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Relay model of a fading neuron[1]

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Abstract. This study is a continuation of M. M. Preobrazhenskaya's work "Relay System of Differential Equations with Delay as a Perceptron Model", which aimed to combine approaches related to artificial neural networks and modeling biological neurons using differential equations with delay. The model of a single neuron was proposed, which allows for the existence of special modes called "aging" and "dying" behavior of the neuron. The study found a certain range of parameters where the "dying" mode of the neuron exists and numerically demonstrated the existence of the "aging" mode. *Purpose.* We will unify the concepts of "aging" and "dying" neurons into the term "freezing" neuron. For this neuron, we will analytically construct a solution and find the range of parameters for its existence and stability, which will extend the results of the reference article. *Methods.* To study this model, an auxiliary equation obtained by exponential substitution in the original equation is considered. Then, the method of step integration of a differential equation with delay and the introduction of additional functions are used. *Results.* A solution of the "freezing" neuron type for the original model is constructed, and the range of parameters for the existence and stability of this solution is described. *Conclusion.* The study obtained an extension of results for solutions of a special type in the model proposed by M. M. Preobrazhenskaya.

Keywords: phenomenological model of a neuron, differential equation with delay, relay equation, step method, parameter range, stability.

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[1]The paper presents materials of a talk given at the conference "Nonlinear days in Saratov for young scientists – 2023".

Introduction

This study is a continuation of the work of [1], the idea of which was to combine approaches related to artificial neural networks and modeling of biological neurons using differential equations.

Currently, neuromodeling is a popular task and is of great interest in science. Choosing a convenient, biologically plausible neuron model is difficult. In this regard, phenomenological models are actively being developed that more or less accurately reflect its dynamics, but are simple enough for analytical or numerical research. More details about the construction and study of neuron models using differential equations with delay are written in [2].

The article [1] studied a phenomenological model of a neural network in the form of a perceptron, which is a system of differential-difference equations where unknown functions describe changes in the normalized membrane potentials of neurons over time. The model involves three types of neurons: sensory, associative and responsive.

A model is proposed for the responding neuron

$$\dot{R} = \lambda\Big[\mathcal{F}\Big(R(t-h)\Big) + \mathcal{H}\Big(X_*(t)\Big)\Big]R(t). \tag{1}$$

Here $R(t)$ is the normalized membrane potential, $\lambda > 0$ is the rate of electrical processes in a nerve cell, $\mathcal{F}$ and $\mathcal{H}$ are threshold functions,

$$\mathcal{F}(u) = \begin{cases} 1, & 0 < u \leqslant 1, \\ -\tilde{\alpha}, & u > 1, \end{cases}$$

$$\mathcal{H}(u) = \begin{cases} -\tilde{\eta}, & 0 < u \leqslant \theta, \\ \tilde{\xi}, & u > \theta, \end{cases}$$

$X_*(t) = e^{\lambda x_*(t)}$, $x_*(t)$ is periodic function with period T_*, at the ends of the period and at some point t_* takes on a zero value, before point t_* in the period it has positive values, and after t_* are negative values until the end of the period:

$$x_*(t) \begin{cases} = 0, & t = 0, \\ > 0, & 0 < t < t_*, \\ = 0, & t = t_*, \\ < 0, & t_* < t < T_*, \end{cases}$$

parameters $\tilde{\eta}, \tilde{\xi}, \tilde{\alpha}, \theta$ are positive values, $h > 0$ is delay.

The equation (1) is a modification of the generalized Hutchinson equation [3]

$$\dot{u} = \lambda f(u(t-1))u, \tag{2}$$

proposed in the article [4]. Here $u = u(t) \geqslant 0$, $\lambda \gg 1$, the function $f(x)$ is infinitely differentiable on the half-axis $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$ and such that $f(0) = 1$, $f(x) \to -\tilde{\alpha}$ for $x \to +\infty$. In this article, it is proven for the equation (2) that for a large parameter λ the equation has an asymptotically orbitally stable relaxation cycle $u_*(t, \lambda) > 0$ of period $\tilde{T}(\lambda)$, satisfying limit relations

$$\lim_{\lambda\to\infty} \tilde{T}(\lambda) = T_0, \qquad \max_{0\leqslant t\leqslant \tilde{T}(\lambda)} |\tilde{x}(t,\lambda) - x_0(t)| = O(1/\lambda), \quad \lambda \to \infty,$$

where $T_0 = (1+\tilde{\alpha})t_0,\ t_0 = 1 + 1/\tilde{\alpha},\ \tilde{x}(t,\lambda) = (1/\lambda)\ln u_*(t,\lambda)$, and the T_0-periodic function $x_0(t)$

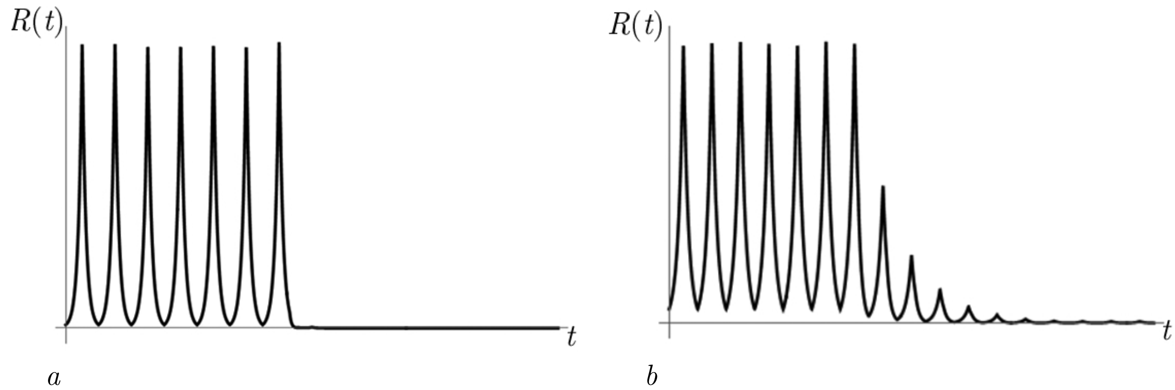

Fig 1. Schematic representation of “dying” (a) and “aging” (b) neurons, respectively

is given by the equalities

$$x_0(t) = \begin{cases} t, & t \in [0,1], \\ 1 - \tilde{\alpha}(t-1), & t \in [1, t_0+1], \\ t - T_0, & t \in [t_0+1, T_0], \end{cases}$$
$$x_0(t+T_0) \equiv x_0(t).$$

Currently, the equation (2) underlies a number of phenomenological neuromodels.

In the work [1] for the equation (1) the existence of solutions of a special form was proven: on some initial interval there is any predetermined number of equally high bursts, after which the solution immediately becomes periodic with small oscillations. Hereinafter, by the term “high bursts” we mean the values of the function $R(t)$ of the order of e^{λ}, by the term “small oscillations” — of the order of $e^{-\lambda}$. Such a neuron is called “dying”. It was also shown numerically that there are solutions where, after an initial interval with equally high bursts, the solution gradually transitions to small oscillations; the bursts decrease their amplitude for some time. Such a neuron is called “aging”.

In this work, an extension of these results was obtained, namely, the existence and stability of the solution was analytically shown and the range of parameters was found in which the solution from high periodic bursts passes to small oscillations immediately or gradually with a decrease in amplitude, which combines the types of behavior “aging” and “dying” neuron (Fig. 1). We will call this generalization a “freezing” neuron, where by the term “freezing” we mean the transition to small oscillations.

1. Formulation of the problem

For the equation (1), we prove the existence and stability of regimes of a special type (Fig. 2): solutions that have any predetermined number of equally high bursts, after which a gradual attenuation of the bursts occurs and oscillations with small amplitude are established.

To do this, in the equation (1) we make an exponential substitution $R(t) = e^{\lambda r(t)}$. We

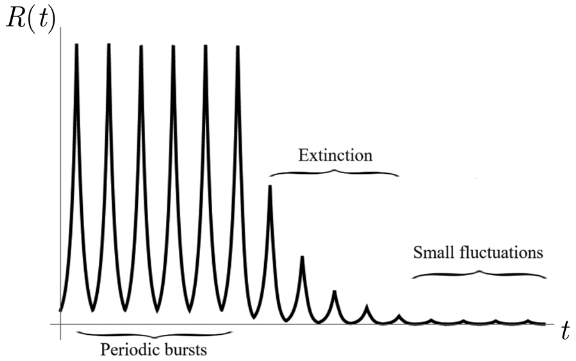

Fig 2. Schematic representation of a special type of function $R(t)$

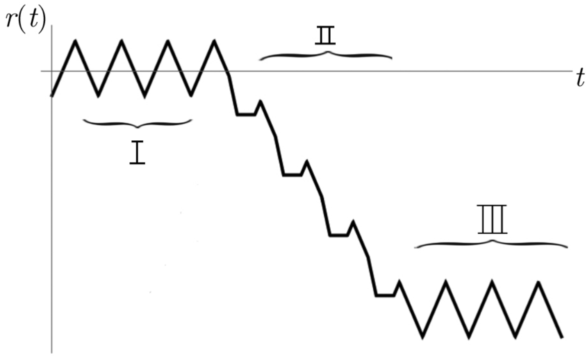

Fig 3. Schematic representation of a special type of function $r(t)$

obtain the relay equation

$$\dot{r} = F(r(t-h)) + H(x_*(t)), \tag{3}$$

$$F(x) = \mathcal{F}(e^{\lambda x}) = \begin{cases} 1, & x \leqslant 0, \\ -\tilde{\alpha}, & x > 0, \end{cases} \tag{4}$$

$$H(x) = \mathcal{H}(e^{\lambda x}) = \begin{cases} -\tilde{\eta}, & x \leqslant 0, \\ \tilde{\xi}, & x > 0. \end{cases} \tag{5}$$

The equation (3) is the central object of study in this work.

For the equation (3), a special mode corresponds to the existence of the following solution (Fig. 3): periodic with positive and negative values in the interval up to a certain point (see I in Fig. 3), after which there is a decrease in positive values and lengths of positive segments (see II in Fig. 3), and from a certain moment a periodic solution with only negative values is established (see III in Fig. 3).

Let us introduce a set of initial functions

$$S \overset{\text{def}}{=} \{\varphi(t) \in C[-h, 0],\ \varphi(t) < 0,\ \varphi(0) = -d\}, \tag{6}$$

where d is a positive parameter, the restrictions on which will be specified further.

Let us redesignate the parameters for greater convenience in further calculations:

$$\alpha := 1 + \tilde{\alpha}, \quad \xi := 1 + \tilde{\xi}, \quad \eta := 1 - \tilde{\eta}.$$

2. Result

Theorem 1. *For any $n \in \mathbb{N}$ there is a parameter range α, ξ, η, h*

$$nT_* \leqslant h < (n+1)T_*, \tag{7}$$

$$\xi > 1, \quad \eta = \frac{-\xi t_*}{T_* - t_*}, \quad \alpha > \max\left\{\frac{\xi t_*}{nT_*}, 1\right\}, \tag{8}$$

such that the equation (3) with an initial function from the set S for

$$0 < d < \xi t_* \tag{9}$$

has a stable solution of the following form.

1. *On the segment* $[0, h+\frac{d}{\xi}]$ *coinciding with the* T_**-periodic function*

$$r^{(1)}(t) = \begin{cases} \xi t - d, & 0 < t < t_*, \\ \eta(t-t_*) + \xi t_* - d, & t_* \leqslant t \leqslant T_*, \end{cases} \tag{10}$$

$$r^{(1)}(t+T_*) = r^{(1)}(t),$$

placing n *full periods on this interval, and this function has positive and negative values on the period.*

2. *On the segment* $[h+\frac{d}{\xi}, h-\frac{\xi t_*-d}{\eta}+t_*+(n-1)T_*]$ *coinciding with the function*

$$r^{(2)}(t) = \begin{cases} r^{(1)}(t) - \alpha t + c_{k_1+}, & h+\dfrac{d}{\xi}+k_1T_* \leqslant t \leqslant h-\dfrac{\xi t_*-d}{\eta}+t_*+k_1T_*, \\ r^{(1)}(t) + c_{k_2-}, & h-\dfrac{\xi t_*-d}{\eta}+t_*+k_2T_* \leqslant t \leqslant h+\dfrac{d}{\xi}+(k_2+1)T_*, \end{cases} \tag{11}$$

where

$$c_{k_1+} = k_1\frac{\alpha(\xi-\eta)}{\eta\xi}(\xi t_*-d) + \alpha(h+\frac{d}{\xi}+k_1T_*), \tag{12}$$

$$c_{k_2-} = (k_2+1)\frac{\alpha(\xi-\eta)}{\eta\xi}(\xi t_*-d), \tag{13}$$

$k_1 = 0, 1, \ldots, n-1,\ k_2 = 0, 1, \ldots, n-2.$

Moreover, from a certain moment $\tilde{t} \in [h+\frac{d}{\xi}, h-\frac{\xi t_*-d}{\eta}+t_*+(n-1)T_*]$ *all values of the function* $r^{(2)}(t)$ *become negative.*

On the segment $[h-\frac{\xi t_*-d}{\eta}+t_*+(n-1)T_*, h+\tilde{t}]$ *the solution of the type of the previous paragraph continues*

$$r^{(2)}(t) = \begin{cases} r^{(1)}(t) - \alpha t + v_{i+}, & a_i \leqslant t \leqslant b_i, \\ r^{(1)}(t) + v_{i-}, & b_i \leqslant t \leqslant a_{i+1}, \end{cases}$$

where $i = 1, 2, \ldots, l,\ l < n,\ v_{i-} \leqslant c_{(n-2)-}$ *and* $v_{i+} \leqslant c_{(n-1)+}$ *are some constants,*
$h-\frac{\xi t_*-d}{\eta}+t_*+(n-1)T_* < a_i, b_i < \tilde{t}$.

3. *From the moment* $\tilde{t}+h$ *the solution coincides with the function*

$$r^{(3)}(t) = \begin{cases} \xi t - d - \tilde{\Delta}, & 0 < t < t_*, \\ \eta(t-t_*) + \xi t_* - d - \tilde{\Delta}, & t_* \leqslant t \leqslant T_*, \end{cases} \tag{14}$$

$$r^{(3)}(t+T_*) = r^{(3)}(t),$$

where $\tilde{\Delta} > \xi t_* - d$. *The solution coincides with a* T_**-periodic function with only negative values on the period.*

Note that the solution described in theorem 1 coincides with the solution corresponding to the solution of a special type of equation (3). More specifically, the function $r^{(1)}(t)$ corresponds to the first part — a periodic solution with positive and negative values on the interval up to some point (see I in Fig. 3). The function $r^{(2)}(t)$ corresponds to the second part of the solution, where the positive values and lengths of positive segments decrease (see II in Fig. 3). The function $r^{(3)}(t)$ corresponds to the third part, on which a periodic solution with only negative values is established (see III in Fig. 3).

Theorem 2 (Corollary of the theorem 1). *For any $n \in \mathbb{N}$ there is a region of parameters α, ξ, η, h defined in the theorem 1 such that the model (1) has stable modes such as a "freezing" neuron.*

Let us move on to proving the theorems.

3. Equation Study (3)

We will construct a solution to the equation (3), corresponding to the solution of a special type of equation (1), described in the Introduction. We will form restrictions on the parameters during the construction of the solution. For convenience, we divide the construction of the function $r(t)$ into stages. The construction of a periodic part with positive and negative values will be called stage 1, the proof of the presence of a transition process will be called stage 2, and the establishment of a periodic part with only negative values will be called stage 3.

Let us choose a natural number n corresponding to the number of bursts at stage 1. We choose the delay $h > 0$ in this way: $nT_* \leqslant h < (n+1)T_*$, which corresponds to the formula (7).

The functions $F(x)$ and $H(x)$, defined by the equalities (4) and (5), are piecewise constant, hence the function $r(t)$ will be piecewise linear. There are only 4 combinations of pairs of function values $F(x)$ and $H(x)$; consider all possible solutions.

3.1. Types of solution to the equation (3) on an interval. Each of the functions $F(x)$ and $H(x)$ takes only two values 1 and $-\alpha+1$, $\xi-1$ and $\eta-1$, respectively. Let's consider some segment $[t_1, t_2]$ on the positive semi-axis, on which the right side of the equation (3) is constant. There are only 4 types of such segments. Let us find the values of the function $r(t)$ on a given segment in general form, denoting the value of the function $F(x)$ by $\tilde{f}$, and the value of the function $H(x)$ by $\tilde{h}$. To do this, let's solve the Cauchy problem on this interval, assuming that the initial value is known, let's denote it $r_1 = r(t_1)$. Then the Cauchy problem looks like this:

$$\dot{r} = \tilde{f} + \tilde{h}, \quad r(t_1) = r_1.$$

Consequently, the solution on the interval $[t_1, t_2]$ with the initial value r_1 will have the form

$$r(t) = (\tilde{f} + \tilde{h})(t - t_1) + r_1. \tag{15}$$

Now we define each of the four types of segments depending on the specific values of the functions $F(r(t-h))$ and $H(x_*(t))$ on the segment $[t_1, t_2]$.

The first type of the segment corresponds to the assumption

$$\begin{cases} [t_1, t_2] \in (kT_*, kT_* + t_*), & k \in \mathbb{N} \cup \{0\}, \\ r(t) \leqslant 0, & t \in [t_1 - h, t_2 - h], \end{cases} \tag{16}$$

with it $F(r(t-h)) = 1$, $H(x_*(t)) = \xi - 1$, therefore, the solution will have the form

$$r(t) = \xi(t - t_1) + r_1. \tag{17}$$

Similarly, the *second type* of the segment corresponds to the assumption

$$\begin{cases} [t_1, t_2] \in [kT_* + t_*, (k+1)T_*], & k \in \mathbb{N} \cup \{0\}, \\ r(t) \leqslant 0, & t \in [t_1 - h, t_2 - h]. \end{cases} \tag{18}$$

Then $F(r(t-h)) = 1$, $H(x_*(t)) = \eta - 1$, therefore, the solution will have the form

$$r(t) = \eta(t - t_1) + r_1. \tag{19}$$

The third type of the segment also corresponds to the assumption

$$\begin{cases} [t_1, t_2] \in (kT_*, kT_* + t_*), & k \in \mathbb{N} \cup \{0\}, \\ r(t) > 0, & t \in [t_1 - h, t_2 - h], \end{cases} \tag{20}$$

at which $F(r(t-h)) = -\alpha + 1$, $H(x_*(t)) = \xi - 1$, therefore, the solution will have the form

$$r(t) = (-\alpha + \xi)(t - t_1) + r_1. \tag{21}$$

Assumption for *fourth type* segment

$$\begin{cases} [t_1, t_2] \in [kT_* + t_*, (k+1)T_*], & k \in \mathbb{N} \cup \{0\}, \\ r(t) > 0, & t \in [t_1 - h, t_2 - h]. \end{cases} \tag{22}$$

Then $F(r(t-h)) = -\alpha + 1$, $H(x_*(t)) = \eta - 1$, therefore, the solution will have the form

$$r(t) = (\eta - \alpha)(t - t_1) + r_1. \tag{23}$$

3.2. Secondary functions. In this section, we introduce and study two auxiliary functions associated with the function $r(t)$. Consider the functions $r_-(t)$ and $r_+(t)$, such as

$$\dot{r}_-(t) = 1 + H(x_*(t)), \tag{24}$$

$$\dot{r}_+(t) = -\alpha + 1 + H(x_*(t)). \tag{25}$$

Equations (24) and (25) are variants of the equation (3) with a constant (1 or $-\alpha + 1$) value of the function $F(r(t-h))$. For the functions $r_-(t)$ and $r_+(t)$ we introduce the initial conditions

$$r_-(0) = -d, \quad r_+(0) = -d.$$

Note that for $t \in (kT_*, kT_* + t_*)$, $k \in \mathbb{N} \cup \{0\}$ the function $H(x_*(t)) = \xi - 1$. Then the Cauchy problems on a given interval for the equations (24) and (25) with initial values denoted by a_{1-} and a_{1+} look like this, respectively:

$$\begin{aligned} \dot{r}_- &= \xi, & r_-(kT_*) &= a_{1-}, \\ \dot{r}_+ &= -\alpha + \xi, & r_+(kT_*) &= a_{1+}. \end{aligned}$$

Therefore, the solutions here will have the form

$$r_-(t) = \xi(t - kT_*) + a_{1-}, \quad r_+(t) = (-\alpha + \xi)(t - kT_*) + a_{1+},$$

which coincide, respectively, with the *first and third types* of solutions to the equation (3) при $t_1 = kT_*$, $r_1 = a_{1-}$ и $r_1 = a_{1+}$.

Similarly, for $t \in [kT_* + t_*, (k+1)T_*]$, $k \in \mathbb{N} \cup \{0\}$ function $H(x_*(t)) = \eta - 1$. This means that the Cauchy problems on this interval for the equations (24) and (25) with initial conditions denoted by a_{2-} and a_{2+}, respectively, look like this

$$\begin{aligned} \dot{r}_- &= \eta, & r_-(kT_* + t_*) &= a_{2-}, \\ \dot{r}_+ &= \eta - \alpha, & r_+(kT_* + t_*) &= a_{2+}. \end{aligned}$$

Therefore, the solutions will have the form

$$r_-(t) = \eta(t - kT_* - t_*) + a_{2-}, \quad r_+(t) = (\eta - \alpha)(t - kT_* - t_*) + a_{2+},$$

which coincide with the *second and fourth types* of solutions to the equation (3) with $t_1 = kT_* + t_*$, $r_1 = a_{2-}$ and $r_1 = a_{2+}$ accordingly.

Remark 1. *The piecewise linear function $r(t)$ on each linear segment can be represented as $r(t) = r_*(t) + c$, where the symbol "$*$" means one of the possible indices "$-$" or "$+$", and c is some constant depending on the segment of consideration. Moreover, the symbol "$*$" on a certain segment $[t_3, t_4]$ will carry the meaning of the index "$-$" if $r(t) \leqslant 0$ for $t \in [t_3 - h, t_4 - h]$, and the meaning of the index "$+$" if $r(t) > 0$ for $t \in [t_3 - h, t_4 - h]$.*

Let us write down the complete solutions of the equations (24) and (25) respectively

$$r_-(t) = \begin{cases} \xi(t - kT_*) + r_-(kT_*), & kT_* \leqslant t < kT_* + t_*, \\ \eta(t - kT_* - t_*) + r_-(kT_* + t_*), & kT_* + t_* \leqslant t \leqslant (k+1)T_*, \end{cases} \tag{26}$$

$$r_+(t) = \begin{cases} (-\alpha + \xi)(t - kT_*) + r_+(kT_*), & kT_* \leqslant t < kT_* + t_*, \\ (\eta - \alpha)(t - kT_* - t_*) + r_+(kT_* + t_*), & kT_* + t_* \leqslant t \leqslant (k+1)T_*, \end{cases} \tag{27}$$

где $k \in \mathbb{N} \cup \{0\}$, $r_-(0) = r_+(0) = -d$.

Let us study how the difference in the values of the introduced functions grows.

Lemma 1. *Consider an arbitrary segment $[\tilde{t}_1, \tilde{t}_2]$ on the positive semi-axis. On its ends it's true: $r_-(\tilde{t}_2) - r_+(\tilde{t}_2) = \alpha(\tilde{t}_2 - \tilde{t}_1) + r_-(\tilde{t}_1) - r_+(\tilde{t}_1)$.*

Proof. Subtract from the left and right sides of the equality (24) the left and right sides of the equality, respectively (25)

$$\dot{r}_-(t) - \dot{r}_+(t) = 1 + H(x_*(t)) - \Big(-\alpha + 1 + H(x_*(t)) \Big) = \alpha,$$

integrate on the interval $[\tilde{t}_1, \tilde{t}_2]$, substitute the value $t = \tilde{t}_2$ into the resulting function and obtain the statement of the lemma 1. □

3.3. Construction of the function $r(t)$ at stage 1. We assume that on the interval $[-h, 0]$ the function $r(t)$ coincides with one of the functions of the set S described by the formula (6). Note that up to the point $t = nT_*$ the function $r(t - h)$ coincides with the function $\varphi(t - h)$ from the initial set S, since $nT_* \leqslant h$. Then the values of the function $r(t - h) < 0$ for $t \in [0, nT_*]$, and the function $F(r(t - h)) = 1$ for the same values of t. Therefore, taking into account the equality $r(0) = r_-(0) = -d$, the functions $r(t)$ and $r_-(t)$ coincide on the interval $[0, nT_*]$ and have the form (26).

To achieve the special form of the function $r(t)$ described in the problem statement, it is necessary that the following conditions be met at the 1st stage of construction.

1. Periodicity of the function $r(t)$ for $t \leqslant nT_*$. This condition is satisfied if the initial values of the Cauchy problems are equal on each interval $(kT_*, kT_* + t_*)$, $k = 0, 1, \ldots, n-1$. We obtain the necessary condition $r(kT_*) = r(0) = -d$, $k = 0, 1, \ldots, n$, equality is achievable by introducing a constraint $r(T_*) = -d \Leftrightarrow \eta T_* + (\xi - \eta)t_* = 0 \Leftrightarrow$

$$\eta = \frac{-\xi t_*}{T_* - t_*}. \tag{28}$$

2. Increasing values of the function $r(t)$ on intervals $(kT_*, kT_* + t_*)$ and decreasing values on intervals $[kT_* + t_*, (k+1)T_*],\ k = 0, 1, \ldots, n-1$. This condition is equivalent to a positive value of the coefficient ξ (satisfied automatically, since $\tilde{\xi} = \xi - 1 > 0$) and a negative value of the coefficient η, which is true from the constraint (28).
3. The existence of positive and negative values on a period. We have negative values from the condition that the function $r(t)$ is periodic. Positive values must be achieved at the ends of the segments on which the function $r(t)$ increases. From the second condition it is known that these are points $kT_* + t_*$, $k = 0, 1, \ldots, n-1$, also from the periodicity condition it is sufficient that $r(t_*) > 0 \Leftrightarrow \xi t_* - d > 0$, that is

$$0 < d < \xi t_*. \tag{29}$$

Thus, the conditions for the existence of a function $r(t)$ of a special form at the first stage are (28) and (29), which correspond to the restrictions (8) and (9) theorem 1. Let us apply the constraint (28) to the formula (26) and obtain a refined form of the solution to the equation (24) for $t \geqslant 0$

$$r_-(t) = \begin{cases} \xi t - d, & 0 \leqslant t < t_*, \\ \eta(t - t_*) + \xi t_* - d, & t_* \leqslant t \leqslant T_*, \end{cases} \tag{30}$$
$$r_-(t + T_*) = r_-(t),$$

that is, the function $r_-(t)$ is T_*-periodic over the entire domain of definition. From the coincidence of the functions $r(t)$ and $r_-(t)$ at the first stage it also follows that the function $r(t)$ for $t \in [0, nT_*]$ has the form

$$r(t) = \begin{cases} \xi t - d, & 0 \leqslant t < t_*, \\ \eta(t - t_*) + \xi t_* - d, & t_* \leqslant t \leqslant T_*, \end{cases} \tag{31}$$
$$r(t + T_*) = r(t).$$

Taking into account the form of the solution (30) of the equation (24) and applying the lemma 1, we obtain a refined form of the solution of the equation (25)

$$r_+(t) = \begin{cases} (-\alpha + \xi)t - d, & 0 \leqslant t < t_*, \\ (\eta - \alpha)(t - t_*) + (-\alpha + \xi)t_* - d, & t_* \leqslant t \leqslant T_*, \end{cases} \tag{32}$$
$$r_+(t + T_*) = r_+(t) - \alpha T_*.$$

Auxiliary fact. *For any $t > 0$ on the interval $[0, t]$ the lemma 1 is true*

$$r_-(t) - r_+(t) = \alpha(t - 0) + r_-(0) - r_+(0) = \alpha t - d - (-d) = \alpha t,$$

it follows that the formulas connecting the functions are true$r_-(t)$ и $r_+(t)$

$$r_+(t) = r_-(t) - \alpha t, \quad r_-(t) = r_+(t) + \alpha t.$$

3.4. Construction of the function $r(t)$ at the 2nd stage. In this section we will consider the change in the behavior of the function $r(t)$ at the 2nd stage of construction. Note that with a constant value of the function $F(r(t-h)) = 1$, the solution to the equation (3) would coincide with the solution (31) on the entire axis, but this is not true, since how the function $F(r(t-h))$ changes its value to $-\alpha+1$ if its argument $r(t-h)$ becomes positive. In this regard, let us consider the influence of positive-valued segments of the 1st stage on the construction of a solution to the equation (3) at the 2nd stage. To do this, we will find the zeros of the function $r(t)$ at the 1st stage. Note that under the introduced restrictions (28) and (29), the function $r(t)$ takes negative values at points $t = kT_*$, $k = 0, 1, \ldots, n$ and positive values at points $t = t_* + kT_*$, $k = 0, 1, \ldots, n-1$. On intervals (kT_*, kT_*+t_*) and segments $[t_*+kT_*, (k+1)T_*]$, $k = 0, 1, \ldots, n-1$ function $r(t)$ is linear, therefore, there will be exactly one zero on these intervals and segments. Let's find their values.

The zeros of the function $r(t)$ on the intervals $(kT_*, kT_* + t_*)$, $k = 0, 1, \ldots, n-1$ will be denoted by $t_k^{(1)}$, let's find their values by solving the equations $\xi(t_k^{(1)} - kT_*) - d = 0$, we get

$$t_k^{(1)} = \frac{d}{\xi} + kT_*, \quad k = 0, 1, \ldots, n-1. \tag{33}$$

The zeros of the function $r(t)$ on the intervals $[t_* + kT_*, (k+1)T_*]$, $k = 0, 1, \ldots, n-1$ will be denoted by $t_k^{(2)}$. We find their values by solving the equations $\eta(t_k^{(2)} - kT_* - t_*) + \xi t_* - d = 0$, we get

$$t_k^{(2)} = \frac{\xi t_* - d}{-\eta} + t_* + kT_*, \quad k = 0, 1, \ldots, n-1. \tag{34}$$

Note that all zeros of the function $r(t)$ at stage 1 are strictly ordered

$$t_0^{(1)} < t_0^{(2)} < t_1^{(1)} < t_1^{(2)} < \cdots < t_{n-1}^{(1)} < t_{n-1}^{(2)}.$$

At stage 1, the function $r(t)$ takes positive values on the segments: $[t_0^{(1)}, t_0^{(2)}]$, $[t_1^{(1)}, t_1^{(2)}]$, $\ldots [t_{n-1}^{(1)}, t_{n-1}^{(2)}]$. Note that $r(t-h) < 0$ for $t < h + t_0^{(1)}$, since the function $r(t)$ first takes on a non-negative value at the point $t_0^{(1)}$. Therefore, up to the point $t = h + t_0^{(1)}$ the function $F(r(t-h)) = 1$, which means that the solution to the equation (3) coincides with the solution equation (24) to a given point, which corresponds to the formula (10) of the theorem 1, and $r(h + t_0^{(1)}) = r_-(h + t_0^{(1)})$. We will consider the point $h + t_0^{(1)}$ to be the end of the first stage. At the second stage $[h + t_0^{(1)}, h + t_{n-1}^{(2)}]$ we divide all points into two types.

1. Points on segments $[h+t_k^{(1)}, h+t_k^{(2)}]$, $k = 1, 2, \ldots, n-1$. On them the function $r(t-h) > 0 \Rightarrow F(r(t-h)) = -\alpha+1$, then, according to the remark from paragraph 3.2, it is true that on these intervals the function $r(t)$ can be represented as $r(t) = r_+(t)+c_{k+}$, $k = 1, 2, \ldots, n-1$, where c_{k+} are some constants.
2. Points on segments $[h + t_k^{(2)}, h + t_{k+1}^{(1)}]$, $k = 1, 2, \ldots, n-2$. On these segments the function $r(t-h) \leqslant 0 \Rightarrow F(r(t-h)) = 1$, then similarly to the previous case it is true that the function $r(t)$ can be represented in the form $r(t) = r_-(t) + c_{k-}$, $k = 1, 2, \ldots, n-2$, where c_{k-} are some constants.

Let us clarify the values of the constants c_{k-} and c_{k+}. As shown above on $[h+t_0^{(1)}, h+t_0^{(2)}]$ the function $r(t)$ can be represented as $r(t) = r_+(t)+c_{0+}$. Moreover, it is known that $r(h+t_0^{(1)}) = r_-(h + t_0^{(1)}) = r_+(h + t_0^{(1)}) + \ alpha(h + t_0^{(1)})$ (the last equality follows from **auxiliary fact**).

From this we obtain that $c_{0+} = \alpha(h + t_0^{(1)})$, function $r(t) = r_+(t) + \alpha(h + t_0^{(1)})$ and the value at the last point of the segment $r(h + t_0^{(2)}) = r_+(h + t_0^{(2)}) + \alpha(h + t_0^{(1)})$.

Similarly, on the interval $[h + t_0^{(2)}, h + t_1^{(1)}]$ we have $r(t) = r_-(t) + c_{0-}$. Then, using an auxiliary fact, we obtain $r(h+t_0^{(2)}) = r_+(h+t_0^{(2)}) + (1+\alpha)(h+t_0^{(1)}) = r_-(h+t_0^{(2)}) - \alpha(t_0^{(2)} - t_0^{(1)})$. Hence it is correct $c_{0-} = -\alpha(t_0^{(2)} - t_0^{(1)})$.

Lemma 2. *For the constants c_{k+} and c_{k-} the following formulas are correct:* $c_{0+} = \alpha(h + t_0^{(1)})$

$$c_{k+} = -\sum_{i=0}^{k-1} \alpha(t_i^{(2)} - t_i^{(1)}) + \alpha(h + t_k^{(1)}), \quad k = 1, 2, \ldots, n-1, \tag{35}$$

$$c_{k-} = -\sum_{i=0}^{k} \alpha(t_i^{(2)} - t_i^{(1)}), \quad k = 0, 1, 2, \ldots, n-2. \tag{36}$$

Proof. Let us carry out the proof using the method of mathematical induction (MMI). The basic statements of the MMI were proven above for $k = 0$. Suppose that for some $k = m < n-2$ the statement is true. Then on $[h+t_m^{(1)}, h+t_m^{(2)}]$ the function $r(t) = r_+(t) - \sum_{i=0}^{m-1} \alpha(t_i^{(2)} - t_i^{(1)}) +$ $+ \alpha(h + t_m^{(1)})$, and on $[h + t_m^{(2)}, h + t_{m+1}^{(1)}]$ true $r(t) = r_-(t) - \sum_{i=0}^{m} \alpha(t_i^{(2)} - t_i^{(1)})$.

Let us check the statements for $k = m + 1$. As shown above on $[h + t_{m+1}^{(1)}, h + t_{m+1}^{(2)}]$ the function $r(t)$ can be represented as $r(t) = r_+(t) + c_{(m+1)+}$. From the induction hypothesis and the auxiliary fact, $r(h+t_{m+1}^{(1)}) = r_-(h+t_{m+1}^{(1)}) + c_{m-} istrue = r_+(h+t_{m+1}^{(1)}) + \alpha(h+t_{m+1}^{(1)}) + c_{m-}$. From here we get that $c_{(m+1)+} = \alpha(h + t_{m+1}^{(1)}) + c_{m-} = -\sum_{i=0}^{m} \alpha(t_i^{(2)} - t_i^{(1)}) + \alpha(h + t_{m+1}^{(1)})$ are confirms the formula (35) lemma 2. Function $r(t) = r_+(t) + c_{(m+1)+}$ and the value at the last point of the segment $r(h + t_{m+1}^{(2)}) = r_+(h + t_{m+1}^{(2)}) + c_{(m+1)+}$.

Similarly, on the interval $[h+t_{m+1}^{(2)}, h+t_{m+2}^{(1)}]$ we have $r(t) = r_-(t) + c_{(m+1)-}$. Then, using the previous step and an auxiliary fact, we obtain $r(h + t_{m+1}^{(2)}) = r_+(h + t_{m+1}^{(2)}) + c_{(m+1)+} =$ $= r_-(h + t_{m+1}^{(2)}) - \alpha(h + t_{m+1}^{(2)}) + c_{(m+1)+}$. Hence $c_{(m+1)-} = -\sum_{i=0}^{m} \alpha(t_i^{(2)} - t_i^{(1)}) + \alpha(h + t_{m+1}^{(1)}) -$ $-\alpha(h+t_{m+1}^{(2)}) = -\sum_{i=0}^{m+1} \alpha(t_i^{(2)} - t_i^{(1)})$ are confirms the formula (36) of the lemma 2. The statement of Lemma 2 is proven. Note that the formulas (35) and (36) confirm the formulas (12) and (13) of the theorem 1. □

Thus, we obtain the form of solution of the equation (3) at stage 2 for $t \in [h+t_0^{(1)}, h+t_{n-1}^{(2)}]$ (see Fig. 4)

$$r(t) = \begin{cases} r_+(t) + c_{k+}, & h + t_k^{(1)} \leqslant t \leqslant h + t_k^{(2)}, \\ r_-(t) + c_{k-}, & h + t_k^{(2)} \leqslant t \leqslant h + t_{k+1}^{(1)}. \end{cases} \tag{37}$$

Consider the value of the function $r(t)$ at the last point of the second stage

$$r(h + t_{n-1}^{(2)}) = r_+(h + t_{n-1}^{(2)}) + c_{(n-1)+} = r_-(h + t_{n-1}^{(2)}) - \sum_{i=0}^{n-1} \alpha(t_i^{(2)} - t_i^{(1)}).$$

We obtain a general formula for the difference between the values of the function $r_-(t)$ and $r(t)$ at the last point of the second stage; we denote it Δ:

$$\Delta = r_-(h + t_{n-1}^{(2)}) - r(h + t_{n-1}^{(2)}) = \sum_{i=0}^{n-1} \alpha(t_i^{(2)} - t_i^{(1)}). \tag{38}$$

Let us substitute into the formula (38) the values of the zeros of the first stage (33) and (34) of the function $r(t)$ and the value of the parameter η from (28):

$$\Delta = n\frac{\alpha(\eta-\xi)}{\eta\xi}\Big(\xi t_* - d\Big) = n\frac{\alpha T_*}{\xi t_*}\Big(\xi t_* - d\Big) = \gamma(\xi t_* - d), \quad \text{где} \quad \gamma = n\frac{\alpha T_*}{\xi t_*}. \tag{39}$$

3.5. Construction of the function $r(t)$ at stage 3. At the 3rd stage of the solution, it is necessary to obtain negative values of the function $r(t)$ for all t. Using **auxiliary fact** and formula (37) we express the function $r(t)$ only in terms of the function $r_-(t)$

$$r(t) = \begin{cases} r_-(t) - \alpha t + c_{k_1+}, & h + t^{(1)}_{k_1} \leqslant t \leqslant h + t^{(2)}_{k_1}, \\ r_-(t) + c_{k_2-}, & h + t^{(2)}_{k_2} \leqslant t \leqslant h + t^{(1)}_{k_2+1}, \end{cases}$$

$k_1 = 0, 1, \ldots, n-1$, $k_2 = 0, 1, \ldots, n-2$. Let us apply the lemma 2 to the resulting expression for the coefficients c_{k+} and c_{k-}, we obtain

$$r(t) = \begin{cases} r_-(t) - \alpha(t - h - t^{(1)}_0), & h + t^{(1)}_0 \leqslant t \leqslant h + t^{(2)}_0, \\ r_-(t) - \alpha(t - h - t^{(1)}_{k_3}) + c_{(k_3-1)-}, & h + t^{(1)}_{k_3} \leqslant t \leqslant h + t^{(2)}_{k_3}, \\ r_-(t) + c_{k_4-}, & h + t^{(2)}_{k_4} \leqslant t \leqslant h + t^{(1)}_{k_4+1}, \end{cases}$$

here $k_3 = 1, \ldots, n-1$, $k_4 = 0, 1, \ldots, n-2$, and all $c_{k_i-} < 0$ according to the formulas (36), $i = 1, 2, 3, 4$. From the previous paragraph it is also known that $r(h + t^{(2)}_{n-1}) = r_-(h + t^{(2)}_{n-1}) - \Delta$. Let us introduce restrictions under which $r(h + t^{(2)}_{n-1}) < 0$. To do this, it is necessary that $r_-(h + t^{(2)}_{n-1}) - \Delta < 0$. The maximum value of the function $r_-(t)$ is achieved at points $kT_* + t_*$, $k \in \mathbb{N} \cup \{0\}$ and is equal to $r_{\max -} = \xi t_* - d$. Since we do not know the exact location of the point $h + t^{(2)}_{n-1}$ relative to the period of the function $r_-(t)$, then a sufficient condition for the negative value at this point is the negative value at the maximum function point: $r_{\max -} - \Delta < 0$. Let us substitute into this inequality the values of the parameters $r_{\max -}$ from what was written above and Δ from the equality (39) and make some transformations

$$\xi t_* - d - n\frac{\alpha T_*}{\xi t_*}\Big(\xi t_* - d\Big) < 0,$$

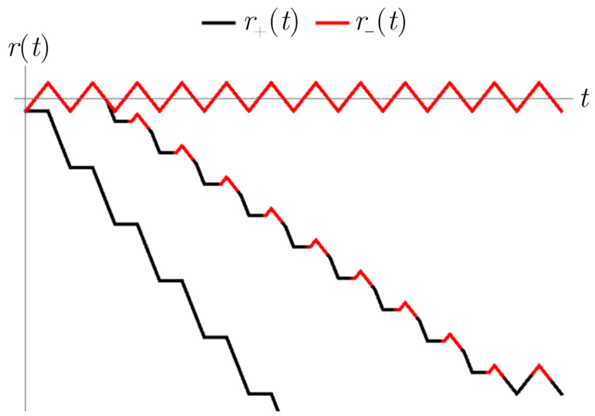

Fig 4. Schematic representation of the functions $r_-(t)$, $r(t)$, $r_+(t)$ at stage 2 (color online)

we get a limitation on the parameter α:

$$\alpha > \frac{\xi t_*}{nT_*}. \tag{40}$$

Note that for sufficiently large n this condition is satisfied automatically, since $\alpha = \tilde{\alpha} + 1 > 1$. This condition corresponds to the restrictions (8) of the theorem 1.

Lemma 3. *When the constraint (40) is satisfied, the function $r(t) < 0$ for all $t > h + t^{(2)}_{n-1}$.*

Proof. Let for all $t \in [t^{(2)}_{n-1}, h + t^{(2)}_{n-1}]$ be a function $r(t) \leqslant 0$. Then at the 3rd stage for $t \in [h + t^{(2)}_{n-1}, 2h + t^{(2)}_{n-1}]$ the function $r(t-h) \leqslant 0$ and $F(r(t-h)) = 1$, from which it follows that the function $r(t)$ can be represented as $r(t) = r_-(t) - c_-$, where c_- is some constant. From paragraph 3.4 it is known that $r(h+t^{(2)}_{n-1}) = r_-(h+t^{(2)}_{n-1}) - \Delta$, therefore, for $t \in [h+t^{(2)}_{n-1}, 2h+t^{(2)}_{n-1}]$ function $r(t) = r_-(t) - \Delta$. According to the introduced condition of Lemma 3, the maximum value of the function $r_-(t)$ is less than Δ, therefore, for $t \in [h+t^{(2)}_{n-1}, 2h+t^{(2)}_{n-1}]$ function $r(t) < 0$. Then we note that all the reasoning for the segment $[t^{(2)}_{n-1}, h + t^{(2)}_{n-1}]$ can now be repeated for the segment $[h + t^{(2)}_{n-1}, 2h + t^{(2)}_{n-1}]$ and further for each segment $[kh + t^{(2)}_{n-1}, (k+1)h + t^{(2)}_{n-1}]$, $k \in \mathbb{N}$. This means that the function $r(t) < 0$ for all $t > h + t^{(2)}_{n-1}$.

Let now on the interval $[t^{(2)}_{n-1}, h+t^{(2)}_{n-1}]$ there exist l pairs ($l \in \mathbb{N}$, $l < n$) such ordered points $\{a_i,\ b_i\}$, $i = 1, \ldots, l$ such that for $t \in (a_i, b_i)$ the function $r(t) > 0$ (these points necessarily exist in pairs on a given segment, since the values at its ends are less than or equal to zero). Then, similarly to Section 3.4 and Lemma 2, we obtain a formula for $r(t)$ for $t \in [h + t^{(2)}_{n-1}, 2h + t^{(2)}_{n-1}]$

$$r(t) = \begin{cases} r_-(t) - \Delta, & h + t^{(2)}_{n-1} \leqslant t \leqslant h + a_1, \\ r_-(t) - \alpha(t - h - a_v) - \Delta - \sum_{i=1}^{v-1} \alpha(b_i - a_i), & h + a_v \leqslant t \leqslant h + b_v, \\ r_-(t) - \Delta - \sum_{i=1}^{w} \alpha(b_i - a_i), & h + b_w \leqslant t \leqslant h + a_{w+1}, \\ r_-(t) - \Delta - \sum_{i=1}^{l} \alpha(b_i - a_i), & h + b_l \leqslant t \leqslant 2h + t^{(2)}_{n-1}, \end{cases} \tag{41}$$

here $v = 1, \ldots, l$, $w = 1, \ldots, l-1$. Therefore, for $t \in [h + t^{(2)}_{n-1}, 2h + t^{(2)}_{n-1}]$ the function $r(t) \leqslant r_-(t) - \Delta < 0$. This means that with this segment we can carry out reasoning in the first case (there are no positive values of the function $r(t)$ on it) and then for each segment $[kh+t^{(2)}_{n-1}, (k+1)h + t^{(2)}_{n-1}]$, $k \in \mathbb{N}$. This means that the function $r(t) < 0$ for all $t > h + t^{(2)}_{n-1}$. ☐

Corollary of the Lemma 3. *Under the constraint (40), the function $r(t)$ for $t > 2h + t^{(2)}_{n-1}$ is a T_*-periodic function, has only negative values and is described by formula*

$$r(t) = r_-(t) - \Delta - \sum_{i=1}^{l} \alpha(b_i - a_i),\ l \in \mathbb{N} \cup \{0\},\ l < n, \tag{42}$$

Proof. By Lemma 3 it is true that for $t > h + t^{(2)}_{n-1}$ the function $r(t) < 0$, which means for $t > 2h + t^{(2)}_{n-1}$ function $r(t-h) < 0$ and function $F(r(t-h)) = 1$. Then for $t > 2h + t^{(2)}_{n-1}$ the function $r(t)$ can be represented as $r(t) = r_-(t) + \text{const}$. From the formula (42) we obtain that $r(2h+t^{(2)}_{n-1}) = r_-(2h+t^{(2)}_{n-1}) - \Delta - \sum_{i=1}^{l} \alpha(b_i - a_i), where l \in \mathbb{N} \cup \{0\},\ l < n$ (if $l = 0$, then the last term is simply missing). Then for $t > 2h + t^{(2)}_{n-1}$ the function $r(t) = r_-(t) - \Delta - \sum_{i=1}^{l} \alpha(b_i - a_i)$, where $l \in \mathbb{N} \cup \{0\}, l < n$. Moreover, the function $r_-(t)$ is a T_*-periodic function, and when adding the constant $-\Delta$ it is negative-valued. The result guarantees the validity of the formula (14) of the theorem 1. $\square$

Thus, a function of a special form $r(t)$ is constructed step by step.

1. At the first stage, we obtained a formula for the function $r(t)$ (31). On it it is a T_*-periodic function with positive and negative values.
2. At the second stage, we obtained the formula (37) for the function $r(t)$. A transition process takes place there.
3. At the third stage, we obtained formulas (41) and (42) for the function $r(t)$. On it, by Lemma 3 and its corollary, the function $r(t)$ goes into a periodic mode with only negative values.

3.6. Study of the solution of the equation (3) for stability. Let us prove the stability of the constructed solution to the equation (3). Let $\|\cdot\|$ is standard norm in $C[-h,0]$, that is

$$\|r(t)\| = \max_{t\in[-h,0]} |r(t)|.$$

We will follow the definition of stability for a differential equation with a deviating argument from the work [5]. Let us choose $\varepsilon > 0$ and show the existence of a $\delta(\varepsilon) > 0$ such that for the initial functions $\psi(t) \in C[-h,0]$ and $\varphi(t) \in S$ inequality $\|\psi(t) - \varphi(t)\| < \delta(\varepsilon)$ entails the inequality $\|r_\psi(t) - r_\varphi(t)\| < \varepsilon$ for solutions constructed from them for $t > 0$. We will look for $\delta(\varepsilon)$ in the form $M\varepsilon$, that is, we will prove the existence of a constant M such that the required $\delta(\varepsilon) = M\varepsilon$. Let us fix $\varphi(t) \in S$, Let us denote the perturbation of this initial function by $\theta(t) = \psi(t) - \varphi(t)$, and assume that $\|\theta(t)\| < M\varepsilon$ and the constant $M > 0$ is at our disposal. Let us denote the value of the function $\theta(t)$ at the zero point via $\tilde\theta \stackrel{\text{def}}{=} \theta(0)$. From the assumed smallness of the norm of the addition $\theta(t)$ it follows that the introduced constant $\tilde\theta$ satisfies the inequality $|\tilde\theta| < M\varepsilon$.

Consider the solution with the initial function $\psi(t) = \varphi(t)+\theta(t)$. We are interested in small ε, so we will assume that the inequality $\psi(t) < 0$ on $[-h,0]$ holds (since $\|\psi(t) - \varphi(t)\| < \delta(\varepsilon)$, then δ can be chosen from the condition $\delta < \min_{t\in[-h,0]} |\varphi(t)|)$. Then the function $\psi(t)$ differs from functions from the set S only by a "corrected" value at point zero: $\psi(0) = -d + \tilde\theta$. Consequently, the solution to the equation (3) with the initial function $\psi(t)$ differs from the solution with the initial function from the set S only by replacing the parameter $-d$ with $-d + \tilde{theta}$. Then at the first stage

$$r_\psi(t) = \begin{cases} \xi t - d + \theta, & 0 < t < t_*, \\ \eta(t - t_*) + \xi t_* - d + \theta, & t_* \leqslant t \leqslant T_*, \end{cases} \tag{43}$$
$$r_\psi(t + T_*) = r_\psi(t).$$

The fulfillment of $\xi t - d + \theta > 0$ follows from the constraint (29) and the smallness of ε. Note also that from the formulas (31) and (43) are true $|r_\varphi(t) - r_\psi(t)| = |\tilde\theta|$ for all $t \in [0, nT_*]$, therefore $\|r_\psi(t) - r_\varphi(t)\| = |\tilde\theta|$ at the first stage.

To estimate the norm of the difference between the values of the functions $r_\psi(t)$ and $r_\varphi(t)$ at the second stage it is necessary to obtain the value Δ_ψ. Let us carry out reasoning similar to paragraph 3.5 and obtain

$$\Delta_\psi = \gamma(\xi t_* - d + \tilde\theta).$$

We emphasize separately that the constraint (40) does not depend on the parameter d, so its execution remains true. The difference between the values of the functions $r_\psi(t)$ and $r_\varphi(t)$

increases with increasing parameter t at stage 2 (by construction), and therefore reaches a maximum at the end points of the stage, then there is

$$\|r_\varphi(t) - r_\psi(t)\| = \max_{t\in[nT_*, h+t^{(2)}_{n-1}]} |r_\varphi(t) - r_\psi(t)| \leqslant |\theta + \Delta - \Delta_\psi| \leqslant (1+\gamma)|\tilde{\theta}|.$$

Now let's look at the third stage. All values on it are strictly negative. Note that from the moment a periodic solution with only negative values is established, for both functions $r_\psi(t)$ and $r_\varphi(t)$ the difference in their values will remain constant. Until this moment, the influence of positive segments from the second stage can be no more than n pieces, and the lengths are strictly less than the lengths of segments with positive values of the function $r(t)$ of the first stage. Hence,

$$\|r_\varphi(t) - r_\psi(t)\| = \max_{t\in[h+t^{(2)}_{n-1}, 3h]} |r_\varphi(t) - r_\psi(t)| < (1+2\gamma)|\tilde{\theta}|.$$

It's obvious that

$$(1+2\gamma)|\tilde{\theta}| > (1+\gamma)|\tilde{\theta}| > |\tilde{\theta}|,$$

whence it follows that the constant M must be chosen based on restrictions on the norm of the difference between the functions $r_\psi(t)$ and $r_\varphi(t)$ at the third stage. So it is necessary

$$(1+2\gamma)|\tilde{\theta}| < (1+2\gamma)M\varepsilon < \varepsilon,$$

then the constant M must be chosen from the condition

$$1 + 2\gamma \leqslant \frac{1}{M},$$

for example, $M = \dfrac{1}{1+2\gamma}$.

Thus, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) = M\varepsilon$ such that the stability condition for the solution of the equation (3) is satisfied. Therefore, theorem 1 is completely proven.

3.7. Numerical results. In this section we provide illustrations of solutions obtained numerically in the "WOLFRAM MATHEMATICA" application for specific parameters that satisfy the found restrictions specified in the theorem 1.

For example, graphs of functions $r(t)$ and $R(t)$ for $d = 2$, $\tilde{\alpha} = 0.06$, $\tilde{\xi} = 2$, $t_* = 1$, $T_* = 2$, $n = 6$, $h = 12.2$, $\lambda = 1$ in Fig. 5.

Another example is the graphs of the functions $r(t)$ and $R(t)$ for $d = 2$, $\tilde{\alpha} = 2.5$, $\tilde{\xi} = 2$, $t_* = 1.5$, $T_* = 3$, $n = 6$, $h = 19.8$, $\lambda = 1$ in Fig. 6.

Conclusion

In this work, an extension of the results of [1] is obtained in the sense that the existence and stability of the solution is analytically shown and the range of parameters is found step by step for which the solution passes from high periodic bursts to small oscillations, that is, there are solutions of the form "freezing" neuron.

The theorem 1 has been proven for the equation (3). For the equation (1), theorem 2 has been proven, which is a consequence of theorem 1. The results of the work are numerically illustrated.

In the future, we plan to consider combining several "freezing" neurons into a network, including their interaction in various forms of the circuit: cycle, fully connected system, and

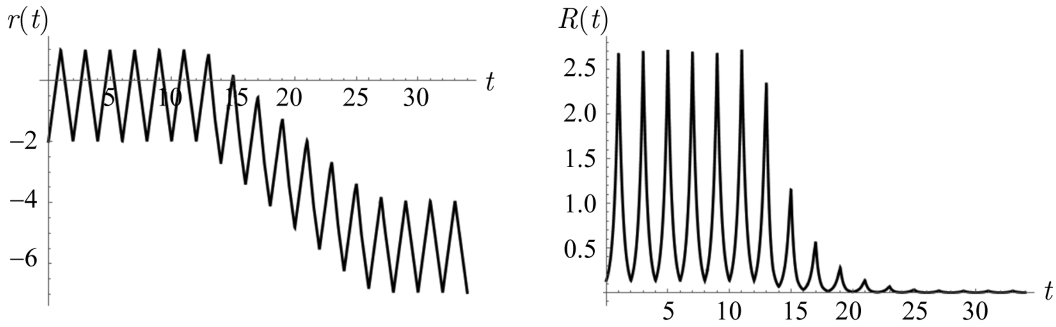

Fig 5. Graphs of functions $r(t)$ and $R(t)$ demonstrating the behavior of an "aging" neuron

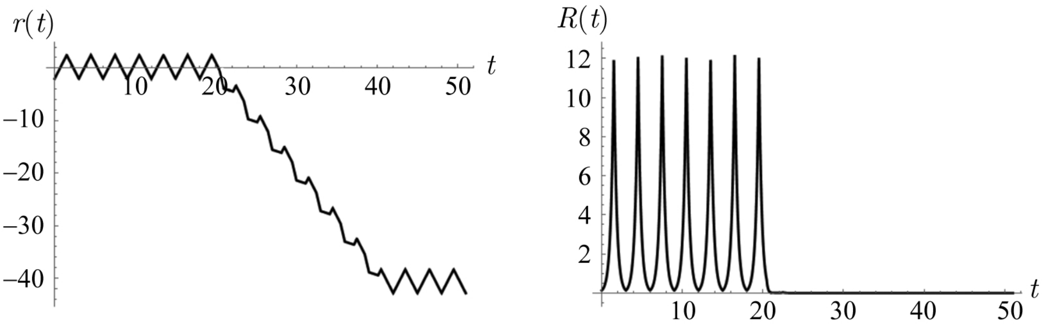

Fig 6. Graphs of functions $r(t)$ and $R(t)$ demonstrating the behavior of a "dying" neuron

others. For chains of "freezing" neurons, it is planned to study the question of the existence and stability of periodic solutions. This research could help create more accurate and efficient artificial neural networks. The work can also find application in mathematical biology and the theory of neural network modeling.

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