Bifurcation in Dynamical Systems. Deterministic Chaos. Quantum Chaos

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Multistability near the boundary of noise-induced synchronization in ensembles of uncoupled chaotic systems

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Abstract. The *aim* of this work is to study the possibility of the existence of multistability near the boundary of noise-induced synchronization in chaotic continuous and discrete systems. Ensembles of uncoupled Lorenz systems and logistic maps being under influence of a common source of white noise have been chosen as an object under study. *Methods*. The noise-induced synchronization regime detection has been performed by means of direct comparison of the system states being under influence of the common noise source and by calculation of the synchronization error. To determine the presence of multistability near the boundary of this regime, the multistability measure has been calculated and its dependence on the noise intensity has been obtained. In addition, for fixed moments of time, the basins of attraction of the synchronous and asynchronous regimes have been received for one of the systems driven by noise for fixed initial conditions of the other system. The *result* of the work is a proof of the presence of multistability near the boundary of noise-induced synchronization. *Conclusion*. It is shown that the regime of intermittent noise-induced synchronization, as well as the regime of intermittent generalized synchronization, is characterized by multistability, which manifests itself in this case as the existence in the same time interval of the synchronous behavior in one pair of systems being under influence of a common noise source, whereas in the other pair the asynchronous behavior is observed. The found effect is typical for both flow systems and discrete maps being under influence of a common noise source. It can find an application in the information and telecommunication systems for improvement the methods for secure information transmission based on the phenomenon of chaotic synchronization.

Keywords: noise-induced synchronization, generalized synchronization, multistability, white noise, intermittency.

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In the regimern world, the phenomenon of synchronization, which belongs to the fundamental concepts of the theory of nonlinear dynamics and chaos, plays an important role. This phenomenon is widespread in nature, science, technology and society, see, for example, the monographs [1–4] and review articles [5–9].

One of the types of chaotic synchronous behavior is the noise-induced synchronization regime. It implies the establishment of identical oscillations in two or more uncoupled chaotic systems identical in control parameters due to the influence of a common noise source on them [10, 11]. Near the boundary of this regime, intermittent behavior takes place, the characteristics of which obey the same laws as for the intermittent generalized synchronization regime [12].

Relatively recently, it was found that the regime of intermittent generalized synchronization is characterized by multistability, implying in this context the existence of synchronous behavior in one pair of systems under the influence of a common chaotic signal at the same time interval, while the other pair has asynchronous behavior [13,14]. Since the regimes of generalized synchronization and noise-induced synchronization are essentially the same regime and differ only in the nature of the external signal acting on the systems [15], it can be expected that multistability will also occur near the noise-induced synchronization boundary.

In this paper, for the first time, the possibility of the existence of multistability near the boundary of induced by noise is investigated. Chaotic systems with continuous (Lorentz oscillators) and discrete (logistic maps) time under the influence of a common noise source are selected as objects of research.

Lorentz oscillators are described by the following systems of equations:

$$\begin{aligned} \dot{x}_i &= \sigma(y_i - x_i) + \varepsilon\xi, \\ \dot{y}_i &= rx_i - y_i - x_i z_i + \varepsilon\xi, \\ \dot{z}_i &= -bz_i + x_i y_i + \varepsilon\xi, \end{aligned} \tag{1}$$

where ξ is white Gaussian noise identical for all equations of the system, $\langle\xi_i(t)\rangle = 0$, $\langle\xi_i(t)\xi_j(t)\rangle = \delta_{ij}\delta(t - t')$, $\forall i, j$, ε is the noise intensity, $\mathbf{x}_i = (x_i, y_i, z_i)$ are state vectors of systems, affected by noise, $i = 1, 2$ in the case of classical noise-induced synchronization, $\sigma = 10$, $r = 28$, $b = 8/3$ are control parameters. Solving a system of equations (1) was carried out numerically using the Euler method adapted for stochastic differential equations, with a time step $h = 0.001$, similar to how it was done in the work [10].

For logistic maps, the equations have the following form:

$$x^i_{n+1} = f(x^i_n, \lambda) + \varepsilon(f(\xi_n, \lambda) - f(x^i_n, \lambda)), \tag{2}$$

where ξ_n is Gaussian noise with mean $\mu = 0.5$ and standard deviation $\sigma = 0.12$, $f(x, \lambda) = \lambda x(1 - x)$, $\lambda = 3.75$ — control parameter, $i = 1.2$ in the case of classical synchronization induced by noise, ε is an intensity of noise influence.

With the selected values of the control parameters, the noise-induced synchronization regime occurs in interacting Lorentz systems at the noise intensity $\varepsilon = 6.5$, and in logistic mappings — at $\varepsilon = 0.1625$. Below the boundary of this synchronous regime, as noted above, there is intermittent behavior. At the same time, the signal representing the difference between the states of the systems affected by noise looks like an alternation of synchronous (laminar) and asynchronous (turbulent) phases, and the presence of one or another phase of behavior at a fixed time may depend on the choice of initial conditions of the analyzed systems, which indicates the presence of multistability in this case. To prove the above, at fixed points in time, basins of attraction of one of the systems under the influence of noise were constructed under fixed initial conditions of the other system. Such basins of attraction are shown in Fig. 1 for Lorenz systems (1) at $\varepsilon = 5.75$ and in Fig. 2 for logistic mapp (2) with $\varepsilon = 0.153$. The blue color corresponds to the phases of synchronous behavior (when the states of both systems under the influence of noise turn out to be identical), the green color corresponds to asynchronous phases of behavior. The white color corresponds to the going of the image point to infinity. It can be seen that for

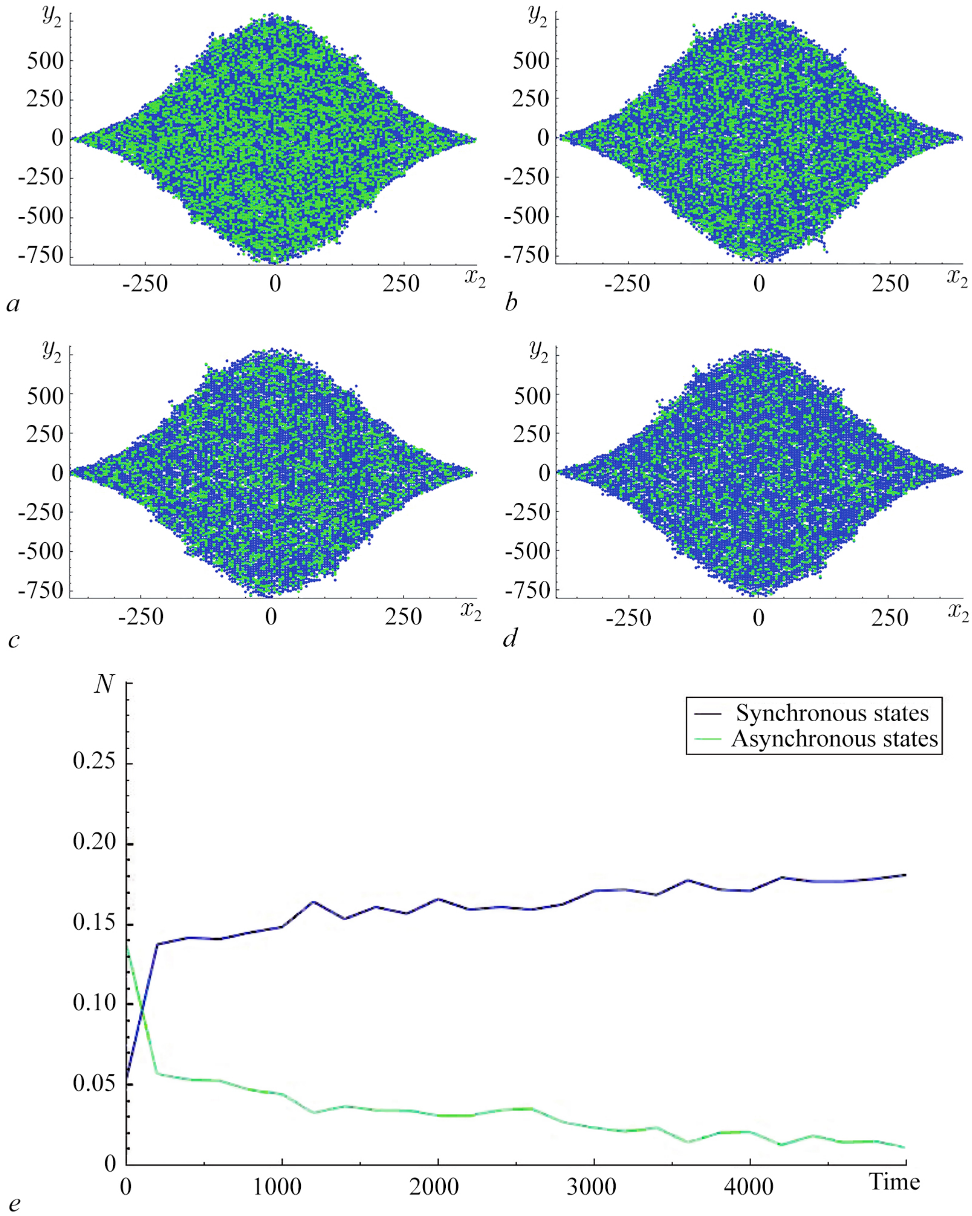

Fig 1. a–d — Basins of attraction of synchronous and asynchronous states of one Lorenz system from (1) being in the intermittent noise-induced synchronization regime for the value of noise intensity $\varepsilon = 5.75$ on the plane of initial conditions (x_2, y_2) $(z_2 = 1.1)$ obtained in different moments of time: $t = 1000$ (a), 2000 (b), 3000 (c), 4000 (d). Blue color corresponds to the realization for a fixed moment of time the identical behavior in Lorenz systems being under influence of the common noise, green color refers to the nonidentical (asynchronous) behavior of such systems. White color corresponds to the going the representation point to infinity. e — Time dependences of the normalized sizes of the synchronous and asynchronous clusters obtained for the same Lorenz systems for $\varepsilon = 5.75$ (color online)

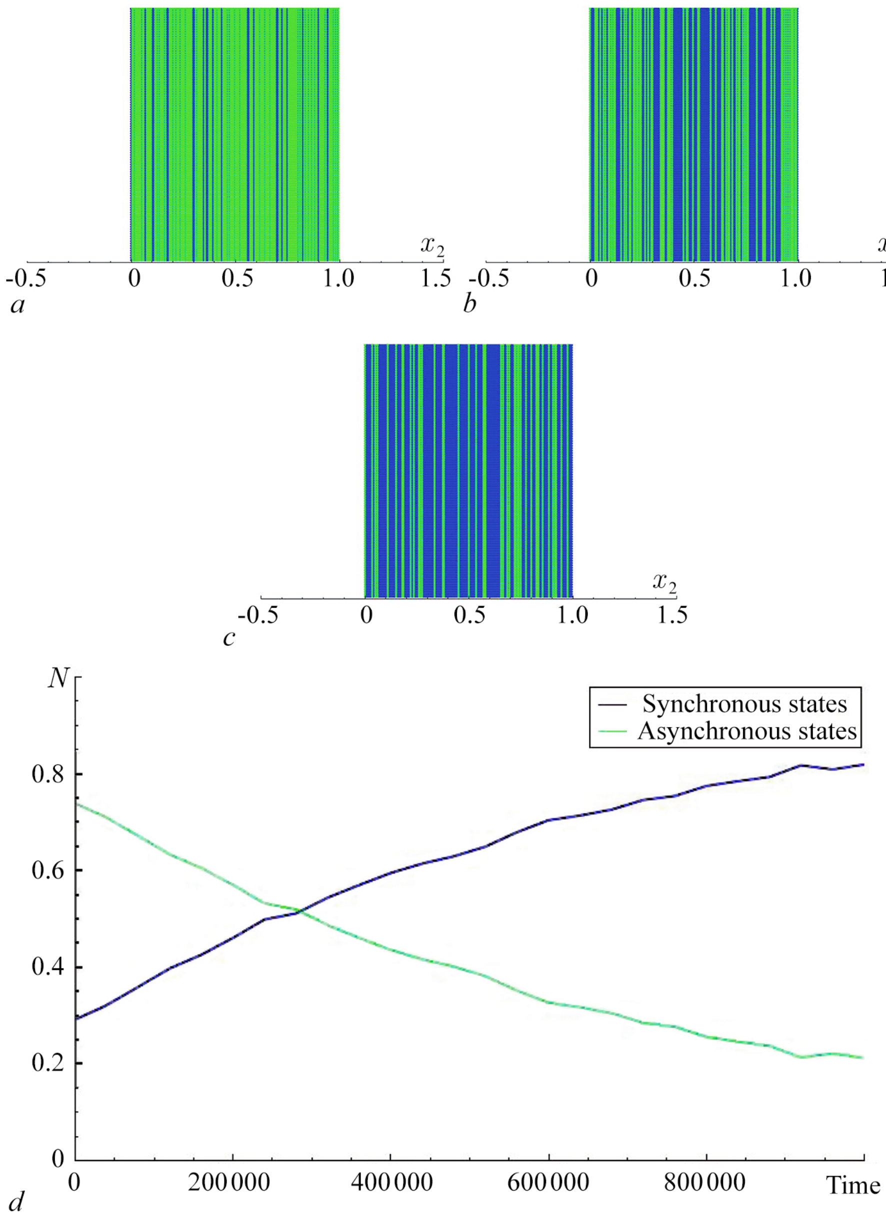

Fig 2. a–c — Linear basins of attraction of synchronous and asynchronous states of one logistic map from (2) being in the intermittent noise-induced synchronization regime for the value of noise intensity $\varepsilon = 0.153$ on the line of initial conditions x_2 obtained in different moments of time: $t = 10000$ (a), 500000 (b), 1000000 (c). Blue color corresponds to the realization for a fixed moment of time the identical states in logistic maps being under influence of the common noise, green color refers to the nonidentical (asynchronous) behavior of such systems. White color corresponds to the going the representation point to infinity. d — Time dependences of the normalized sizes of the synchronous and asynchronous clusters obtained for the same logistic maps for $\varepsilon = 0.153$ (color online)

both systems under consideration, multistability takes place at fixed time points in the regime of intermittent synchronization induced by noise. It is important to note that the number of synchronous and asynchronous states changes quite smoothly over time, which is confirmed by the dependences of the normalized sizes of synchronous and asynchronous clusters on time, shown in Fig. 1, *e* and 2, *d*. It can be seen that both for Lorenz systems and for logistic mapp, both clusters always coexist, while the size of the synchronous cluster turns out to be larger than the asynchronous one.

To quantify the degree of multistability by analogy with the works of [13,14] it is necessary to move from considering two systems to an ensemble of systems identical in control parameters, starting from different initial conditions, equally distributed over the attractors of interacting systems under the influence of common noise. Such ensembles are described by the systems (1) and (2) for $i = 1, 2 \dots N$, where N is the number of elements of the ensemble. To diagnose the synchronous regime in this case, it is necessary to compare the states of systems under the influence of noise, according to the principle of «each with each», and for each pair of interacting systems at each time to calculate the difference of their states by the formula

$$D = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}, \quad i, j = 1, \dots N, \quad i \neq j, \tag{3}$$

after that using the formula

$$P_a = 1 - \sum_{i=1}^{N} \frac{n}{N(N-1)} \tag{4}$$

(where n is the number of systems in the same state with the i-th oscillator) it is necessary to estimate the probability of detecting asynchronous regime. In this case, the time-averaged probability of detecting a turbulent phase, calculated according to the formula, will act as a measure of multistability.

$$\mathrm{P} = \lim_{T \to \infty} \int_0^T P_a(t) dt \tag{5}$$

when the intensity of noise influence changes. This measure is equal to 0 when all interacting systems are in synchronous regime at each moment of time, and is equal to 1 when at each moment of time all systems exhibit non-identical behavior [13]. If $\mathrm{P} \in (0, 1)$ at a given value of the intensity of the noise effect, multistability takes place in the interacting systems [13,14].

In Fig. 3, 4 the dependences of the multistability measure on the intensity of noise influence obtained for ensembles of $N = 50$ Lorentz systems (1) and $N = 50$ logistic mapp (2) that are under the influence of noise. It can be seen from the figures that as the intensity of noise influence increases, the measure of multistability in both systems gradually decreases from 1 to 0 almost all the time.The exception is a small area near $\varepsilon = 0.01$ in Fig. 4, where the system dynamics change occurs., which corresponds to the transition from asynchronous state to noise-induced synchronization regime. It is important to note that near the boundary of the synchronous regime, the measure of multistability in both cases turns out to be positive, which indicates the presence of multistability in the intermittent synchronization induced by noise.

Thus, in this paper, using the example of Lorenz systems and logistic maps under the influence of a common noise source, it is shown that multistability is characteristic of the regime of intermittent synchronization induced by noise, as well as for the intermittent generalized synchronization regime. The obtained results are confirmed by constructing maps of the attraction basins of synchronous and asynchronous regimes and by calculating the measure of multistability depending on the magnitude of the noise intensity. It is important to note that despite the fact that the main results of the work were obtained on regimel systems, it can be expected that

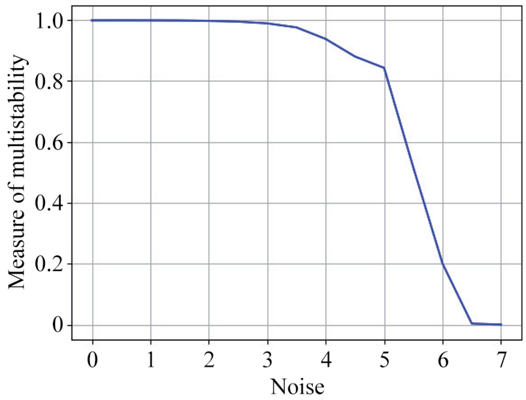

Fig 3. Dependence of the multistability measure P on the noise intensity ε obtained for the ensemble of Lorenz systems (1) being under influence of the common noise source

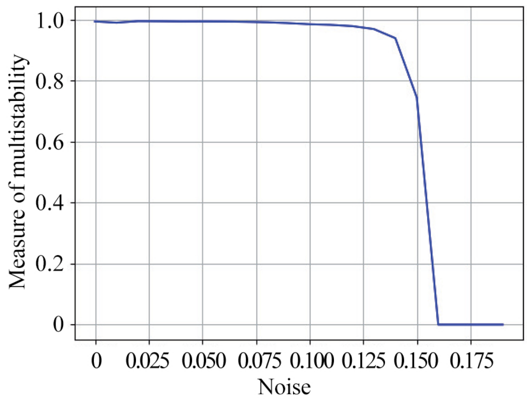

Fig 4. Dependence of the multistability measure P on the noise intensity ε obtained for the ensemble of logistic maps (2) being under influence of the common noise source

similar patterns will be observed in real radio engineering systems, which will allow using the discovered effect to improve the methods of secure information transmission, which are based on the phenomenon of chaotic synchronization.

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