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Stochastic stability of an autoresonance model with a center–saddle bifurcation[1]

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Abstract. The *purpose* of this work is to investigate the effect of stochastic perturbations of the white noise type on the stability of capture into autoresonance in oscillating systems with a variable pumping amplitude and frequency such that a center–saddle bifurcation occurs in the corresponding limiting autonomous system. The another purpose is determine the dependence of the intervals of stochastic stability of the autoresonance on the noise intensity. *Methods.* The existence of autoresonant regimes with increasing amplitude is proved by constructing and justificating asymptotic solutions in the form of power series with constant coefficients. The stability of solutions in terms of probability with respect to noise is substantiated using stochastic Lyapunov functions. *Results.* The conditions are described under which the autoresonant regime is preserved and disappears when the parameters pass through bifurcation values. The dependence of the intervals of stochastic stability of autoresonance on the degree of damping of the noise intensity is found. It is shown that more stringent restrictions are required to preserve the stability of solutions for the bifurcation values of the parameters. *Conclusion.* At the level of differential equations describing capture into autoresonance, the effect of damped stochastic perturbations on the center–saddle bifurcation is studied. The results obtained indicate the possibility of using damped oscillating perturbations for stable control of nonlinear systems.

Keywords: autoresonance, asymptotics, stability, bifurcation, stochastic perturbation.

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Introduction

The paper considers a model system of differential equations that describes autoresonance capture in nonlinear oscillating systems with small chirped pumping [1]. The phenomenon of autoresonance, associated with stable adjustment of the system phase to the pump phase and a significant increase of the oscillation amplitude, has a wide range of applications and has been actively studied recently [2]. The paper discusses a special case when the pump amplitude and frequency are matched in such a way that when the parameters vary, a center–saddle bifurcation occurs for resonant solutions in the corresponding limiting autonomous system. The influence of deterministic perturbations on such a bifurcation was discussed in [3], where the conditions under which the corresponding bifurcation is preserved or destroyed are described. In this case, the influence of stochastic disturbances was not considered. In this paper, we study the existence and stability of autoresonance with respect to stochastic disturbances when parameters pass through bifurcation values.

1. Formulation of the problem

We consider a non-autonomous system of two nonlinear differential equations

$$\begin{aligned} &\frac{d\rho}{d\tau} + \alpha(\tau)\rho = \beta(\tau)\sin\phi, \\ &\left(\frac{d\phi}{d\tau} - \rho^2 + \lambda(\tau)\right)\rho = \beta(\tau)\cos\phi \end{aligned} \tag{1}$$

with smooth functions $\alpha(\tau)$, $\beta(\tau)$ and $\lambda(\tau)$ defined for all $\tau > 0$ and having the following asymptotic behavior at infinity:

$$\alpha(\tau) \sim \tau^{-1}\sum_{k=0}^{\infty}\alpha_k\tau^{-k}, \quad \beta(\tau) \sim \tau^{b-1}\sum_{k=0}^{\infty}\beta_k\tau^{-k}, \quad \lambda(\tau) \sim \tau^{2b}\sum_{k=0}^{\infty}\lambda_k\tau^{-k}, \quad \tau\to\infty,$$

where α_k, β_k, $\lambda_k \in \mathbb{R}$, $\alpha_0, \lambda_0 \in \mathbb{R}_+$, $\beta_0 = 1$ and $2b \in \mathbb{Z}_+$. System (1) arises when studying the phenomenon of autoresonance in a wide class of nonlinear oscillating systems with small chirped pumping and weak dissipation [1]. The function $\alpha(\tau)$ is associated with dissipation in the system, $\beta(\tau)$ and $\lambda(\tau)$ — with the amplitude and frequency of the disturbance, respectively. The solutions of the system $\rho(\tau)$ and $\phi(\tau)$ play the role of the amplitude and phase detuning of the nonlinear oscillator. Of interest are the solutions $\rho(\tau)\to\infty$ and $\phi(\tau) = \mathcal{O}(1)$ as $\tau\to\infty$, which correspond to synchronization of the oscillator phase with the perturbation phase and capturing the system into autoresonance. In this case, solutions with $\rho(\tau) = \mathcal{O}(1)$ and $|\phi(\tau)|\to\infty$ as $\tau\to\infty$ correspond to the phenomenon of phase drift and the absence of autoresonance . As a simple example leading to system (1), consider

$$\frac{d^2x}{dt^2} + U'(x) = -A(t)\frac{dx}{dt} + \tilde{\varepsilon}B(t)\cos\Lambda(t), \tag{2}$$

where $A(t) \equiv A_0(1+\tilde{\varepsilon}t)^{-1}$, $B(t) \equiv B_0(1+\tilde{\varepsilon}t)^{b-1}$, $\Lambda(t) \equiv t - \vartheta t^{2b+1}$, $U(x) = x^2/2 - x^4/4 + \mathcal{O}(x^6)$ at $x\to 0$, $A_0, B_0, \vartheta, \tilde{\varepsilon} \in \mathbb{R}_+$. The right side of the equation represents a perturbation with small parameters $0 < A_0, \tilde{\varepsilon} \ll 1$. Note that the autonomous system corresponding to (2) with $\tilde{\varepsilon} = A_0 = 0$ has a Lyapunov-stable equilibrium $(0, 0)$ of center type (see, for example, [4, §4.1]). In this case, solutions to the perturbed equation with $\tilde{\varepsilon} \neq 0$, $A_0 \neq 0$ and initial data near the point

$(0,0)$, for which the energy $E(t) \equiv (x'(t))^2/2 + U(x(t))$ increases significantly with time, and the phase $\Psi(t) = \arctan(x'(t)/x(t))$ adjusts to the phase of the disturbance $\Psi(t) - \Lambda(t) = \mathcal{O}(1)$, correspond to the capture into autoresonance. For an asymptotic description of such solutions at the initial stage of the capture, we introduce slow and fast variables $\tau = \varepsilon B_0 t/(2\kappa)$ and $\zeta = \Lambda(t)$ with $\kappa = (4B_0/3)^{1/3}$. It is easy to check that the substitution $x(t) = \tilde\varepsilon^{1/3}\kappa\rho(\tau)\cos(\phi(\tau) - \zeta) + \mathcal{O}(\tilde\varepsilon)$ as $\tilde\varepsilon \to 0$ into equation (2) and averaging over the fast variable (see, for example, [5]) lead to system (1) with $\alpha(\tau) \equiv \kappa A(t) B_0^{-1}\tilde\varepsilon^{-2/3}$, $\beta(\tau) \equiv B(t)B_0^{-1}$ and $\lambda(\tau) \equiv \vartheta(1+2b)(2\kappa\tilde\varepsilon^{-2/3}B_0^{-1})^{2b+1}\tau^{2b}$. A similar transition to systems of type (1) takes place when studying autoresonance in infinite-dimensional systems described by nonlinear partial differential equations [1]. Note that systems of the form (1) arise, in particular, in problems of controlling the dynamics of domain walls in ferromagnetic films in a weak external magnetic field [6].

This paper studies the influence of stochastic disturbances on the stability of autoresonant solutions of system (1). We will consider the perturbed system in the form

$$\begin{aligned} &\frac{d\rho}{d\tau} + \alpha(\tau)\rho = [\beta(\tau) + \varepsilon\sigma_1(\tau)\xi_1(\tau)]\sin\phi, \\ &\left(\frac{d\phi}{d\tau} - \rho^2 + \lambda(\tau)\right)\rho = [\beta(\tau) + \varepsilon\sigma_1(\tau)\xi_1(\tau)]\cos\phi + \varepsilon\sigma_2(\tau)\xi_2(\tau)\rho, \end{aligned} \tag{3}$$

where $\xi_1(\tau)$ and $\xi_2(\tau)$ are independent stochastic processes defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. It is assumed that $\mathbb{E}[\xi_i(\tau)] = 0$ and $\mathbb{E}[\xi_i(\tau)\xi_i(\tau')] = \delta(\tau - \tau')$ for all $i \in \{1, 2\}$, where $\delta(\tau)$ is the Dirac δ function. Deterministic functions $\sigma_1(\tau)$ and $\sigma_2(\tau)$ with parameter $\varepsilon \in \mathbb{R}_+$ are used to control the noise intensity. Let us set $\xi_i(\tau) = \dot W_i(\tau)$, where $W_1(\tau), W_2(\tau)$ are independent Wiener processes. Then system (3) can be considered in the form Itô stochastic differential equations [7, Ch. 5]. The purpose of the work is to describe the conditions under which the capture into autoresonance persists in perturbed system with a probability close to unity.

2. Resonant solutions of an unperturbed system

Let us select the component in the amplitude that grows over time and make the substitution

$$\rho(\tau) = \sqrt{\lambda(\tau)} + \tau^{-\frac{1}{2}}R(s(\tau)), \quad \phi(\tau) = \Psi(s(\tau)), \quad s(\tau) = \frac{2}{q}\tau^{\frac{q}{2}}, \quad q = 2b+1 \tag{4}$$

in unperturbed system (1). Then for the new variables $R(s)$, $\Psi(s)$ the system takes the form

$$\frac{dR}{ds} = F(R, \Psi, s), \quad \frac{d\Psi}{ds} = G(R, \Psi, s), \tag{5}$$

where

$$F(R, \Psi, s(\tau)) \equiv \tau^{-\frac{q-3}{2}}\left(\beta(\tau)\sin\Psi - \alpha(\tau)\sqrt{\lambda(\tau)} - \frac{\lambda'(\tau)}{2\sqrt{\lambda(\tau)}}\right) + \tau^{-\frac{q-2}{2}}\left(\frac{1}{2}\tau^{-1} - \alpha(\tau)\right)R,$$

$$G(R, \Psi, s(\tau)) \equiv 2\tau^{-b}\sqrt{\lambda(\tau)}R + \tau^{-\frac{q}{2}}R^2 + \frac{\beta(\tau)\tau^{-\frac{q-2}{2}}\cos\Psi}{\sqrt{\lambda(\tau)} + \tau^{-\frac{1}{2}}R}.$$

Note that

$$\lambda^{\frac{m}{2}} \sim \tau^{bm}\lambda_0^{\frac{m}{2}}\sum_{k=0}^\infty \tau^{-k}\zeta_k^m, \quad \beta\lambda^{-\frac{1}{2}} \sim \lambda_0^{-\frac{1}{2}}\sum_{k=0}^\infty \tau^{-k}\gamma_k, \quad \tau^{-\frac{q-3}{2}}\left(\alpha\sqrt{\lambda} + \frac{\lambda'}{2\sqrt{\lambda}}\right) \sim \sum_{k=0}^\infty \tau^{-k}\mu_k$$

at $\tau \to \infty$, where ζ_k^m, γ_k, $\mu_k = \text{const}$. In particular, $\zeta_0^m = \gamma_0 = 1$, $\mu_0 = \sqrt{\lambda_0}(1+b) > 0$, $\zeta_1^m = m\lambda_1/\lambda_0$, $\gamma_1 = \beta_1 + \zeta_1^{-1}$, $\mu_1 = \sqrt{\lambda_0}\alpha_1 + \lambda_1/\sqrt{4\lambda_0}$. It is easy to verify that the asymptotic expansions take place $F(R,\Psi,s) \sim \sum_{k=0}^{\infty} s^{-k/q} F_k(R,\Psi)$, $G(R,\Psi,s) \sim \sum_{k=0}^{\infty} s^{-k/q} G_k(R,\Psi)$ as $s \to \infty$, where

$$F_k(R,\Psi) \equiv f_{k/2}(\Psi) + R\nu_{(k-q)/2}, \quad G_k(R,\Psi) \equiv R\eta_{k/2} + g_{k-q}(R,\Psi),$$

$$f_k(\Psi) \equiv (\beta_k \sin\Psi - \mu_k)\left(\frac{2}{q}\right)^{\frac{2k}{q}}, \quad \nu_k = \left(\frac{1}{2}\delta_{k,0} - \alpha_k\right)\left(\frac{2}{q}\right)^{\frac{2k}{q}-1},$$

$$g_k(R,\Psi) \equiv \delta_{k,0}\frac{2R^2}{q} + \sum_{ql+2(m+n)=k} (-1)^l R^l \cos\Psi \lambda_0^{-\frac{l+1}{2}} \zeta_m^{-l}\gamma_n \left(\frac{2}{q}\right)^{1+\frac{k}{q}}, \quad \eta_k = 2\sqrt{\lambda_0}\zeta_k^1 \left(\frac{2}{q}\right)^{\frac{2k}{q}}.$$

Note that $q \in \mathbb{Z}_+$ and $q \geqslant 2$. It is assumed that $f_k(\Psi) \equiv g_k(R,\Psi) \equiv 0$ and $\nu_k = \eta_k = 0$ if $k \notin \mathbb{N}_0$. Thus, system (5) is asymptotically autonomous [8]. The corresponding limit system

$$\frac{dR}{ds} = \sin\Psi - \mu_0, \quad \frac{d\Psi}{ds} = \sqrt{4\lambda_0}R \tag{6}$$

has two fixed points: $z_s = (0, \arcsin\mu_0)$ is a saddle and $z_c = (0, \pi - \arcsin\mu_0)$ is a center if $\mu_0 \in (0,1)$. At $\mu_0 = 1$ the saddle and the center merge into a degenerate fixed point $z_0 = (0,1)$, which disappears at $\mu_0 > 1$. If $\mu_0 > 1$, then all trajectories of the limit system turn out to be unbounded.

Note that the functions $\tilde{F}(R,\Psi,s) \equiv F(R,\Psi,s) - F_0(R,\Psi)$ and $\tilde{G}(R,\Psi,s) \equiv G(R,\Psi,s) - G_0(R,\Psi)$ play the role of damped disturbances of system (6). It is easy to verify that in the vicinity of the saddle such additions do not lead to a qualitative change in the behavior of trajectories (see, for example, [3]). In this case, the dynamics near the center and the degenerate point depend on the disturbance parameters.

Let us give a definition of the stability of resonant solutions with increasing amplitude, which will be used below.

Definition 1. *A solution $\rho_*(\tau)$, $\psi_*(\tau)$ of system (1) is called stable if $\forall\, \varepsilon > 0$ there exist $\delta_0 > 0$ and $\tau_0 > 0$ such that for any ϱ_0 and φ_0: $|\rho_*(\tau_0) - \varrho_0| + |\psi_*(\tau_0) - \varphi_0| \leqslant \delta_0$, for solving $\rho(\tau)$, $\psi(\tau)$ system (1) with initial data $\rho(\tau_0) = \varrho_0$, $\psi(\tau_0) = \varphi_0$ the inequality holds*

$$\sup_{\tau \geqslant \tau_0}\left\{\tau^{\frac{1}{2}}|\rho(\tau) - \rho_*(\tau)| + |\psi(\tau) - \psi_*(\tau)|\right\} \leqslant \varepsilon.$$

Let us first consider the behavior of trajectories near the point z_c. Fair

Theorem 1. *Let $0 < \sqrt{\lambda_0}(1+b) < 1$ and $\alpha_0 > \dfrac{1}{2} - \dfrac{4(b+1)}{(2b+1)^2}$. Then system (1) has a stable solution $\rho_c(\tau) \equiv \sqrt{\lambda(\tau)} + \tau^{-1/2}R_c(s(\tau))$, $\phi_c(\tau) \equiv \Psi_c(s(\tau))$, where $s(\tau) = (2/q)\tau^{q/2}$,*

$$R_c(s) \sim \sum_{k=1}^{\infty} s^{-\frac{k}{q}} r_k, \quad \Psi_c(s) \sim \psi_0 + \sum_{k=1}^{\infty} s^{-\frac{k}{q}}\psi_k, \quad s \to \infty, \tag{7}$$

with coefficients $r_k, \psi_k = \text{const}$, $\psi_0 = \pi - \arcsin\mu_0$, $q = 2b+1$.

Proof. Substituting (7) into system (5) and grouping expressions with the same powers of s lead to a system of recurrent equations:

$$\sqrt{4\lambda_0}r_k = \mathcal{A}_k, \quad -\sqrt{1-\mu_0^2}\psi_k = \mathcal{B}_k, \quad k \geqslant 1, \tag{8}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ are expressed in terms of $r_1, \psi_1, \dots, r_{k-1}, \psi_{k-1}$. In particular,

$$\mathcal{A}_1 = -G_1(0,\psi_0), \quad \mathcal{A}_2 = -G_2(0,\psi_0) - r_1\partial_R G_1(0,\psi_0) - \psi_1\partial_\Psi G_1(0,\psi_0),$$

$$\mathcal{B}_1 = -F_1(0,\psi_0), \quad \mathcal{B}_2 = \mu_0\psi_1^2/2 - F_2(0,\psi_0) - r_1\partial_R F_1(0,\psi_0) - \psi_1\partial_\Psi F_1(0,\psi_0).$$

Since $\lambda_0 \neq 0$ and $0 < \mu_0 < 1$, then system (8) is solvable. To prove the existence of a solution to system (7), we define the functions $R_N(s) \equiv \sum_{k=1}^N s^{-k/q} r_k$, $\Psi_N(s) \equiv \psi_0 + \sum_{k=1}^N s^{-k/q}\psi_k$ with some integer $N \in \mathbb{N}$. It follows from the construction that

$$R_N'(s) - F(R_N(s), \Psi_N(s), s) = \mathcal{O}(s^{-\frac{N+1}{q}}), \quad \Psi_N'(s) - G(R_N(s), \Psi_N(s), s) = \mathcal{O}(s^{-\frac{N+1}{q}})$$

as $s \to \infty$. Substitution $R(s) = R_N(s) + r(s)$, $\Psi(s) = \Psi_N(s) + \psi(s)$ in (5) leads to the system

$$\frac{dr}{ds} = \mathcal{F}_N(r,\psi,s), \quad \frac{d\psi}{ds} = \mathcal{G}_N(r,\psi,s), \tag{9}$$

where $\mathcal{F}_N(r,\psi,s) \equiv F(R_N(s)+r, \Psi_N(s)+\psi, s) - R_N'(s)$ and $\mathcal{G}_N(r,\psi,s) \equiv G(R_N(s)+r, \Psi_N(s)+ + \psi, s) - \Psi_N'(s)$. It is easy to check that

$$\mathcal{F}_N = \sum_{k=0}^{q} s^{-\frac{k}{q}} \left\{ f_{k/2}(\psi + \Psi_N) - f_{k/2}(\Psi_N) + \delta_{k,q}\nu_0 r \right\} + \mathcal{O}(d)\mathcal{O}(s^{-\frac{q+1}{q}}) + \mathcal{O}(s^{-\frac{N+1}{q}}),$$

$$\mathcal{G}_N = \sum_{k=0}^{q} s^{-\frac{k}{q}} \left\{ r\eta_{k/2} + \delta_{k,q}\big(g_0(r + R_N, \psi + \Psi_N) - g_0(R_N, \Psi_N)\big) \right\} + \mathcal{O}(d)\mathcal{O}(s^{-\frac{q+1}{q}}) + \mathcal{O}(s^{-\frac{N+1}{q}})$$

as $s \to \infty$ and $d := d(r,\psi) \equiv \sqrt{r^2 + \psi^2} \to 0$. As a Lyapunov function, consider $V(r,\psi,s) \equiv V_c(r,\psi,s;\Psi_N(s),\vartheta)$, where

$$V_c(r,\psi,s;\Psi_N,\vartheta) = \sum_{k=0}^{q} s^{-\frac{k}{q}} \left\{ \eta_{k/q}\frac{r^2}{2} - \int_0^\psi f_{k/2}(\phi + \Psi_N)\,d\phi + \psi f_{k/2}(\Psi_N) \right\} + s^{-1}\vartheta r\psi.$$

Note that $V(r,\psi,s) = (\eta_0 r^2 + \sqrt{1-\mu_0^2}\psi^2)/2 + \mathcal{O}(d^3) + \mathcal{O}(d^2)\mathcal{O}(s^{-1/q})$ as $s \to \infty$ and $d \to 0$, where $\eta_0 = \sqrt{4\lambda_0} > 0$. The derivative of the function $V(r,\psi,s)$ on the trajectories of system (9) has the following form:

$$\frac{dV}{ds}\Big|_{(9)} = s^{-1}\left(-A_\vartheta r^2 - B_\vartheta\psi^2 + \mathcal{O}(d^3) + \mathcal{O}(s^{-\frac{1}{q}})\mathcal{O}(d^2)\right) + \mathcal{O}(d)\mathcal{O}(s^{-\frac{N+1}{q}})$$

as $s \to \infty$ and $d \to 0$ with parameters $A_\vartheta = \eta_0(q(2\alpha_0 - 1)/4 - \vartheta)$ и $B_\vartheta = \sqrt{1-\mu_0^2}(\vartheta + 2\mu_0/\sqrt{q^2\lambda_0})$. Let us choose the parameter $\vartheta = \vartheta_c$ that satisfies the inequalities $-2\mu_0/\sqrt{q^2\lambda_0} < \vartheta_c < q(2\alpha_0 - 1)/4$, then $A_\vartheta > 0$ and $B_\vartheta > 0$. Consequently, there are $d_1 > 0$ and $s_1 > 0$ such that

$$m_- d^2 \leqslant V(r,\psi,s) \leqslant m_+ d^2, \quad \frac{dV}{ds}\Big|_{(9)} \leqslant -s^{-1}Cd^2 + s^{-\frac{N+1}{q}}Dd \tag{10}$$

as $s \geqslant s_1$ and $d \leqslant d_1$ with positive parameters m_-, m_+, $C = \min\{A_\vartheta, B_\vartheta\}/2$ and D . Let us choose $N \geqslant q$, then for any $\varepsilon \in (0, d_1)$ there are

$$\delta_\varepsilon = \min\left\{d_1, \frac{2s_\varepsilon^{-1/q}}{C}, \varepsilon\sqrt{\frac{m_-}{2m_+}}\right\}, \quad s_\varepsilon = \max\left\{s_1, \left(\frac{4}{C\varepsilon}\right)^q\right\}$$

such that $dV/ds|_{(9)} = -s^{-1}(Cd^2 - s_\varepsilon^{-1/q}\delta_\varepsilon^{-1}Dd^2) \leqslant -s^{-1}Cd^2/2 < 0$ for all $s \geqslant s_\varepsilon$ and (r,ψ) such that $\delta_\varepsilon \leqslant d(r,\psi) \leqslant \varepsilon$. Hence and from the inequalities $\sup_{d\leqslant\delta_\varepsilon} V(r,\psi,s) \leqslant m_+\delta_\varepsilon^2 \leqslant m_-\varepsilon^2 = \inf_{d=\varepsilon} V(r,\psi,s)$ for all $s \geqslant s_\varepsilon$ it follows that any solution of system (9) with initial data $d(r(s_\varepsilon),\psi(s_\varepsilon)) \leqslant \delta_\varepsilon$ does not leaves the ε-neighborhood of zero $d(r(s),\psi(s)) \leqslant \varepsilon$ as $s \geqslant s_\varepsilon$. Moreover, from (10) it follows that $dV/ds|_{(9)} \leqslant s^{-1-(N+1-q)/q}\varepsilon D$ for all $s \geqslant s_\varepsilon$ and $d \leqslant \varepsilon$. Integrating the last inequality, we get $d(r(s),\psi(s)) = \mathcal{O}(s^{-(N+1-q)/(2q)})$ as $s \to \infty$ for any $N \geqslant q$. This implies the existence of a stable solution to system (9) with asymptotics (7). Taking into account the substitution (4), we obtain the proof of theorem 1. □

If $\mu_0 = 1$, system (8) turns out to be unsolvable, and an asymptotic solution in the form (7) near the degenerate point z_0 cannot be constructed. In this case, depending on the disturbance parameters, the appearance of either a stable regime with trajectories tending to equilibrium of limit system (6) or an unstable regime with infinitely growing trajectories is possible.

We have

Theorem 2. *Let* $\sqrt{\lambda_0}(b+1) = 1$, $\beta_1 > \mu_1$ *and* $\alpha_0 > \dfrac{1}{2} - \dfrac{4b-1}{(2b+1)^2}$. *Then system* (1) *has a stable solution* $\rho_0(\tau) \equiv \sqrt{\lambda(\tau)} + \tau^{-1/2}R_0(s(\tau))$, $\phi_0(\tau) \equiv \Psi_0(s(\tau))$, *where* $s(\tau) = (2/q)\tau^{q/2}$,

$$R_0(s) \sim \sum_{k=1}^{\infty} s^{-\frac{k}{q}} r_k, \quad \Psi_0(s) \sim \frac{\pi}{2} + \sum_{k=1}^{\infty} s^{-\frac{k}{q}}\psi_k, \quad s \to \infty, \tag{11}$$

with coefficients $r_k, \psi_k = \mathrm{const}$, $\psi_1 = \sqrt{2(\beta_1 - \mu_1)}(2/q)^{1/q}$, $q = 2b+1$.

Proof. Substituting (11) into system (5) and equating the expressions for the same powers of s lead to the equation

$$\frac{\psi_1^2}{2} = (\beta_1 - \mu_1)\left(\frac{2}{q}\right)^{\frac{2}{q}}. \tag{12}$$

The remaining coefficients r_k, ψ_k are determined from the system of equations

$$\sqrt{4\lambda_0}r_k = \mathcal{A}_k, \quad -\psi_1\psi_{k+1} = \mathcal{C}_{k+1}, \quad k \geqslant 1, \tag{13}$$

where $\mathcal{A}_k$ and $\mathcal{C}_k$ are expressed in terms of $r_1, \psi_1, \ldots, r_{k-1}, \psi_{k-1}$. For example,

$$\mathcal{A}_1 = -G_1\left(0,\frac{\pi}{2}\right), \quad \mathcal{A}_2 = -G_2\left(0,\frac{\pi}{2}\right) - r_1\partial_R G_1\left(0,\frac{\pi}{2}\right) - \psi_1\partial_\Psi G_1\left(0,\frac{\pi}{2}\right),$$
$$\mathcal{C}_2 = -\delta_{q,2}\nu_0 r_1, \quad \mathcal{C}_3 = -\frac{\psi_1^4}{24} + \frac{\psi_2^2}{2} + \beta_1\frac{\psi_1^2}{2}\left(\frac{2}{q}\right)^{\frac{2}{q}} - \delta_{q,3}\nu_0 r_1 + (\mu_2 - \beta_2)\left(\frac{2}{q}\right)^{\frac{4}{q}}.$$

Since $\lambda_0 \neq 0$ and $\beta_1 > \mu_1$, then there is a solution to system (12), (13), which depends on the choice of the root of equation (12).

Consider the functions $R_N(s) \equiv \sum_{k=1}^N s^{-k/q} r_k$, $\Psi_N(s) \equiv \pi/2 + \sum_{k=1}^N s^{-k/q}\psi_k$. Substitution $R(s) = R_N(s) + s^{-3/(2q)}r(s)$, $\Psi(s) = \Psi_N(s) + s^{-1/q}\psi(s)$ in (5) leads to system (9) with

$$\mathcal{F}_N(r,\psi,s) \equiv s^{\frac{3}{2q}}\left(F(R_N(s) + s^{-\frac{3}{2q}}r, \Psi_N(s) + s^{-\frac{1}{q}}\psi, s) - R_N'(s)\right) + s^{-1}\frac{3r}{2q},$$
$$\mathcal{G}_N(r,\psi,s) \equiv s^{\frac{1}{q}}\left(G(R_N(s) + s^{-\frac{3}{2q}}r, \Psi_N(s) + s^{-\frac{1}{q}}\psi, s) - \Psi_N'(s)\right) + s^{-1}\frac{\psi}{q}.$$

It is easy to check that

$$\mathcal{F}_N = \sum_{k=0}^{q+2} s^{-\frac{2k-3}{2q}} \left\{ f_{k/2}(s^{-\frac{1}{q}}\psi + \Psi_N) - f_{k/2}(\Psi_N) \right\} + s^{-1}\left(\nu_0 + \frac{3}{2q}\right) r + \\ + \mathcal{O}(d)\mathcal{O}(s^{-\frac{q+1}{q}}) + \mathcal{O}(s^{-\frac{2N-1}{2q}}) = s^{-\frac{1}{2q}}(-\psi_1\psi + \mathcal{O}(d^2) + \mathcal{O}(s^{-\frac{1}{q}})),$$

$$\mathcal{G}_N = \sum_{k=0}^{q} s^{-\frac{2k+1}{2q}} r\eta_{k/2} + s^{-1+\frac{1}{q}} \left(\left(g_0(s^{-\frac{3}{2q}} r + R_N, s^{-\frac{1}{q}}\psi + \Psi_N) - g_0(R_N, \Psi_N) \right) \right) + s^{-1}\frac{\psi}{q} + \\ + \mathcal{O}(d)\mathcal{O}(s^{-\frac{q+1}{q}}) + \mathcal{O}(s^{-\frac{N}{q}}) = s^{-\frac{1}{2q}}(\eta_0 r + \mathcal{O}(s^{-\frac{1}{q}}))$$

as $s \to \infty$ and $d(r,\psi) \to 0$.

Let $\psi_1 > 0$. In this case, the limit system corresponding to (9) has a fixed point of the center type. Consider the Lyapunov function for system (9) in the form $V(r,\psi,s) \equiv V_0(r,\psi,s;\Psi_N(s),\vartheta)$, where ϑ — some parameter and

$$V_0(r,\psi,s,\Psi_N,\vartheta) \equiv \sum_{k=0}^{q} s^{-\frac{k}{q}} \eta_{k/q} \frac{r^2}{2} - \sum_{k=0}^{q+2} s^{-\frac{k-2}{q}} \left\{ \int\limits_0^{\psi} f_{k/2}(s^{-\frac{1}{q}}\phi + \Psi_N)\, d\phi - \psi f_{k/2}(\Psi_N) \right\} + \\ + s^{-\frac{2q-1}{2q}} \vartheta r \psi.$$

notice, that $V(r,\psi,s) = (\eta_0 r^2 + \psi_1\psi^2)/2 + \mathcal{O}(d^3) + \mathcal{O}(d^2)\mathcal{O}(s^{-1/q})$ as $s \to \infty$ and $d \to 0$, where $\eta_0 = \sqrt{4\lambda_0} > 0$. The derivative of the function $V(r,\psi,s)$ on the trajectories of system (9) has the form

$$\left.\frac{dV}{ds}\right|_{(9)} = s^{-1}\left(-A_\vartheta r^2 - B_\vartheta \psi^2 + \mathcal{O}(d^3) + \mathcal{O}(s^{-\frac{1}{q}})\mathcal{O}(d^2)\right) + \mathcal{O}(d)\mathcal{O}(s^{-\frac{2N-1}{2q}})$$

as $s \to \infty$ and $d \to 0$ with parameters $A_\vartheta = \eta_0(q(2\alpha_0 - 1)/4 - 3/(2q) - \vartheta)$, $B_\vartheta = \psi_1(2\mu_0/\sqrt{q^2\lambda_0} - q^{-1} + \vartheta)$. Let us choose $\vartheta = \vartheta_0$ satisfying the inequalities $(\sqrt{\lambda_0} - 2)/\sqrt{q^2\lambda_0} < \vartheta_0 < (q^2(2\alpha_0 - 1) - 6)/(4q)$, then $A_\vartheta > 0$ and $B_\vartheta > 0$. Consequently, there are $d_1 > 0$ and $s_1 > 0$ such that $m_- d^2 \leqslant V(r,\psi,s) \leqslant m_+ d^2$, $dV/ds|_{(9)} \leqslant -s^{-1}Cd^2 + s^{-2N-1/(2q)}Dd$ as $s \geqslant s_1$ and $d \leqslant d_1$ with positive parameters m_-, m_+, $C = \min\{A_\vartheta, B_\vartheta\}/2$ and D . Let us choose $N \geqslant q+1$, then, repeating the reasoning of theorem 1, we obtain a proof of theorem 2. □

Note that the choice of a negative root of the equation (12) corresponds to a saddle-type fixed point in the limit system. In this case, damped disturbances do not significantly affect the behavior of nearby trajectories and the asymptotic regime corresponding to (11) with $\psi_1 < 0$ turns out to be unstable.

If $\mu_0 = 1$ and $\beta_1 < \mu_1$, then an asymptotic solution in the form (11) is not constructed. Moreover, in this case the trajectories of system (5) behave in the same way as the solutions of limiting system (6) for $\mu_0 > 1$ and are unbounded [3].

3. Stochastic stability of resonant solutions

This section discusses the stability of autoresonant solutions of system (1) with respect to stochastic disturbances for $0 < \mu_0 < 1$ and for $\mu_0 = 1$. It is known that even small stochastic disturbances can lead to a loss of stability of solutions [9, Ch. 10] and the emergence of new stable states [10]. Let us describe the conditions under which the stability of autoresonance is guaranteed to be preserved in probability at least over asymptotically large time intervals.

Note that the substitution (4) reduces system (3) to the following form (see [7, §8.5]):

$$
\begin{aligned}
&dR = F(R,\Psi,s)\,ds + \varepsilon\sigma_{1,1}(R,\Psi,s)\,dw_1(s),\\
&d\Psi = G(R,\Psi,s)\,ds + \varepsilon\sigma_{2,1}(R,\Psi,s)\,dw_1(s) + \varepsilon\sigma_{2,2}(R,\Psi,s)\,dw_2(s),
\end{aligned} \tag{14}
$$

where $(w_1(s), w_2(s))$ is some two-dimensional Wiener process,

$$
\sigma_{1,1}(R,\Psi,s(\tau)) \equiv \tau^{\frac{4-q}{4}}\sigma_1(\tau)\sin\Psi, \quad \sigma_{2,1}(R,\Psi,s(\tau)) \equiv \frac{\tau^{\frac{2-q}{4}}\sigma_1(\tau)\cos\Psi}{\sqrt{\lambda(\tau)}+\tau^{-1/2}R},
$$

$$
\sigma_{2,2}(R,\Psi,s(\tau)) \equiv \tau^{\frac{2-q}{4}}\sigma_2(\tau).
$$

From Theorem 1 it follows that system (14) for $\varepsilon = 0$ and $0 < \mu_0 < 1$ has a stable solution $R_c(s)$, $\Psi_c(s)$ with asymptotics (7). Let us show that the solution remains stable with respect to stochastic perturbations at $\varepsilon \neq 0$ under certain restrictions on the class of perturbations $\mathcal{K}_{a_1,a_2} := \{(\sigma_1(\tau),\sigma_2(\tau)) : \sigma_1(\tau) = \mathcal{O}(\tau^{a_1}), \sigma_2(\tau) = \mathcal{O}(\tau^{a_2}) \text{ at } \tau\to\infty\}$. Let us define the function $d(r,\psi) \equiv \sqrt{r^2+\psi^2}$. Then, we have

Theorem 3. *Let* $0 < \sqrt{\lambda_0}(1+b) < 1$, $\alpha_0 > \dfrac{1}{2} - \dfrac{4(b+1)}{(2b+1)^2}$ *and* $(\sigma_1(\tau),\sigma_2(\tau)) \in \mathcal{K}_{a_1,a_2}$ *with parameters* $a_1 \leqslant -1 + (2b+1)K/4$, $a_2 \leqslant -1/2 + (2b+1)K/4$, $K \leqslant 1$. *Then there exists* $\tau_0 > 0$ *such that for any* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *there are* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that any solution* $\rho(\tau)$, $\psi(\tau)$ *systems* (3) *with* $d(\rho(\tau_0)-\rho_c(\tau_0), \psi(\tau_0)-\psi_c(\tau_0)) \leqslant \delta_1$ *and* $0 < \varepsilon < \delta_2$ *satisfies the estimate*

$$
\mathbb{P}\left(\sup_{0\leqslant\tau-\tau_0\leqslant\mathcal{T}} d\left(\tau^{\frac{1}{2}}(\rho(\tau)-\rho_c(\tau)), \psi(\tau)-\psi_c(\tau)\right) \geqslant \varepsilon_2\right) \leqslant \varepsilon_1 \tag{15}
$$

with parameter $\mathcal{T} = \varepsilon^{-1}$ *for* $0 < K \leqslant 1$, $\mathcal{T} = s(\tau_0)(\exp\varepsilon^{-1} - 1)$ *for* $K = 0$ *and* $\mathcal{T} = \infty$ *for* $K < 0$.

Proof. Substitution $R(s) = R_c(s) + r(s)$, $\Psi(s) = \Psi_c(s) + \psi(s)$ in (14) leads to the system

$$
\begin{aligned}
&dr = \mathcal{F}(r,\psi,s)\,ds + \varepsilon\tilde\sigma_{1,1}(r,\psi,s)\,dw_1(s),\\
&d\psi = \mathcal{G}(r,\psi,s)\,ds + \varepsilon\tilde\sigma_{2,1}(r,\psi,s)\,dw_1(s) + \varepsilon\tilde\sigma_{2,2}(r,\psi,s)\,dw_2(s),
\end{aligned} \tag{16}
$$

where $\mathcal{F}(r,\psi,s) \equiv F(R_c(s)+r, \Psi_c(s)+\psi, s) - F(R_c(s),\Psi_c(s),s)$, $\mathcal{G}(r,\psi,s) \equiv G(R_c(s)+r,\Psi_c(s)+\psi,s) - G(R_c(s),\Psi_c(s),s)$, $\tilde\sigma_{i,j}(r,\psi,s) \equiv \sigma_{i,j}(R_c(s)+r,\Psi_c(s)+\psi,s)$ and $(R_c(s),\Psi_c(s))$ is the solution of system (5) with asymptotic behavior (7). It is easy to check that

$$
\begin{aligned}
&\mathcal{F} = \sum_{k=0}^{q} s^{-\frac{k}{q}}\left\{f_{k/2}(\psi+\Psi_c) - f_{k/2}(\Psi_c) + \delta_{k,q}\nu_0 r\right\} + \mathcal{O}(d)\mathcal{O}(s^{-1-\frac{1}{q}}),\\
&\mathcal{G} = \sum_{k=0}^{q} s^{-\frac{k}{q}}\left\{r\eta_{k/2} + \delta_{k,q}\big(g_0(r+R_c,\psi+\Psi_c) - g_0(R_c,\Psi_c)\big)\right\} + \mathcal{O}(d)\mathcal{O}(s^{-1-\frac{1}{q}}),\\
&\tilde\sigma_{i,j} = \mathcal{O}(s^{-1+K})
\end{aligned}
$$

as $s\to\infty$ and $d\to 0$.

Let's define the operator

$$
\mathcal{L} = \partial_s + \mathcal{F}\partial_r + \mathcal{G}\partial_\psi + \frac{\varepsilon^2}{2}\left(\tilde\sigma_{1,1}^2\partial_r^2 + 2\tilde\sigma_{1,1}\tilde\sigma_{2,1}\partial_r\partial_\psi + (\tilde\sigma_{2,1}^2+\tilde\sigma_{2,2}^2)\partial_\psi^2\right),
$$

associated with (16) and playing a key role in the study of stochastic stability (see [11, §3.6]). Consider the auxiliary function $V(r,\psi,s)\equiv V_c(r,\psi,s;\Psi_c(s),\vartheta_c)$ with the parameter ϑ_c defined in the proof of the theorem 1. Note that there are $d_0>0$ and $s_0>0$ such that

$$\begin{aligned} &m_- d^2(r,\psi)\leqslant V(r,\psi,s)\leqslant m_+ d^2(r,\psi),\\ &\mathcal{L}V(r,\psi,s)\leqslant -s^{-1}Cd^2(r,\psi)+s^{-1+K}\varepsilon^2 M \end{aligned} \tag{17}$$

as $s\geqslant s_0$ and $d(r,\psi)\leqslant d_0$ with positive constants m_-, m_+, C and M. Then the Lyapunov function for system (16) can be taken in the following form [12, 13]: $U(r,\psi,s)\equiv V(r,\psi,s)+$ $+\varepsilon^2 M\theta_K(s)$ c

$$\theta_K(s)=\begin{cases} s_0^{-1+K}(\mathcal{T}+s_0-s), & 0<K\leqslant 1,\\ \log(\mathcal{T}+s_0)-\log s, & K=0,\\ \int\limits_s^{\mathcal{T}+s_0}\varsigma^{K-1}\,d\varsigma, & K<0.\end{cases}$$

Note that

$$U(r,\psi,s)\geqslant m_- d^2(r,\psi),\quad \mathcal{L}U(r,\psi,s)\leqslant 0 \tag{18}$$

for all $(r,\psi,s)\in\mathcal{D}(d_0,s_0,\mathcal{T}):=\{(r,\psi,s): d\leqslant d_0, 0\leqslant s-s_0\ leq\mathcal{T}\}$. Let us fix the parameters $\varepsilon_1\in(0,d_0)$ and $\varepsilon_2>0$. Let $(r(s),\psi(s))$ be a solution to system (16) for $d(r(s_0),\psi(s_0))\leqslant\delta_1$ and $0<\varepsilon<\delta_2$. Let us denote by $s_{\mathcal{D}}$ the moment of the first exit of trajectories from the region $\mathcal{D}(\delta_1,s_0,\mathcal{T})$, and set $\varsigma_s\equiv\min\{s_{\mathcal{D}},s\}$. Then $(r(\varsigma_s),\psi(\varsigma_s),\varsigma_s)$ is a process stopped at the moment of the first exit from the region $\mathcal{D}(\delta_1,s_0,\mathcal{T})$. Moreover, from (18) it follows that $U(r(\varsigma_s),\psi(\varsigma_s),\varsigma_s)$ is a non-negative supermartingale [11, §5.2], and the estimates hold

$$\begin{aligned}\mathbb{P}(\sup_{0\leqslant s-s_0\leqslant\mathcal{T}} d(r(s),\psi(s))>\varepsilon_1)&=\mathbb{P}(\sup_{s\geqslant s_0} d(r(\varsigma_s),\psi(\varsigma_s))>\varepsilon_1)\\ &\leqslant\mathbb{P}(\sup_{s\geqslant s_0} U(r(\varsigma_s),\psi(\varsigma_s),\varsigma_s)>m_-\varepsilon_1^2)\leqslant\frac{U(r(s_0),\psi(s_0),s_0)}{m_-\varepsilon_1^2}.\end{aligned} \tag{19}$$

The last estimate follows from Doob's inequality for supermartingales. Note that $U(r(s_0),\psi(s_0),s_0)\leqslant$ $m_+\delta_1^2+\varepsilon^2 M\theta_K(s_0)$. Let us choose $\delta_1=\varepsilon_2\sqrt{\varepsilon_2 m_-/(2m_+)}$ and

$$\delta_2=\begin{cases} m_+\delta_1^2 M^{-1}s_0^{1-K}, & 0<K\leqslant 1,\\ m_+\delta_1^2 M^{-1}, & K=1,\\ m_+\delta_1^2 M^{-1}|K|s_0^{-K}, & K<0.\end{cases}$$

From here, from (4) and (19) the estimate (15) follows. □

Theorem 4. *Let $\sqrt{\lambda_0}(1+b)=1$, $\beta_1>\mu_1$, $\alpha_0>\dfrac{1}{2}-\dfrac{4b-1}{(2b+1)^2}$ and $(\sigma_1(\tau),\sigma_2(\tau))\in\mathcal{K}_{a_1,a_2}$ with parameters $a_1\leqslant -7/4+(2b+1)K/4$, $a_2\leqslant -1+(2b+1)K/4$, $K\leqslant 1$. Then there exists $\tau_0>0$ such that for any $\varepsilon_1>0$ and $\varepsilon_2>0$ there are $\delta_1>0$ and $\delta_2>0$ such that any solution $\rho(\tau)$, $\psi(\tau)$ of system (3) with $d(\rho(\tau_0)-\rho_0(\tau_0),\psi(\tau_0)-\psi_0(\tau_0))\leqslant\delta_1$ and $0<\varepsilon<\delta_2$ satisfies the estimate*

$$\mathbb{P}\left(\sup_{0\leqslant\tau-\tau_0\leqslant\mathcal{T}} d\left(\tau^{\frac{5}{4}}(\rho(\tau)-\rho_0(\tau)),\tau^{\frac{1}{2}}(\psi(\tau)-\psi_0(\tau))\right)\geqslant\varepsilon_2\right)\leqslant\varepsilon_1$$

with parameter $\mathcal{T} = \varepsilon^{-1}$ *for* $0 < K \leqslant 1$, $\mathcal{T} = s(\tau_0)(\exp\varepsilon^{-1} - 1)$ *for* $K = 0$ *and* $\mathcal{T} = \infty$ *for* $K < 0$.

Proof. Substitution $R(s) = R_0(s) + s^{-3/(2q)}r(s)$, $\Psi(s) = \Psi_0(s) + s^{-1/q}\psi(s)$ in (14) for $\mu_0 = 1$ leads to system (16) with

$$\mathcal{F}(r,\psi,s) \equiv s^{\frac{3}{2q}}\left(F\left(R_0(s) + s^{-\frac{3}{2q}}r, \Psi_0(s) + s^{-\frac{1}{q}}\psi, s\right) - F(R_0(s),\Psi_0(s),s)\right) + s^{-1}\frac{3r}{2q},$$

$$\mathcal{G}(r,\psi,s) \equiv s^{\frac{1}{q}}\left(G\left(R_0(s) + s^{-\frac{3}{2q}}r, \Psi_0(s) + s^{-\frac{1}{q}}\psi, s\right) - G(R_0(s),\Psi_0(s),s)\right) + s^{-1}\frac{\psi}{q},$$

$$\tilde\sigma_{1,1}(r,\psi,s) \equiv s^{\frac{3}{2q}}\sigma_{1,1}\left(R_0(s) + s^{-\frac{3}{2q}}r, \Psi_0(s) + s^{-\frac{1}{q}}\psi, s\right),$$

$$\tilde\sigma_{2,j}(r,\psi,s) \equiv s^{\frac{1}{q}}\sigma_{2,j}\left(R_0(s) + s^{-\frac{3}{2q}}r, \Psi_0(s) + s^{-\frac{1}{q}}\psi, s\right),$$

where $R_0(s)$, $\Psi_0(s)$ are the solution of system (5) for $\mu_0 = 1$ with asymptotics (11). It is easy to check that

$$\begin{aligned}
\mathcal{F} &= \sum_{k=0}^{q+2} s^{-\frac{2k-3}{2q}}\left\{f_{k/2}(\Psi_0 + s^{-\frac{1}{q}}\psi) - f_{k/2}(\Psi_0)\right\} + s^{-1}\left(\nu_0 + \frac{3}{2q}\right)r + \mathcal{O}(d)\mathcal{O}(s^{-\frac{q+1}{q}}),\\
\mathcal{G} &= \sum_{k=0}^{q} s^{-\frac{2k+1}{2q}} r\eta_{k/2} + s^{-\frac{q-1}{q}}\left(\left(g_0(R_0 + s^{-\frac{3}{2q}}r, \Psi_0 + s^{-\frac{1}{q}}\psi) - g_0(R_0,\Psi_0)\right)\right) + \\
&\qquad + s^{-1}\frac{\psi}{q} + \mathcal{O}(d)\mathcal{O}(s^{-\frac{q+1}{q}}),\\
\tilde\sigma_{i,j} &= \mathcal{O}(s^{-1+K})
\end{aligned}$$

as $s \to \infty$ and $d(r,\psi) \to 0$. Note that the function $V(r,\psi,s) \equiv V_0(r,\psi,s;\Psi_0(s),\vartheta_0)$ with the parameter ϑ_0 defined in the theorem 2, satisfies (17). In this case, the construction of the Lyapunov function for stochastic system (16) and further justification are carried out in the same way as in the proof of the theorem 3. □

Conclusion

Thus, the conditions are described under which the autoresonant regime is preserved and disappears when the pump parameters pass through bifurcation values in the corresponding limit system. The influence of damped stochastic disturbances is studied and the dependence of the intervals of stochastic stability of autoresonance on the degree of noise intensity attenuation is found. It is shown that to maintain the stability of solutions at appropriate bifurcation values of the parameters, more stringent restrictions are required.

The results obtained expand the possibility of using the autoresonance phenomenon for stable control of nonlinear dynamics. The possibility of a significant change in the energy of oscillating systems using a small chirped disturbance in the presence of weak dissipation and noise has been proven. In particular, it is shown that stochastic perturbations do not destroy capture into autoresonance when the pump parameters pass through bifurcation values.

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