons.

Summary and conclusion

The numerical simulation of nonlinear wave process in the walls of an annular channel indicates the need of accounting the presence of viscous fluid in the channel in the study of longitudinal deformation waves propagation. Excitation of the strain wave in the outer shell at the initial moment of time leads to the appearance of the strain wave in the opposite shell. In other words, energy transfer from one shell to the other occurs via the liquid. This process is accompanied by a decrease in the amplitude of the wave in the outer shell, which leads to a decrease in the rate of propagation of the deformation wave in this shell. At the same time, the amplitude of the wave in the opposite shell increases. Due to fluctuations in amplitudes and velocities, their amplitudes be come equal in cause of time. In addition, for the case where a solitary strain wave is excited in each shell at the initial moment of time, calculations have shown that these waves are solitons. The results obtained can be used for the development of non-destructive methods of control of pipelines with viscous liquids used in devices, machines and units, as well as the control of working processes.

References

1. Nariboli GA. Nonlinear longitudinal dispersive waves in elastic rods. J. Math. Phys. Sci. 1970;
4:64–73.
2. Nariboli GA, Sedov A. Burgers's-Korteweg-De Vries equation for viscoelastic rods and plates. J. Math. Anal. Appl. 1970;32(3):661–677. DOI: 10.1016/0022-247X(70)90290-8.
3. Erofeev VI, Klyueva NV. Solitons and nonlinear periodic strain waves in rods, plates, and shells (a review). Acoustical Physics. 2002;48(6):643–655. DOI: 10.1134/1.1522030.
4. Zemlyanukhin AI, Mogilevich LI. Nonlinear waves in inhomogeneous cylindrical shells: A new evolution equation. Acoustical Physics. 2001;47(3):303–307. DOI: 10.1007/BF03353584.
5. Zemlyanukhin AI, Andrianov IV, Bochkarev AV, Mogilevich LI. The generalized Schamel equation in nonlinear wave dynamics of cylindrical shells. Nonlinear Dynamics. 2019;98(1): 185–194. DOI: 10.1007/s11071-019-05181-5.
6. Bochkarev SA, Matveenko VP. Stability of coaxial cylindrical shells containing a rotating fluid. Computational Continuum Mechanics. 2013;6(1):94–102. DOI: 10.7242/1999-6691/2013.6.1.12.

7. Mogilevich L, Ivanov S. Longitudinal waves in two coaxial elastic shells with hard cubic nonlinearity and filled with a viscous incompressible fluid. In: Dolinina O, Bessmertny I, Brovko A, Kreinovich V, Pechenkin V, Lvov A, Zhmud V, editors. Recent Research in Control Engineering and Decision Making. ICIT 2020. Vol. 337 of Studies in Systems, Decision and Control. Cham: Springer; 2021. P. 14–26. DOI: 10.1007/978-3-030-65283-8_2.
8. Païdoussis MP. Fluid-Structure Interactions: Slender Structures and Axial Flow. 2nd edition. London: Academic Press; 2014. 867 p. DOI: 10.1016/C2011-0-08057-2.
9. Amabili M. Nonlinear Vibrations and Stability of Shells and Plates. New York: Cambridge University Press; 2008. 374 p. DOI: 10.1017/CBO9780511619694.
10. Samarskii AA. The Theory of Difference Schemes. Boca Raton: CRC Press; 2001. 786 p. DOI: 10.1201/9780203908518.
11. Il'yushin AA. Continuum Mechanics. Moscow: Moscow University Press; 1990. 310 p. (in Russian).
12. Jones RM. Deformation Theory of Plasticity. Blacksburg: Bull Ridge Publishing; 2009. 622 p.
13. Kauderer H. Nichtlineare Mechanik. Berlin: Springer-Verlag; 1958. 684 s. (in German). DOI: 10.1007/978-3-642-92733-1.
14. Zemlyanukhin AI, Bochkarev AV, Andrianov IV, Erofeev VI. The Schamel-Ostrovsky equation in nonlinear wave dynamics of cylindrical shells. Journal of Sound and Vibration. 2021;491:115752. DOI: 10.1016/j.jsv.2020.115752.
15. Loitsyanskii LG. Mechanics of Liquids and Gases. Vol. 6 of International Series of Monographs in Aeronautics and Astronautics. Oxford: Pergamon Press; 1966. 804 p. DOI: 10.1016/C2013-0-05328-5.
16. Gerdt VP, Blinkov YA, Mozzhilkin VV. Gröbner bases and generation of difference schemes for partial differential equations. Symmetry, Integrability and Geometry: Methods and Applications. 2006;2:051. DOI: 10.3842/SIGMA.2006.051.