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# Turing instability in the one-parameter Gierer–Meinhardt system

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***Abstract.*** The *purpose* of this work is to find the region of necessary and sufficient conditions for diffusion instability on the parameter plane $(\tau, d)$ of the Gierer-Meinhardt system, where $\tau$ is the relaxation parameter, $d$ is the dimensionless diffusion coefficient; to derive analytically the dependence of the critical wave number on the characteristic size of the spatial region; to obtain explicit representations of secondary spatially distributed structures, formed as a result of bifurcation of a spatially homogeneous equilibrium position, in the form of series in degrees of supercriticality. *Methods.* To find the region of Turing instability, methods of linear stability analysis are applied. To find secondary solutions (Turing structures), the Lyapunov-Schmidt method is used in the form developed by V.I. Yudovich. *Results.* Expressions for the critical diffusion coefficient in terms of the eigenvalues of the Laplace operator for an arbitrary bounded region are obtained. The dependence of the critical diffusion coefficient on the characteristic size of the region is found explicitly in two cases: when the region is an interval and a rectangle. Explicit expressions for the first terms of the expansions of the secondary stationary solutions with respect to the supercriticality parameter are constructed in the one-dimensional case, as well as for a rectangle, when one of the wave numbers is equal to zero. In these cases, sufficient conditions for a soft loss of stability are found, and examples of secondary solutions are given. *Conclusion.* A general approach is proposed for finding the region of Turing instability and constructing secondary spatially distributed structures. This approach can be applied to a wide class of mathematical models described by a system of two reaction-diffusion equations.

***Keywords***: Turing instability, reaction-diffusion systems, necessary and sufficient conditions for diffusion instability, critical diffusion coefficient.

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## Introduction

The reaction-diffusion equations are used in mathematical modeling of various natural phenomena, and a large number of publications are devoted to them (see, for example, [1], as well as recent works by [2, 3] and the literature cited in them). As a result of bifurcations of spatially homogeneous states, spatially inhomogeneous regimes arise in reaction-diffusion systems. At the boundary of the region under consideration, various types of boundary conditions can be satisfied. Of particular interest, in our opinion, is the study of reaction-diffusion systems by analytical methods. In [4], explicit asymptotic representations of self-oscillatory modes are found for the infinite-dimensional analog of the Rayleigh system (which can also be considered as a special case of the Fitzhugh-Nagumo system), and in [5]— stationary modes under Dirichlet boundary conditions and mixed boundary conditions. In [6], bifurcations on invariant subspaces of the system were studied for the Rayleigh system under Neumann boundary conditions, in [7], the bifurcation behavior of stationary regimes branching off from the zero equilibrium position of the Fitzhugh-Nagumo system was studied under the same conditions.

The study of diffusion instability, which was started by A. Turing in his classic work [8], subsequently called Turing instability, is continued by many authors to the present day. The monograph [9] initiated the study of a region in the parameter space of the system called the Turing instability region.

In [10], using the example of the Schnakenberg system, an approach is proposed for the analytical description of the Turing instability region in the parameter space of the system, as well as for finding the range of critical wave numbers for which this instability occurs. It is shown that the boundary of the domain of necessary conditions is the envelope of the boundary of the domain of sufficient conditions. In the case of the Schnakenberg system, the points of intersection of two adjacent curves of sufficient conditions lie on a straight line, the slope of which depends on the eigenvalues of the Laplace operator in the region under consideration. It is also shown in [10] that the semi-axis $d > 1$, where $d$ is the diffusion coefficient, can be represented as a union of semi-intervals, each of which corresponds to a critical wave number at which the stability of the equilibrium position of the system is lost.

In [11], the results of [10] are generalized to a certain class of reaction-diffusion systems, which, in addition to the diffusion coefficient, contain two parameters. It is assumed that the coefficients of the system linearized in the vicinity of the equilibrium position are subject to certain restrictions (hypotheses). A replacement of variables is proposed, in which the Turing instability domain takes on some standard form. The Gehrer-Meinhardt system considered in this paper contains only one parameter besides the diffusion coefficient. For the two-parameter Girer-Meinhardt system [12], one of the hypotheses of [11] does not hold.

The Girer-Meinhardt system was proposed in [12], a description of the mathematical model for various parameter values is given in [13]. In this paper, we consider a special case of the general model -the Girer-Meinhardt system with the relaxation parameter $\tau > 0$ [1] in the $m$-dimensional bounded domain $\Omega \subset R^m$ at $t > 0$ with Neumann boundary conditions on the boundary

$$u_t = \Delta u - u + \frac{u^2}{v}, \quad \tau v_t = d\Delta v - v + u^2, \tag{1}$$

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0. \tag{2}$$

Here $u = u(x,t)$— activator, $v = v(x,t)$— inhibitor, $d > 0$ — a dimensionless diffusion coefficient equal to the ratio of the diffusion coefficients of the inhibitor and activator, $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_m^2}$ — the Laplace operator. The system (1), (2) has a single spatially homogeneous equilibrium position

$$(u_0, v_0) = (1, 1). \tag{3}$$

Solutions of a singularly perturbed Gehrer-Meinhardt system with a relaxation parameter were investigated in [14], the Eckhaus instability and zigzag instability were analytically and numerically investigated for the Gehrer-Meinhardt system in [15].

The purpose of this work is to deduce the necessary and sufficient conditions for the instability of the Turing equilibrium position (3), to find the critical diffusion coefficient and its dependence on the characteristic size of the $\Omega$ region, to find secondary Turing structures in the vicinity of the equilibrium position with small deviations of the diffusion coefficient from the critical value. The approach of [10] is used to describe the Turing instability domain. All constructions are analytical in nature; numerical results are presented only to illustrate the theoretical material.

## 1. Necessary conditions for Turing instability

The results of this section are known, and we present them for completeness and for the purpose of introducing notation. A general approach for finding the necessary conditions for Turing instability was developed in [9], for the Gehrer-Meinhardt system with a relaxation parameter, the necessary conditions are formulated in [1]. Dividing the second equation of the system (1) by $\tau$, we introduce notation for the reaction terms

$$f(u,v) = -u + \frac{u^2}{v}, \quad g(u,v) = -\frac{v}{\tau} + \frac{u^2}{\tau} \tag{4}$$

and find the Jacobi matrix J

$$\mathrm{J} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}\Bigg|_{(u_0,v_0)} = \begin{pmatrix} 1 & -1 \\ \dfrac{2}{\tau} & -\dfrac{1}{\tau} \end{pmatrix}. \tag{5}$$

The system (1), (2) in the diffusion-free approximation takes the form

$$\frac{du}{dt} = -u + \frac{u^2}{v}, \quad \frac{dv}{dt} = -\frac{v}{\tau} + \frac{u^2}{\tau}, \tag{6}$$

the corresponding (6) linearized in the neighborhood of $(u_0, v_0)$ the equation has the form

$$\frac{d\boldsymbol{y}}{dt} = \mathrm{J}\boldsymbol{y}, \quad \boldsymbol{y} \in R^2, \tag{7}$$

where J is defined in (5). The eigenvalues of the Jacobi matrix J lie strictly in the left half-plane if and only if

$$\mathrm{Tr}(\mathrm{J}) \equiv f_u + g_v = 1 - \frac{1}{\tau} < 0, \quad \mathrm{Det}(\mathrm{J}) \equiv f_u g_v - f_v g_u = \frac{1}{\tau} > 0. \tag{8}$$

From (8) we obtain the condition of asymptotic stability in the diffusion-free approximation

$$0 < \tau < 1. \tag{9}$$

Now let's consider a linearized system with diffusion (1), (2)

$$u_t = \Delta u + f_u \cdot u + f_v \cdot v, \quad v_t = \frac{d}{\tau}\Delta v + g_u \cdot u + g_v \cdot v, \tag{10}$$

$$\frac{\partial u}{\partial \boldsymbol{n}}\Big|_{\partial\Omega} = \frac{\partial v}{\partial \boldsymbol{n}}\Big|_{\partial\Omega} = 0, \tag{11}$$

where the elements of the Jacobi matrix are given in (5).

Let $\mu_k$ and $\psi_k$ be the eigenvalues and eigenfunctions of the operator $-\Delta$ with Neumann boundary conditions, $k = 0, 1, 2...$

$$\Delta\psi_k + \mu_k\psi_k = 0, \quad x \in \Omega, \quad \frac{\partial\psi_k}{\partial \boldsymbol{n}}\Big|_{\partial\Omega} = 0. \tag{12}$$

The eigenvalue of a linear operator is called simple if the dimension of the generalized eigenspace corresponding to a given eigenvalue is equal to one. In this paper, as in [10], the simplicity of the eigenvalues of $\mu_k$ is assumed.

Let $H$ be a Hilbert space of vector functions $\boldsymbol{w} = (u, v)$ with components $u, v \in L_2(\Omega)$. Let the operator $\mathrm{A}_0 : H \to H$, acting according to the rule $\mathrm{A}_0 = D\Delta$, is defined on the set of vector functions $\boldsymbol{w} = (u, v)$ with components from Sobolev spaces $W_2^2(\Omega)$ satisfying boundary conditions (2), where $D$ is a matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{d}{\tau} \end{pmatrix}. \tag{13}$$

Then the linearized system (10), (11) reduces to the equation in $H$

$$\boldsymbol{w}_t = \mathrm{A}\boldsymbol{w}, \quad \mathrm{A} = \mathrm{A}_0 + \mathrm{J}, \quad \boldsymbol{w} \in H. \tag{14}$$

The spectrum of the operator A is discrete due to the compactness of its resolvent in $H$.

**Definition 1.** *Equilibrium position $(u_0, v_0)$ is called Turing unstable if all the eigenvalues of the linearized problem in the diffusion-free approximation (7) lie strictly in the left half-plane and there is an eigenvalue of the linearized problem with diffusion (14), which lies in the right half-plane.*

Consider a linear spectral problem for the operator A (14) in $H$:

$$\mathrm{A}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}, \quad \boldsymbol{\varphi} \neq 0. \tag{15}$$

We obtain the necessary conditions for the existence of the eigenvalue of the operator A in the right half-plane.

Looking for an eigenfunction $\boldsymbol{\varphi}$ (15) as a series of eigenfunctions of the Laplace operator with vector coefficients

$$\boldsymbol{\varphi} = \sum_{k=0}^{+\infty} \boldsymbol{C_k}\psi_k, \quad \boldsymbol{C_k} = (c_k^1, c_k^2), \tag{16}$$

after substituting the series (16) into (15) and equating the coefficients with the same eigenfunctions $\psi_k$, for any fixed $k$ we obtain a linear system with the matrix $\mathrm{J}_k$, which corresponds to the eigenvalue $\lambda_k$ and the eigenvector $\boldsymbol{C}_k$:

$$\mathrm{J}_k\boldsymbol{C}_k = \lambda_k\boldsymbol{C}_k, \quad \boldsymbol{C}_k \neq 0, \tag{17}$$

where $\mathrm{J}_k$ is defined by the formula

$$\mathrm{J_k} = \begin{pmatrix} f_u - \mu_k & f_v \\ g_u & g_v - \dfrac{d}{\tau}\mu_k \end{pmatrix}\Bigg|_{(u_0,v_0)} = \begin{pmatrix} 1 - \mu_k & -1 \\ \dfrac{2}{\tau} & -\dfrac{1}{\tau} - \dfrac{d}{\tau}\mu_k \end{pmatrix}, \tag{18}$$

moreover, the determinant and trace of the matrix $\mathrm{J_k}$ obey the relations

$$\begin{aligned}\mathrm{Tr}(\mathrm{J}_k) &= \mathrm{Tr}(\mathrm{J}) - \left(1 + \frac{d}{\tau}\right)\mu_k < \mathrm{Tr}(\mathrm{J}) < 0,\\ \mathrm{Det}(\mathrm{J}_k) &= \frac{d}{\tau}\mu_k^2 - \left(\frac{d}{\tau}\cdot f_u + g_v\right)\mu_k + \mathrm{Det}(\mathrm{J}).\end{aligned}$$

Since $\mathrm{Tr}(\mathrm{J_k}) < \mathrm{Tr}(\mathrm{J}) < 0$, then the loss of stability of the equilibrium $(u_0, v_0)$ can occur only if the determinant is equal to zero: $\mathrm{Det}(\mathrm{J_k}) = 0$. Since $\mathrm{Det}(\mathrm{J}) > 0$ (8), then $k > 0$ and, accordingly, $\mu_k > 0$.

Let $h(\mu)$ denote the polynomial

$$h(\mu) = d\mu^2 - (d-1)\mu + 1. \tag{19}$$

The equality $\mathrm{Det}(\mathrm{J}_k) = 0$ is possible if and only if $h(\mu_k) = 0$.

In order for a linearized system with diffusion (8), (9) to have an eigenvalue in the right half-plane, it is necessary that the trinomial $h(\mu)$ had positive roots. To do this, the discriminant of the trinomial must be non-negative, and the second coefficient must be negative. From these conditions we get a restriction on the diffusion coefficient

$$d \geqslant \left(1 + \sqrt{2}\right)^2. \tag{20}$$

Together, the inequalities (9) and (20) lead to the necessary conditions for Turing instability on the parameter plane $(\tau, d)$ [1]:

$$0 < \tau < 1, \quad d \geqslant \left(1 + \sqrt{2}\right)^2. \tag{21}$$

## 2. Sufficient Turing instability conditions

Applying the [10] approach, we obtain constraints on the system parameters under which the linearized reaction-diffusion system (8), (9) has an eigenvalue in the right half-plane. Let's take into account the discreteness of the spectrum of the operator A. Let's express from the equation $h(\mu_k) = 0$, where $h(\mu)$ is set in (19), the diffusion coefficient $d$:

$$d_k = \frac{\mu_k + 1}{\mu_k(1 - \mu_k)}. \tag{22}$$

The condition of the positivity of the diffusion coefficient (22) leads to a restriction

$$\mu_k < 1. \tag{23}$$

If this condition is not met, then Turing instability does not occur. In the one-dimensional case $\Omega = (0, \ell)$, when $\mu_k = \left(\frac{\pi k}{\ell}\right)^2$, we arrive at a lower estimate for the size of the region: $\ell > \pi k$.

Let's introduce a notation for $k \in N$

$$\gamma_k = \mu_k + \mu_{k+1} + \mu_k\mu_{k+1}. \tag{24}$$

Note that the expression (24) is also involved in describing the domain of sufficient Turing instability conditions for the Schnakenberg system [10], brusselator and other systems [11].

Using elementary calculations, the following statement is proved.

**Statement 1.** *Let* $\mu_k < 1$ *and* $\mu_{k+1} < 1$. *The equality* $d_k = d_{k+1}$ *holds if and only if* $\gamma_k = 1$, *and the inequality* $d_k < d_{k+1}$ *is equivalent to the inequality* $\gamma_k > 1$.

Note that for the function

$$d(y) = \frac{y+1}{y(1-y)}, \quad 0 < y < 1, \tag{25}$$

the value $y = \sqrt{2} - 1$ is the point of the global minimum, with $d(\sqrt{2}-1) = (\sqrt{2}+1)^2$. Therefore, for $d_k$ (22) the inequality holds: $d_k \geqslant (1+\sqrt{2})^2$.

**Definition 2.** *The critical value of the wave number is a number* $k$ *for which the eigenvalue* $\mu_k$ *is the root of the polynomial* $h(\mu)$: $h(\mu_k) = 0$, *while for* $d<d_k$ *all eigenvalues of the linearized system with diffusion* (14) *lie strictly in the left half-plane, and for* $d > d_k$ *there is an eigenvalue of the system* (14) *in the right half-plane. When these conditions are met,* $d_k$ *is called the critical diffusion coefficient.*

To find the critical diffusion coefficient, we need the following auxiliary statement, which is proved similarly to statement 1.

**Statement 2.** *Let* $1 \leqslant k < m$, $\mu_k < 1$ *and* $\mu_m < 1$, *we introduce the notation*

$$\gamma_{k,m} = \mu_k + \mu_m + \mu_k\mu_m.$$

*The equality* $d_k = d_m$ *holds if and only if* $\gamma_{k,m} = 1$, *and the inequality* $d_k < d_m$ *is equivalent to the inequality* $\gamma_{k,m} > 1$.

Statements 1 and 2 are valid for an arbitrary bounded domain $\Omega$ in which the Laplace operator has simple eigenvalues $\mu_k$ (12). We describe an algorithm for finding the critical wave number $k$ and establish the dependence of the critical diffusion coefficient $d_k$ on the characteristic size of the region $\Omega$. First, let's do the reasoning for the one-dimensional case $\Omega = (0, \ell)$.

**2.1. A one-dimensional case.** In this case, $\mu_k = \left(\frac{\pi k}{\ell}\right)^2$. Using $\ell_{k,k+1}$, we denote the length of the segment for which $d_k = d_{k+1}$, $k \in N$. We find $\ell_{k,m}$ from the equation $\gamma_{k,m} = 1$:

$$\ell^2_{k,k+1} = \frac{\pi^2}{2}\left(\sqrt{(k^2+(k+1)^2)^2 + 4k^2(k+1)^2} + k^2 + (k+1)^2\right). \tag{26}$$

Similarly, following statement 2, for $1 \leqslant k < m$, using $\ell_{k,m}$ we denote the length of the segment for which $d_k = d_m$. We find $\ell_{k,m}$ from the equation $\gamma_{k,m} = 1$:

$$\ell^2_{k,m} = \frac{\pi^2}{2}\left(\sqrt{(k^2+m^2)^2 + 4k^2m^2} + k^2 + m^2\right). \tag{27}$$

Obviously, for $m < n$, the inequality $\ell_{k,m} < \ell_{k,n}$ holds. Let's find the value of the critical diffusion coefficient depending on the characteristic size of the region.

Approximate values of the boundaries of the segment $\ell_{k,k+1}$ corresponding to the first few critical values of the wave number $k$, are shown in the table. 1.

**Statement 3.** *Let* $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$, *where* $k \in N$ *and* $\ell_{0,1} = \pi$. *Then*

$$d_k = \min_m d_m, \tag{28}$$

*where the minimum is taken for all* $m$ *for which the expression* $d_m$ *is defined .*

Table 1. Boundaries of the segment $\ell_{k,k+1}$, corresponding to the critical values of the wave number $k$

| The critical wave number | Segment | Boundaries |
|---|---|---|
| $k = 1$ | $\ell \in (\pi, \ell_{1,2})$ | $\ell_{1,2} = 2.38779 \cdot \pi$ |
| $k = 2$ | $\ell \in (\ell_{1,2}, \ell_{2,3})$ | $\ell_{2,3} = 3.91738 \cdot \pi$ |
| $k = 3$ | $\ell \in (\ell_{2,3}, \ell_{3,4})$ | $\ell_{3,4} = 5.46148 \cdot \pi$ |
| $k = 4$ | $\ell \in (\ell_{3,4}, \ell_{4,5})$ | $\ell_{4,5} = 7.00999 \cdot \pi$ |
| $k = 5$ | $\ell \in (\ell_{4,5}, \ell_{5,6})$ | $\ell_{5,6} = 8.56046 \cdot \pi$ |
| $k = 6$ | $\ell \in (\ell_{5,6}, \ell_{6,7})$ | $\ell_{6,7} = 10.11195 \cdot \pi$ |
| $k = 7$ | $\ell \in (\ell_{6,7}, \ell_{7,8})$ | $\ell_{7,8} = 11.66406 \cdot \pi$ |
| $k = 8$ | $\ell \in (\ell_{7,8}, \ell_{8,9})$ | $\ell_{8,9} = 13.21656 \cdot \pi$ |
| $k = 9$ | $\ell \in (\ell_{8,9}, \ell_{9,10})$ | $\ell_{9,10} = 14.76934 \cdot \pi$ |
| $k = 10$ | $\ell \in (\ell_{9,10}, \ell_{10,11})$ | $\ell_{10,11} = 16.3223 \cdot \pi$ |

**Proof.** Let's prove statement 3. Let's consider the interval $\ell \in (\ell_{0,1}, \ell_{1,2})$. For $\ell \in (\pi, 2\pi]$ only the value $d_1$ is defined. For $\ell \in (2\pi, \ell_{1.2})$ $d_1$ and $d_2$ are defined, but the inequality $\gamma_1 > 1$ implies that $d_1 < d_2$. Similar reasoning is carried out for the first few values of $k$.

Now let $k > 1$, $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$. As $\ell$ increases, the functions $\mu_k(\ell)$ and also $\gamma_k(\ell)$ decrease, and on the interval $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$ several diffusion coefficients $d_m$ can be defined. Let's show that $d_k$ is the minimum of them.

First, we prove that $d_m > d_k$ for $m < k$. Indeed, the inequality $d_m > d_k$ is equivalent to the inequality $\gamma_{m,k} < 1$, which, in turn, is equivalent to $\ell > \ell_{m,k}$. Since $\max\limits_{m<k} \ell_{m,k} = \ell_{k-1,k}$, the inequality holds at the specified interval.

Next, let's make sure that $d_k < d_n$ for $n > k$, Indeed, this inequality is equivalent to the following: $\gamma_{k,n} > 1$, which holds for $\ell < \ell_{k,n}$. Since $\min\limits_{n>k} \ell_{k,n} = \ell_{k,k+1}$, the required inequality is proved. □

It follows from statement 3 that the formula (28) gives the critical value of the diffusion coefficient $d_k$, where $k$ is the critical wave number. In the variables $(\tau, d)$ for $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$, the domain of sufficient Turing instability conditions has the form

$$0 < \tau < 1, \quad d \geqslant d_k. \tag{29}$$

Let $d_k(\ell) = d\left(\left(\frac{\pi k}{\ell}\right)^2\right)$, where the function $d_k(y)$ is defined in (25). To describe the relative position of the curves, $d_k(\ell)$ let's prove the following statement.

**Statement 4.** *On each interval $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$, $k \in N$ the function $d_k(\ell)$ has a unique point of minimum $\ell = \ell_*$, with $d_k(\ell_*) = (1 + \sqrt{2})^2$ and*

$$\ell_*^2 = \left(1 + \sqrt{2}\right)\left(\pi k\right)^2; \quad \ell_{k-1,k}^2 \leqslant \left(1 + \sqrt{2}\right)\left(\pi k\right)^2 \leqslant \ell_{k,k+1}^2. \tag{30}$$

**Proof.** To prove statement 4, we will replace the variables $\xi = \frac{1}{\mu}$ in (25). Then

$$d(\xi) = \frac{\xi(\xi + 1)}{\xi - 1}. \tag{31}$$

It is easy to verify that $\xi_* = 1 + \sqrt{2}$ is the point of the global minimum of the function $d(\xi)$, the corresponding critical value of the segment length is $\ell_*^2 = (\pi k)^2 \xi_*$.

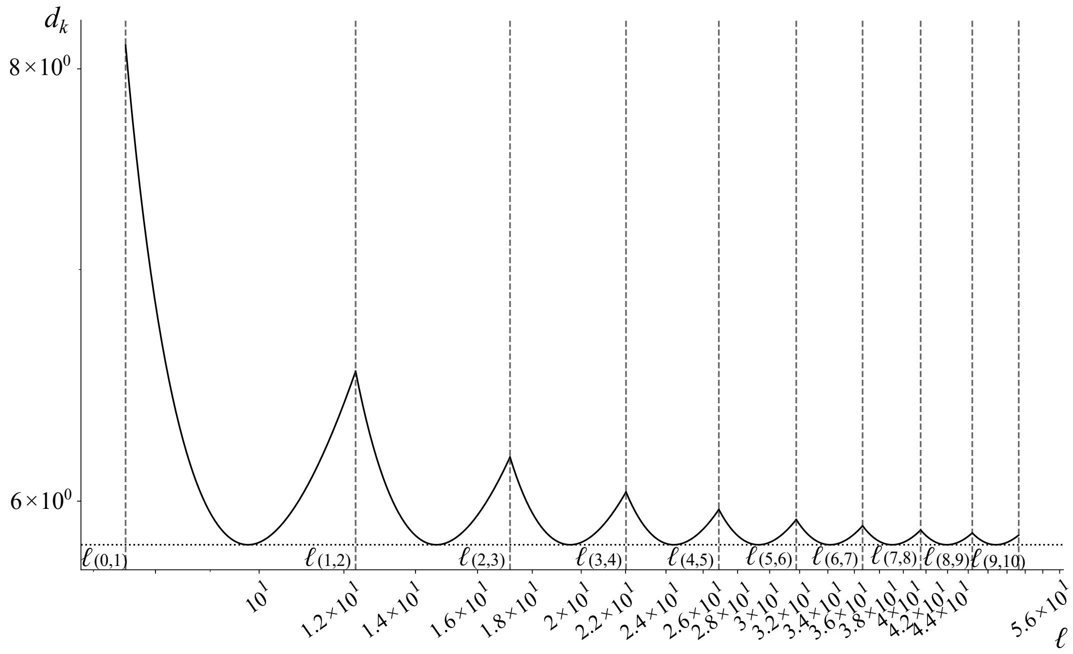


Fig. 1. Dependence of the critical diffusion coefficient $d_k(\ell)$ on the length of the segment (on a logarithmic scale)

Next, from (26) we get the inequality

$$\frac{\ell^2_{k,k+1}}{(\pi k)^2} = \frac{1}{2}\left(\sqrt{\left(1+\frac{(k+1)^2}{k^2}\right)^2 + 4\frac{(k+1)^2}{k^2}} + 1 + \frac{(k+1)^2}{k^2}\right) \geqslant 1+\sqrt{2}. \tag{32}$$

Similarly

$$\frac{\ell^2_{k-1,k}}{(\pi k)^2} = \frac{1}{2}\left(\sqrt{\left(1+\frac{(k-1)^2}{k^2}\right)^2 + 4\frac{(k-1)^2}{k^2}} + 1 + \frac{(k-1)^2}{k^2}\right) \leqslant 1+\sqrt{2}. \tag{33}$$

The formula (30) follows from (32) and (33). □

**2.2. The rectangle case.** Let now $\Omega = (0,a)\times(0,b)$. Since, by assumption, the eigenvalues of the Laplace operator are simple, we consider the case of incommensurable squares of the sides of a rectangle. Let, for definiteness,

$$b^2 = \frac{a^2}{\sqrt{2}}. \tag{34}$$

By introducing the notation $z = \left(\frac{a}{\pi}\right)^2$, we write out the eigenvalues of the Laplace operator:

$$\lambda_{m,n} = \left(\frac{\pi}{a}\right)^2 (m^2+\sqrt{2}n^2) = \frac{m^2+\sqrt{2}n^2}{z}, \quad m,n = 0,1,\ldots. \tag{35}$$

Let's arrange $\lambda_{m,n}$ in ascending order:

$$\begin{aligned} &\mu_1 = \lambda_{1,0}; \quad \mu_2 = \lambda_{0,1}; \quad \mu_3 = \lambda_{1,1}; \\ &\mu_4 = \lambda_{2,0}; \quad \mu_5 = \lambda_{2,1}; \quad \mu_6 = \lambda_{0,2}; \ldots \end{aligned} \tag{36}$$

Thus,

$$\mu_k = \frac{\nu_k}{z}; \quad \nu_k = m^2 + \sqrt{2}n^2. \tag{37}$$

Using $a_{k,m}$, we denote the length of the side of the rectangle for which $d_k = d_m$. Let's find $a_{k,k+1}$ from the equation $\gamma_k = 1$. We have

$$z^2 - (\nu_k + \nu_{k+1})z - \nu_k\nu_{k+1} = 0. \tag{38}$$

Since $z > 0$, then

$$a^2_{k,k+1} = \frac{\pi^2}{2}\left(\sqrt{(\nu_k + \nu_{k+1})^2 + 4\nu_k\nu_{k+1}} + \nu_k + \nu_{k+1}\right). \tag{39}$$

Let $d_k(a) = d\left(\left(\frac{\pi}{a}\right)^2 \nu_k\right)$, where $d_k(y)$ is defined in (25). Similarly to statement 4, the following statement is proved.

**Statement 5.** *At each interval $a \in (a_{k-1,k}, a_{k,k+1})$, $k \in N$, there is a unique minimum point $a = a_*$ of the function $d_k(a)$, with $d_k(a_*) = (1 + \sqrt{2})^2$ and*

$$a^2_* = \big(1 + \sqrt{2}\big)\pi^2\nu_k; \quad a^2_{k-1,k} \leqslant \big(1 + \sqrt{2}\big)\pi^2\nu_k \leqslant a^2_{k,k+1}. \tag{40}$$

From statements 4 and 5, as well as the conditions (21) and (29), it follows that at each interval $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$ and $a \in (a_{k-1,k}, a_{k,k+1})$, respectively, there is a unique value of the characteristic size of the domain $\ell_*$ (respectively, $a_*$) for which the necessary and sufficient Turing instability conditions coincide. For reaction-diffusion systems with two parameters, for example, for the Schnakenberg system, a similar situation arises [10, 11]. At the point of intersection of the curves of necessary and sufficient Turing instability conditions, these curves touch. For the one-parameter Girer-Meinhardt system, the curve of sufficient conditions is $d_k(\ell)$ (or $d_k(a)$) at the minimum point touches the curve of necessary conditions $d = (1 + \sqrt{2})^2$.

Approximate values of the boundaries of the side length of the rectangle $a_{k,k+1}$, corresponding to the first few values of the wave number $k$, are shown in the table. 2.

Table 2. Boundaries $a_{k,k+1}$ of the side corresponding to the critical values of the wave number $k$

| The critical wave number | Segment | Boundaries |
|---|---|---|
| $k = 1$ | $a \in (\pi, a_{1,2})$ | $a_{1,2} = 1.70341 \cdot \pi$ |
| $k = 2$ | $a \in (a_{1,2}, a_{2,3})$ | $a_{2,3} = 2.13887 \cdot \pi$ |
| $k = 3$ | $a \in (a_{2,3}, a_{3,4})$ | $a_{3,4} = 2.76998 \cdot \pi$ |
| $k = 4$ | $a \in (a_{3,4}, a_{4,5})$ | $a_{4,5} = 3.36546 \cdot \pi$ |
| $k = 5$ | $a \in (a_{4,5}, a_{5,6})$ | $a_{5,6} = 3.65555 \cdot \pi$ |
| $k = 6$ | $a \in (a_{5,6}, a_{6,7})$ | $a_{6,7} = 3.85352 \cdot \pi$ |
| $k = 7$ | $a \in (a_{6,7}, a_{7,8})$ | $a_{7,8} = 4.3402 \cdot \pi$ |
| $k = 8$ | $a \in (a_{7,8}, a_{8,9})$ | $a_{8,9} = 4.74518 \cdot \pi$ |
| $k = 9$ | $a \in (a_{8,9}, a_{9,10})$ | $a_{9,10} = 4.92167 \cdot \pi$ |
| $k = 10$ | $a \in (a_{9,10}, a_{10,11})$ | $a_{10,11} = 5.28149 \cdot \pi$ |

## 3. Secondary stationary solutions

We will be interested in secondary stationary solutions of the initial nonlinear system (1), (2), which arise at a critical value of the diffusion coefficient $d = d_k$ as a result of loss of stability of the equilibrium position $(u_0, v_0)$. When considering Turing instability, such solutions are called Turing structures.

To find Turing structures, we apply the Lyapunov-Schmidt method in the form developed by V. I. Yudovich [16, 17]. This method, along with the central manifold method, is used in problems described by parabolic equations, in particular, the equations of hydrodynamics [18], as well as the reaction-diffusion equations [4–7]. First, linear spectral and linear adjoint problems are considered, then the solution to nonlinear equations is sought in the form of a series in degrees of supercriticality. To find the terms of the series, we obtain linear inhomogeneous equations, from the solvability condition of which we find the coefficients of the branching equation.

**3.1. Linear spectral and linear adjoint problems.** Hereafter $k$ is a critical wave number, the operator $\mathrm{A}$ is defined in (14). Then the adjoint to the operator $\mathrm{A}$ has the form: $\mathrm{A}^* = \mathrm{A}_0 + \mathrm{J}^*$, where $\mathrm{J}^*$ denotes the finite-dimensional operator to which the matrix $\mathrm{J}^*$ transposed to $\mathrm{J}$ (5) corresponds. Let's find the eigenfunctions of the operators $\mathrm{A}$ and $\mathrm{A}^*$ corresponding to the zero eigenvalue:

$$\mathrm{A}\boldsymbol{\varphi}_{\boldsymbol{k}} = 0, \quad \mathrm{A}^*\boldsymbol{\Phi}_k = 0, \quad \boldsymbol{\varphi}_k \neq 0, \quad \boldsymbol{\Phi}_k \neq 0. \tag{41}$$

It follows from (15)–(17) that the eigenfunctions have the form

$$\boldsymbol{\varphi}_k(x) = \boldsymbol{C}_k \psi_k(x), \quad \boldsymbol{\Phi}_k(x) = \boldsymbol{D}_k \psi_k(x), \tag{42}$$

where $\psi_k(x)$ is the eigenfunctions of the Laplace operator (12), and the vector coefficients $\boldsymbol{C}_k$ and $\boldsymbol{D}_k$ are the eigenvectors of the matrices $\mathrm{J}_k$ and $\mathrm{J}_k^*$, respectively:

$$\mathrm{J}_k\boldsymbol{C}_k = 0, \quad \mathrm{J}_k^*\boldsymbol{D}_k = 0, \quad \boldsymbol{C}_k \neq 0, \quad \boldsymbol{D}_k \neq 0. \tag{43}$$

From (43) we find the coordinates of $\boldsymbol{C}_k$ and $\boldsymbol{D}_k$:

$$\boldsymbol{C}_k = (1; 1 - \mu_k), \quad \boldsymbol{D}_k = \left(1; -\frac{\tau}{2}(1 - \mu_k)\right). \tag{44}$$

Note that due to restrictions on the eigenvalues $0 < \mu_k < 1$ and the relaxation parameter $0 < \tau < 1$ the scalar product of the vectors $\boldsymbol{\varphi}_k$ and $\boldsymbol{\Phi}_k$ is positive

$$\int\limits_{\Omega} \boldsymbol{\varphi}_k(x)\boldsymbol{\Phi}_k(x)dx = \left(1 - \frac{\tau}{2}(1 - \mu_k)^2\right) \int\limits_{\Omega} \psi_k^2(x)dx > 0. \tag{45}$$

The condition (45) implies the absence of blocks of dimension greater than one in the matrix representation of the operator $\mathrm{A}$.

**3.2. The perturbation equation.** After changing variables in the vicinity of the equilibrium position $(u_0; v_0) = (1; 1)$ (for convenience, we retain the previous notation $(u; v)$):

$$u \to u + 1; \quad v \to v + 1 \tag{46}$$

from (1) we arrive at the perturbation equation

$$u_t = \Delta u - (u + 1) + \frac{(u + 1)^2}{v + 1}, \quad \tau v_t = d\Delta v - (v + 1) + (u + 1)^2. \tag{47}$$

We transform the nonlinear term in the first equation (1). Using the Taylor series expansion and leaving the terms no higher than the third degree

$$\frac{1}{v+1} = 1 - v + v^2 - v^3 + \ldots,$$
$$-(u+1) + (1+u)^2(1 - v + v^2 - v^3 + \ldots) = u - v + (u-v)^2 - v(u-v)^2 + \ldots,$$

let's reduce the equations (46) to the form

$$u_t = \Delta u + u - v + (u-v)^2 - v(u-v)^2, \quad \tau v_t = d\Delta v + 2u - v + u^2. \tag{48}$$

Let $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$, where $k$ is the critical wave number. Let us introduce the supercriticality parameter $\varepsilon$ using a change of variables

$$d = d_k + \varepsilon^2. \tag{49}$$

Then the perturbation equation (48) takes the form:

$$u_t = \Delta u + u - v + (u-v)^2 - v(u-v)^2, \quad \tau v_t = d_k\Delta v + 2u - v + u^2 + \varepsilon^2\Delta v. \tag{50}$$

We will look for a stationary solution of the system (50) in the form of series in powers of the parameter $\varepsilon$

$$u(x) = \sum_{k=1}^{+\infty} \varepsilon^k u_k(x), \quad v(x) = \sum_{k=1}^{+\infty} \varepsilon^k v_k(x). \tag{51}$$

Let us equate the coefficients at the same powers $\varepsilon$. For $\varepsilon^1$ we arrive at the problem

$$0 = \Delta u_1 + u_1 - v_1, \quad 0 = d_k\Delta v_1 + 2u_1 - v_1, \tag{52}$$

the solution of which is

$$(u_1; v_1) = \beta_1 \boldsymbol{\varphi_k}(x) = \beta_1 \boldsymbol{C_k}\psi_k(x), \tag{53}$$

where $\boldsymbol{\varphi_k}(x)$, $\boldsymbol{C_k}$ are defined in (41), (44), the amplitude $\beta_1$ is not yet defined.

**3.3. Equations for $\varepsilon^2$.** Equating expressions in (50) with $\varepsilon^2$, to find $(u_2, v_2)$ we arrive at the system

$$\Delta u_2 + u_2 - v_2 = -(u_1 - v_1)^2, \quad d_k\Delta v_2 + 2u_2 - v_2 = -u_1^2. \tag{54}$$

Taking into account the expressions $u_1$, $v_1$ (53), we find the vector function on the right-hand side of the system (54)

$$\boldsymbol{f_2}(x) = -(\mu_k^2; 1)\beta_1^2\psi_k^2(x). \tag{55}$$

The solvability condition for the system (54) is that the right-hand side $\boldsymbol{f_2}$ is orthogonal to the eigenvector $\boldsymbol{\Phi_k}(x)$ of the linear adjoint operator $\mathrm{A}^*$:

$$\int\limits_{\Omega} \boldsymbol{f_2}(x)\boldsymbol{\Phi_k}(x)dx = -\beta_1^2\left(\mu_k^2 - \frac{\tau}{2}(1-\mu_k)\right)\int\limits_{\Omega}\psi_k^3(x)dx = 0. \tag{56}$$

The condition (56) is satisfied in the one-dimensional case, as well as in the case of a rectangular parallelepiped due to the fact that the integral of the cube of the eigenfunction is equal to zero. Next we consider the one-dimensional case.

**3.3.1. A nonlinear additive in the one-dimensional case.** In the one-dimensional case, the formula (55) takes the form

$$\boldsymbol{f_2}(x) = -(\mu_k^2; 1)\beta_1^2 \frac{1}{2}\left(1 + \cos\left(\frac{2\pi k}{\ell}x\right)\right). \tag{57}$$

Therefore, the solution to the system (54) has the following structure

$$(u_2; v_2) = \beta_2 \boldsymbol{\varphi_k}(x) + \boldsymbol{z_0} + \boldsymbol{z}(x), \tag{58}$$

where the first term is the solution of the homogeneous equation corresponding to the equation (54), and the second and third are the solutions of the inhomogeneous equation

$$\boldsymbol{z_0} = (C_1^0; C_2^0); \quad \boldsymbol{z}(x) = (C_1^1; C_2^1)\cos\left(\frac{2\pi k}{\ell}x\right). \tag{59}$$

To find $\boldsymbol{z_0}$ from (54), we come to the system

$$C_1^0 - C_2^0 = -\frac{1}{2}\beta_1^2\mu_k^2, \quad 2C_1^0 - C_2^0 = -\frac{1}{2}\beta_1^2, \tag{60}$$

the solution of which is

$$C_1^0 = \frac{1}{2}\beta_1^2(\mu_k^2 - 1), \quad C_2^0 = \frac{1}{2}\beta_1^2(2\mu_k^2 - 1). \tag{61}$$

Now let's find $\boldsymbol{z}(x)$. The coefficients $C_1^1$ and $C_2^1$ satisfy the equations

$$C_1^1(1 - \mu_{2k}) - C_2^1 = -\frac{1}{2}\beta_1^2\mu_k^2, \quad 2C_1^1 - C_2^1(d_k\mu_{2k} + 1) = -\frac{1}{2}\beta_1^2. \tag{62}$$

The determinant of the (62) system has the form

$$\Delta(\mu_k) = d_k\mu_{2k}(\mu_{2k} - 1) + \mu_{2k} + 1. \tag{63}$$

If $\mu_{2k} \geqslant 1$, then it follows from (63) that the determinant of the system is $\Delta(\mu_k) > 0$. If $\mu_{2k} < 1$, then the diffusion coefficient $d_{2k}$ is determined and, transforming the expression (63), we again obtain the positivity of the determinant of the system:

$$\Delta(\mu_k) = \mu_{2k}(\mu_{2k} - 1)(d_k - d_{2k}) > 0, \tag{64}$$

since for $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$ we have $d_k < d_{2k}$.

Next we will need another expression $\Delta(\mu_k)$. Taking into account the relation $\mu_{2k} = 4\mu_k$, we transform (64) to the form

$$\Delta(\mu_k) = \frac{3(4\mu_k^2 + 5\mu_k - 1)}{1 - \mu_k} = \frac{3(\gamma_{k,2k} - 1)}{1 - \mu_k} > 0. \tag{65}$$

Now from (62) we find $C_1^1$ and $C_2^1$

$$C_1^1 = \frac{1}{2}\frac{\beta_1^2}{\Delta(\mu_k)}(\mu_k^2(d_k\mu_{2k} + 1) - 1), \quad C_2^1 = \frac{1}{2}\frac{\beta_1^2}{\Delta(\mu_k)}(2\mu_k^2 - (1 - \mu_{2k})). \tag{66}$$

**3.3.2. A nonlinear additive in the case of a rectangle.** Let $\Omega = (0, a) \times (0, b)$, the sides of the rectangle are connected by the ratio (34), $k$ is the critical value of the wave number. For convenience, we introduce the following notation: $\mu_k = \lambda_{m,n}$ and $\psi_k = \Psi_{m,n}(x_1, x_2)$ — eigenvalues and eigenfunctions of the operator $-\Delta$ with Neumann boundary conditions

$$\Delta\Psi_{m,n} + \lambda_{m,n}\Psi_{m,n} = 0, \quad x \in \Omega, \quad \frac{\partial \Psi_{m,n}}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0. \tag{67}$$

There are three types of eigenvalues and eigenfunctions possible.

**Type 1.** For $m \neq 0$, $n = 0$

$$\mu_k = \left(\frac{\pi}{a}\right)^2 \cdot m^2; \quad \Psi_{m,n} = \cos\left(\frac{\pi m}{a} x_1\right). \tag{68}$$

In this case, the eigenvalue of the problem in the rectangle coincides with the eigenvalue $\mu_m$ in the one-dimensional case for $\ell = a$, and the eigenfunction coincides with the eigenfunction $\psi_m(x_1)$ in the one-dimensional case.

Thus, the coefficients of decompositions of secondary solutions for $\varepsilon^2$ have the form (58), (59), where $\mu_k = \lambda_{m,0}$, $\mu_{2k} = \lambda_{2m,0}$, and its eigenfunction $\cos\left(\frac{2\pi k}{\ell} x\right)$ should be replaced with $\cos\left(\frac{2\pi m}{a} x_1\right)$.

**Type 2.** For $m = 0$, $n \neq 0$

$$\mu_k = \left(\frac{\pi}{b}\right)^2 \cdot n^2; \quad \Psi_{m,n} = \cos\left(\frac{\pi n}{b} x_2\right). \tag{69}$$

In this case, the eigenvalue of the problem in the rectangle coincides with the eigenvalue $\mu_n$ for $\ell = b$ in the one-dimensional case, and the eigenfunction coincides with the eigenfunction $\psi_n(x_2)$ in the one-dimensional case.

Then, as for type 1, the equations for $\varepsilon^2$ has already been solved in the one-dimensional case. The coefficients of decompositions of secondary solutions for $\varepsilon^2$ have the form (58), (59), where $\mu_k = \lambda_{0,n}$, $\mu_{2k} = \lambda_{0,2n}$, and the eigenfunction $\cos\left(\frac{2\pi k}{\ell} x\right)$ should be replaced with $\cos\left(\frac{2\pi n}{b} x_2\right)$.

**Type 3.** For $m \neq 0$, $n \neq 0$

$$\mu_k = \left(\frac{\pi}{a}\right)^2 \cdot m^2 + \left(\frac{\pi}{b}\right)^2 \cdot n^2; \quad \Psi_{m,n} = \cos\left(\frac{\pi n}{b} x_2\right) \cos\left(\frac{\pi n}{b} x_2\right). \tag{70}$$

Note that in the rectangle, the relationship between $k$ and $\nu_k$ is generally unknown.

Obviously, for the eigenfunctions of the first and second types of the equation at $\varepsilon^2$ has already been considered in the one-dimensional case. Let's consider the eigenfunctions of the third type. From equality

$$\Psi_{m,n}^2 = \frac{1}{4}\left(1 + \cos\left(\frac{2\pi m}{a} x_1\right) + \cos\left(\frac{2\pi n}{b} x_2\right) + \cos\left(\frac{2\pi m}{a} x_1\right)\cos\left(2\frac{\pi n}{b} x_2\right)\right) \tag{71}$$

it follows that in the expression (58), the term $\boldsymbol{z_0} = \boldsymbol{C^0}$ has the same form as in (59), and the term $\boldsymbol{z}(x)$ has the following structure

$$\boldsymbol{z}(x) = \boldsymbol{C^1}\cos\left(\frac{2\pi m}{a} x_1\right) + \boldsymbol{C^2}\cos\left(\frac{2\pi n}{b} x_2\right) + \boldsymbol{C^3}\cos\left(\frac{2\pi m}{a} x_1\right)\cos\left(\frac{2\pi n}{b} x_2\right). \tag{72}$$

The constants $\boldsymbol{C^1}$, $\boldsymbol{C^2}$, $\boldsymbol{C^3}$ are found using the same reasoning as for eigenfunctions of the first and second types. In this paper, the third type of eigenfunctions is not considered.

**3.4. The equations for $\varepsilon^3$.** Equating expressions in (50) with $\varepsilon^3$, to find $(u_3, v_3)$ we come to the system

$$\begin{aligned} &\Delta u_3 + u_3 - v_3 = -2(u_1 - v_1)(u_2 - v_2) + v_1(u_1 - v_1)^2 \equiv F_1, \\ &d_k \Delta v_3 + 2u_3 - v_3 = -\Delta v_1 - 2u_1 u_2 \equiv F_2. \end{aligned} \tag{73}$$

**3.4.1. The condition of solvability in the one-dimensional case.** Taking into account the expressions $u_1$, $v_1$ (53) and $u_2$, $v_2$ (58), we find the vector function $\boldsymbol{f_3}(x) = (F_1; F_2)$ of the right-hand side of the system (74):

$$\begin{aligned} F_1 = -2\beta_1\beta_2\mu_k^2\psi_k^2(x) - 2\beta_1\mu_k(C_1^0 - C_2^0)\psi_k(x) - 2\beta_2\mu_k(C_1^1 - C_2^1)\psi_k\psi_{2k} + \\ + \beta_1^3\mu_k^2(1-\mu_k)\psi_k^3(x), \\ F_2 = \beta_1\mu_k(1-\mu_k)\psi_k(x) - 2\beta_1\beta_2\psi_k^2(x) - 2\beta_1 C_1^0\psi_k(x) - 2\beta_1 C_1^1\psi_k(x)\psi_{2k}(x). \end{aligned} \tag{74}$$

The condition for the solvability of the equation for $\varepsilon^3$ is that the right-hand side of the system is orthogonal to the solution of the homogeneous adjoint equation:

$$\int\limits_{\Omega} \boldsymbol{f_3}(x)\boldsymbol{\Phi_k}(x)dx = 0. \tag{75}$$

It has the form

$$\begin{aligned} 2\mu_k(C_1^0 - C_2^0) + \mu_k(C_1^1 - C_2^1) - \frac{3}{4}\beta_1^2\mu_k^2(1-\mu_k) + \\ + \frac{\tau}{2}(1-\mu_k)[\mu_k(1-\mu_k) - 2C_1^0 - C_1^1] = 0. \end{aligned} \tag{76}$$

After substituting the coefficients $C_1^0, C_2^0, C_1^1, C_2^1$ in (76), the solvability condition (75) takes the form

$$\beta_1^2 f(\mu_k) = \tau\mu_k(1-\mu_k)^2. \tag{77}$$

Since the right-hand side of (77) is positive, the sign of $\beta_1^2$ coincides with the sign of the expression $f(\mu_k)$:

$$\begin{aligned} &f(\mu_k) = f_1(\mu_k) + \tau f_2(\mu_k); \\ &f_1(\mu_k) = \mu_k\left(\frac{1}{2}\mu_k^2 + \frac{3}{2}\mu_k\right) - \frac{\mu_k}{\Delta(\mu_k)}\left(d_k\mu_k^2 \cdot \mu_{2k} - \mu_k^2 - \mu_{2k}\right); \\ &f_2(\mu_k) = (1-\mu_k)\left(\mu_k^2 - 1 + \frac{1}{2\Delta(\mu_k)}(\mu_k^2(d_k\mu_{2k}+1) - 1)\right). \end{aligned} \tag{78}$$

**3.4.2. Soft and hard loss of stability.** IIf $\beta_1^2 > 0$, then a soft loss of stability occurs — secondary solutions (Turing structures) exist and are stable in the supercritical region $d > d_k$, where $d_k$ is the critical value of the diffusion coefficient. If $\beta_1^2 < 0$, then there is a hard loss of stability — secondary solutions exist in the subcritical region of $d < d_k$, and they are unstable [16, 17].

Let's find the conditions under which a soft or hard loss of stability occurs. Taking into account (65), we convert (78) to the form

$$f_1(\mu_k) = \frac{\mu_k^2(\mu_k+1)}{6(4\mu_k^2+5\mu_k-1)} \cdot g_1(\mu_k);$$
$$f_2(\mu_k) = \frac{(1-\mu_k^2)}{6(4\mu_k^2+5\mu_k-1)} \cdot g_2(\mu_k), \tag{79}$$

where $g_1(\mu_k)$and $g_2(\mu_k)$ are found by formulas

$$g_1(\mu_k) = 12\mu_k^2 + 29\mu_k - 1;$$
$$g_2(\mu_k) = 24\mu_k^3 + 9\mu_k^2 - 34\mu_k + 5. \tag{80}$$

Let's introduce the notation $y = \mu_k$, $0 < y < 1$. Then $f(y)$ in (77) has the form

$$f(y) = \frac{(1+y)(y^2 g_1(y) + \tau(1-y)g_2(y))}{6(4y^2+5y-1)}. \tag{81}$$

Our goal is to investigate the sign of the function $f(y)$. Using the expression $\mu_k = \left(\frac{\pi k}{\ell}\right)^2$, we will find the limits of the change of $y$ when the length of the segment $\ell \in [\ell_{k-1,k}, \ell_{k,k+1}]$. We have an inequality

$$\left(\frac{\pi k}{\ell_{k,k+1}}\right)^2 \leqslant y \leqslant \left(\frac{\pi k}{\ell_{k-1,k}}\right)^2, \tag{82}$$

where $\ell_{k,m}$ are defined in (27).

For $k = 1$ we have:

$$\left(\frac{\pi}{\ell_{1,2}}\right)^2 \leqslant y \leqslant \left(\frac{\pi}{\ell_{0,1}}\right)^2. \tag{83}$$

Given the expressions $\ell_{0,1} = \pi$ and $\ell_{1,2}$, for $k = 1$ we arrive at the inequality

$$y_* \leqslant y \leqslant 1, \quad y_* = \frac{\sqrt{41}-5}{8} \approx 0.1754, \tag{84}$$

where $y_*$ is the positive root of the equation $4y^2+5y-1 = 0$, which is obtained from the condition $\gamma_1 = 1$. Thus, for $y > y_*$ the denominator in (81) is positive.

For $k = 2$ from (82) and (84) we come to equality

$$4\left(\frac{\pi}{\ell_{2,3}}\right)^2 \leqslant y \leqslant 4y_*; \quad 0.2607 \leqslant y \leqslant 0.7016. \tag{85}$$

The last inequality in (85) is obtained as a result of approximate calculations.

It is easy to see that as the wave number $k$ increases, the length of the change interval $y$ (82) decreases. Taking into account the inequalities (32), (33), it is easy to show that in the limit this interval contracts to a point that belongs to all the considered intervals.

**Statement 6.** *At $k \to \infty$, the change interval of $y$ shrinks to the minimum point $y_0 = \sqrt{2} - 1$ of the function $d(y)$ (25).*

From (81) and (84) we conclude that in order for the expression $\beta_1^2$ to have a plus sign, it is enough that the function

$$G(y) = y^2 g_1(y) + \tau(1-y)g_2(y) \tag{86}$$

is positive when $y$ belongs to the interval (83), and $\tau$ changes in the interval $(0, 1)$.

Note that for $\tau = 0$, the function $G(y)$ is positive. The study of $G(y)$ shows that it is positive for all $y$ belonging to the maximum possible change interval of $y \in [y_*; 1]$ if the parameter $\tau$ is small, namely $\tau \in (0; 0.2059)$.

In addition, there is a change interval of $y \in (y_1; 1)$, where $y_1 \approx 0.47$, at which $G(y)$ is positive for all $\tau \in (0; 1)$.

**3.4.3. The condition of solvability in the two-dimensional case.** For the previously considered eigenvalues of the first and second types, when one of the indices $n$ or $m$ vanishes, the reasoning of the one-dimensional case takes place (74)–(81).

Let's assume for certainty $m \neq 0$, $n = 0$. Then, to find the square of the amplitude, we get the same expression as in the one-dimensional case (77). To determine the type of loss of stability, it is necessary to find the interval of change in the variable $y = \mu_k$.

Using the expression $\mu_k = \frac{\pi^2 \nu_k}{a^2}$, we find the limits of the change of $y$ when the side of the rectangle is $a \in (a_{k-1,k}, a_{k,k+1})$. Instead of (82), we come to the inequality

$$\frac{\pi^2 \nu_k}{a^2_{k,k+1}} \leqslant y \leqslant \frac{\pi^2 \nu_k}{a^2_{k-1,k}}, \tag{87}$$

where $a_{k,k+1}$ are defined in (39).

For $k = 1$ we have:

$$\frac{\pi^2 \nu_1}{a^2_{1,2}} \leqslant y \leqslant \frac{\pi^2 \nu_1}{a^2_{0,1}}. \tag{88}$$

Given the expressions $a_{0,1}$ and $a_{1,2}$, for $k = 1$ we arrive at the inequality

$$y_0 \leqslant y \leqslant 1, \quad y_0 = \frac{\sqrt{3 + 6\sqrt{2}} - (1 + \sqrt{2})}{2\sqrt{2}} \approx 0.3445. \tag{89}$$

Since $y_0 > y_*$, then for $y > y_0$ the denominator in (81) is positive. It is easy to see that statement 6 is also true in the two-dimensional case. The sufficient conditions for a soft loss of stability, formulated at the end of the previous paragraph, are also valid in the two-dimensional case.

**3.4.4. Stationary solutions.** Having considered the higher terms of the decomposition of the solution in degrees of $\varepsilon$, we conclude that in (75) the coefficient $\beta_2 = 0$. Summarizing the results obtained, we come to the statement.

**Statement 7.** *Let $k$ be a critical wave number; in the one-dimensional case, the length of the segment $\ell$ is enclosed in the interval $\ell \in (\ell_{k-1,k}, \ell_{k,k+1})$, in the two-dimensional case for $m \neq 0$, $n = 0$ (68) the side of the rectangle $a$ belongs to the interval $a \in (a_{k-1,k}, a_{k,k+1})$. Then at $\tau \in (0; 0.2059)$ there is a soft loss of stability of the equilibrium position $(1; 1)$ of a nonlinear system and for small $d > d_k$ stable secondary spatially inhomogeneous solutions arise*

$$(u(x); v(x)) = \pm (d - d_k)^{1/2} \beta_1 \boldsymbol{C_k} \cos\left(\frac{\pi k}{\ell} x_1\right) + (d - d_k)(\boldsymbol{z_0} + \boldsymbol{z}(x)) + O((d - d_k)^{3/2}), \tag{90}$$

*where is $\boldsymbol{C_k}$ defined in (44),*

$$\boldsymbol{z_0} = \left(C_1^0; C_2^0\right); \quad \boldsymbol{z}(x) = \left(C_1^1; C_2^1\right) \cos\left(\frac{2\pi k}{\ell} x_1\right), \tag{91}$$

*the coefficients $C_1^0, C_2^0$ are found in (61), and $C_1^1, C_2^1$ are found in (66).*

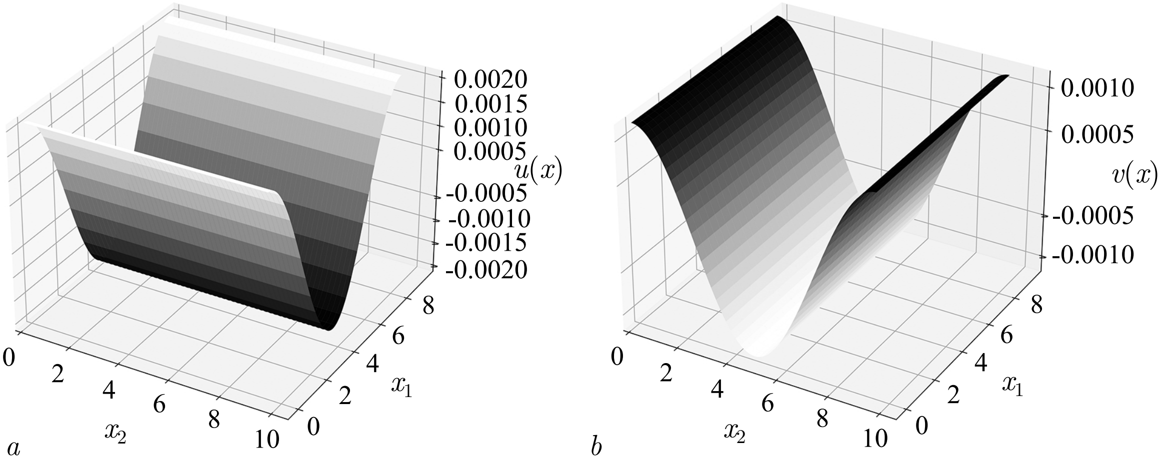


Fig. 2. Components of the secondary solution $u(x)$ $(a)$ и $v(x)$ $(b)$ at $k = 2$, $\ell = 3\pi$

In Fig. 2 an example of a secondary spatially inhomogeneous solution obtained analytically is given in the case when the spatial variable changes in a rectangle.

The numerical calculations are in full agreement with the statements obtained analytically. In Fig. 3 the results of numerical integration of a nonlinear system (48) in the case of a soft loss of stability in the one-dimensional case under the initial condition are presented

$$u(x, 0) = \varepsilon \cos\left(\frac{2}{3}x\right); \quad v(x, 0) = \frac{5}{9}\varepsilon \cos\left(\frac{2}{3}x\right) \tag{92}$$

for a critical wavenumber $k = 2$ and parameter values $\ell = 3\pi$, $\tau = 0.15$, $\varepsilon = 0.1$, $d = d_2 + \varepsilon^2$, where the critical diffusion coefficient is $d_2 = 5.85$ according to the formula (22). The solution of the non -stationary system in a short period of time goes to the stationary regime (90), corresponding to the positive $\beta_1 = 0.2$. If in the initial condition (92) $\varepsilon$ is replaced by $-\varepsilon$, then the mode corresponding to the «minus» sign in the formula (90) is set in the nonlinear system.

If the value of $\tau$ is taken from the interval corresponding to a hard loss of stability, that is, close to one (in Fig. 4 the value of $\tau = 0.95$, the other parameters are the same as in Fig. 3),

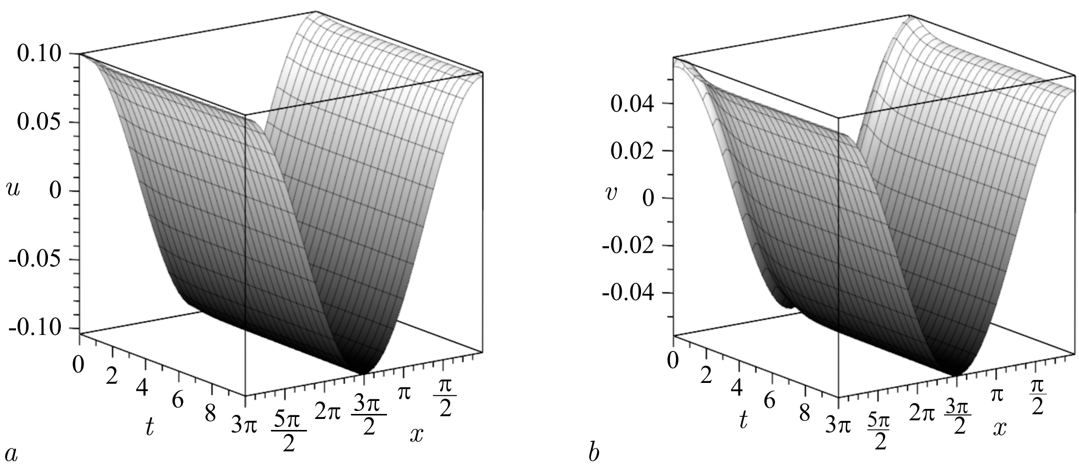


Fig. 3. Numerical solution $u(x, t)$ $(a)$ and $v(x, t)$ $(b)$ of a non-linear non-stationary system for $\tau = 0.15$ with initial conditions close to the stationary state

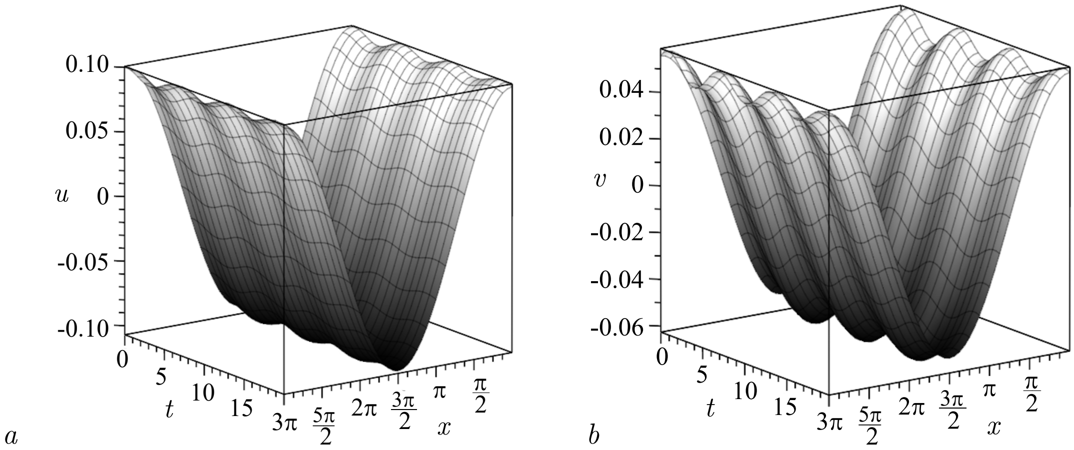


Fig. 4. Numerical solution $u(x,t)$ $(a)$ and $v(x,t)$ $(b)$ of a non-linear non-stationary system for $\tau = 0.95$ with initial conditions close to the stationary state

then in numerical experiments the equilibrium position loses stability, but there is no transition to a stationary mode.

## Conclusion

1. **The Turing instability region.** A region of necessary and sufficient Turing instability conditions is found for the Gehrer-Meinhardt system with a relaxation parameter on the parameter plane $(\tau, d)$, where $\tau$ is the relaxation parameter, and $d$ is the diffusion coefficient.
2. **The critical diffusion coefficient.** An explicit expression of the critical diffusion coefficient is found when the system is considered in an arbitrary bounded domain. It is shown that the critical diffusion coefficient depends on the eigenvalues of the Laplace operator in this domain.The dependence of the critical diffusion coefficient on the characteristic size of the domain in the case of a segment and a rectangle is established. Expressions of the length of the segment and the length of the side of the rectangle are clearly found, at which a «change» of the critical wave number occurs. These expressions are found from the condition that some combination of the eigenvalues of the Laplace operator $\gamma_k$ is equal to one. It is shown that for these domains, for each critical wavenumber, there is a single value of a characteristic size at which the necessary and sufficient Turing instability conditions coincide. This value corresponds to the minimum point of the diffusion coefficient, considered as a function of the length of the segment in the one-dimensional case or the side of the rectangle in the two-dimensional case. A comparison is made with the Turing instability conditions for the Schnakenberg system.
3. **Turing structures.** Using the Lyapunov-Schmidt method, the first few terms of the series for degrees of supercriticality are found explicitly when the diffusion coefficient is in the vicinity of the critical value. The studies are carried out for a segment, as well as for a rectangle in the case when the eigenfunctions of the Laplace operator have the same structure as in the one-dimensional case. Sufficient conditions for soft loss of stability are obtained, and examples of secondary solutions of nonlinear equations are given. The proposed approach is general in nature and can be extended to other reaction-diffusion systems. For example, the Schnakenberg system, Fitzhugh-Nagumo, Gray-Scott, the brucellator model and others.

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